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8-DIMENSIONAL EINSTEIN-THORPE MANIFOLDS

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Abstract

We prove that a compact orientable Einstein-Thorpe manifold of dimension 8 that satisfies $6\chi = |P_2|$ must be flat.

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1. Introduction

The local geometry of a manifold provides us with information about its global topology. For instance, the generalized Gauss-Bonnet theorem [2, 5] states that the Euler-Poincaré characteristic χ of a compact oriented Riemannian manifold M^{4k} can be written as an integral

$$\chi = \frac{2}{V}\frac{[(2k)!]^2}{4k}\int_M \text{trace}\,(*R_{2k} * R_{2k})\, dV,$$

where V is the volume of the Euclidean unit $4k$-sphere, dV is the volume element of M, $*$ is the Hodge $*$-operator, and R_{2k} is called the $2k$th curvature operator. If R_{2k} commutes with $*$, that is, $R_{2k}* = *R_{2k}$, we call this condition a *Thorpe condition* and this metric a *Thorpe metric* and this manifold a *Thorpe manifold.* If Ricci curvature, ric, is a constant multiple of the metric, equivalently, traceless Ricci curvature, ric_0, vanishes then we call this condition an *Einstein condition* and this metric an *Einstein metric* and this manifold an *Einstein manifold.* In the 4-dimensional case, the Thorpe condition is equivalent to the Einstein condition [1]. And so, in 4 dimensions there is another way of stating the Einstein equation, namely the Thorpe condition. Moreover,

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in case of a compact oriented Einstein manifolds of dimension 4 with $2\chi = |P_1|$, Hitchin in [3] has classified these manifolds. On the other hand, Thorpe metrics need not be Einstein in dimensions higher than 4 and the following metrics satisfy the Thorpe condition but they are not Einstein metrics [4]:

(i) $S^{4k} \times H^{4k}$, product metric of standard metrics;
(ii) $CP^2 \times CH^2$, product metric of standard metrics.

The following examples also provide us with Einstein manifolds but not Thorpe manifolds [4]: the canonical quaternion projective space HP^n with $n \geq 3$.

We say that a Riemannian $4k$ manifold is *Einstein-Thorpe* if it is both Einstein and Thorpe. The purpose of the present note is to see what happens to Hitchin's result if both conditions are imposed in dimension eight.

THEOREM 1.1. *Suppose that (M^8, g) is a compact orientable Einstein-Thorpe manifold and that*

$$\chi = \left(\frac{2!2!}{4!}\right) |P_2|.$$

Then (M^8, g) must be a flat manifold.

The crucial ingredient in the proof is Lemma 1.1.

LEMMA 1.1. *Let (M, g) be a Riemannian manifold of dimension* 8. *Then*

$$\text{trace } R_4 = \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{\frac{1}{2}S^2 - 4\,|\text{ric}_0|^2 + 4\,|R|^2\right\}$$

where S is the scalar curvature, ric_0 *is the traceless Ricci curvature and R is the curvature.*

From Lemma 1.1 we can observe that trace R_4 is nonnegative when the Riemannian manifold is Einstein.

2. The pth curvature operator and Thorpe manifolds

Let M be a Riemannian manifold of dimension n and let $\bigwedge^p(M)$ denote the bundle of p-vectors of M; $\bigwedge^p(M)$ is a Riemannian vector bundle, with inner product on the fiber $\bigwedge^p(x)$ over the point x [4]. Let R denote the covariant curvature tensor of M. For each even $p > 0$, we define the pth curvature tensor R_p of M to be the covariant tensor field of order $2p$ given by

$$R_p(u_1, \dots, u_p, v_1, \dots, v_p) = \frac{1}{2^{p/2} p!} \sum_{\alpha, \beta \in S_p} \varepsilon(\alpha)\varepsilon(\beta) R(u_{\alpha(1)}, u_{\alpha(2)}, v_{\beta(1)}, v_{\beta(2)}) \cdots R(u_{\alpha(p-1)}, u_{\alpha(p)}, v_{\beta(p-1)}, v_{\beta(p)}),$$

where $u_i, v_j \in T_x M$ and S_p denotes the group of permutations of $(1, \dots, p)$ and, for $\alpha \in S_p$, $\varepsilon(\alpha)$ is the sign of the permutation α.

The tensor R_p has the following properties: it is alternating in the first p variables, alternating in the last p variables and it is invariant under the operation of interchanging the first p variables with the last p variables. Hence, at each point $x \in M$, R_p can be regarded as a symmetric bilinear form on $\bigwedge^p(x)$. By use of the inner product on $\bigwedge^p(x)$, R_p at x may then be identified with a self-adjoint linear operator R_p on $\bigwedge^p(x)$. Explicitly, this identification is given by

$$\langle R_p(u_1 \wedge \cdots \wedge u_p), v_1 \wedge \cdots \wedge v_p \rangle \equiv R_p(u_1, \dots, u_p, v_1, \dots, v_p)$$

with $u_i, v_j \in T_x M$. From now on, we use the same notations for the pth curvature operators and the pth curvature tensors. The tensor R_p satisfies the Bianchi identity which can be expressed in the following way [5]:

$$\operatorname{Alt} R_p = 0,$$

where Alt is the skew symmetrization operator given by

$$\operatorname{Alt} R_p(v_1, \dots, v_{2p}) = \frac{1}{(2p)!} \sum_{r \in S_{2p}} \varepsilon(r) R_p(v_{r(1)}, \dots, v_{r(2p)})$$

with $v_i \in T_x M$.

When n is a multiple of 4, $p = n/2$ and M is oriented, the Bianchi identity for R_p admits another interpretation in terms of the Hodge star operator on $\bigwedge^p(M)$:

$$\operatorname{Alt} R_p(e_1, \dots, e_n) = \frac{p!p!}{n!} \operatorname{trace} *R_p$$

and hence for the case $p = n/2$, the Bianchi identity for R_p reduces to

$$\operatorname{trace} *R_p = 0.$$

Taking $p = n$, the space $\bigwedge^n(x)$ is one dimensional and hence the self-adjoint linear operator $R_n : \bigwedge^n(x) \to \bigwedge^n(x)$ is a scalar multiple of the identity. More explicitly, when expressed globally, the line bundle homomorphism $R_n : \bigwedge^n(M) \to \bigwedge^n(M)$ is

$$R_n = KI$$

where I is the identity automorphism of $\bigwedge^n(M)$ and K is the Lipschitz-Killing curvature of M. Furthermore, for $x \in M$,

$$K(x) = R_n(e_1, \ldots, e_n, e_1, \ldots, e_n),$$

where $\{e_1, \ldots, e_n\}$ is any orthonormal basis for T_xM. The generalized Gauss-Bonnet theorem [5] expresses the Euler-Poincaré characteristic χ of a compact oriented Riemannian manifold of even dimension n as an integral

$$\chi = \frac{2}{c_n} \int_M K\, dV,$$

where K is the Lipschitz-Killing curvature of M, c_n is the volume of Euclidean unit n-sphere and dV is the volume element of M. Now we show that the Lipschitz-Killing curvature K of M can be expressed in terms of R_p and the Hodge $*$-operator. Let M be an oriented Riemannian manifold of even dimension n, then according to [5], the Lipschitz-Killing curvature K of M is the function whose value at $x \in M$ is

$$\frac{p!(n-p)!}{n!} \operatorname{trace}\left(*R_{n-p} * R_p\right).$$

For an oriented Riemannian manifold of dimension $n = 4k$, we can consider the middle curvature operator R_{2k}, and if this operator satisfies the condition

$$R_{2k}* = *R_{2k},$$

then, since $*^2 = \text{Identity}$, the trace formula for K reduces to

$$K = \frac{[(2k)!]^2}{(4k)!} \operatorname{trace} R_{2k}^2 \geq 0.$$

Next we consider a necessary condition for the existence of a Thorpe metric [5]:

THEOREM 2.1. *Let M be a compact orientable $4k$-dimensional Riemannian manifold which admits a Thorpe metric. Then*

$$\chi \geq \frac{k!k!}{(2k)!} |P_k|,$$

where χ is the Euler characteristic of M and P_k is the kth Pontrjagin number of M. In particular, $\chi \geq 0$. Furthermore, $\chi = 0$ if and only if $R_{2k} = 0$.

PROOF. The de Rham representation for the kth Pontrjagin class of M [2] is the differential $4k$-form

$$\frac{[(2k)!]^3}{(2^k k!)^2 (2\pi)^{2k}} \operatorname{trace}(R_{2k} * R_{2k})\, dV.$$

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Since R_{2k} commutes with $*$ it also commutes with $I \pm *$, where I denotes the identity operator on $\bigwedge^{2k}$. Hence $R_{2k}(I \pm *)$ is self-adjoint and

$$0 \leq \operatorname{trace}[R_{2k}(I \pm *)]^2 = 2\left[\operatorname{trace}(R_{2k})^2 \pm \operatorname{trace}(R_{2k} * R_{2k})\right],$$

and so

$$\operatorname{trace}(R_{2k})^2 \geq |\operatorname{trace}(R_{2k} * R_{2k})|.$$

This means that

$$\chi \geq \frac{k!k!}{(2k)!}|P_k|,$$

and since $K \geq 0$, we have $\chi = 0$ if and only if K is identically zero. $K \equiv 0$ is equivalent to $R_{2k} = 0$ and this completes the proof. □

3. The case of $\chi = \big((2!2!)/4!\big)|P_2|$

In this section we prove that a compact orientable Einstein-Thorpe manifold of dimension 8 that satisfies the above topological equality must be flat.

LEMMA 3.1. *Let (M, g) be a Riemannian manifold of dimension* 8. *Then*

$$\operatorname{trace} R_4 = \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{\frac{1}{2}S^2 - 4|\mathrm{ric}_0|^2 + 4|R|^2\right\}$$

where S is the scalar curvature, ric_0 *is the traceless Ricci curvature and R is the curvature.*

PROOF. For 4-forms $\{e_a \wedge e_b \wedge e_c \wedge e_d\}$ and with the Einstein summation,

$$\begin{aligned}\operatorname{trace} R_4 &= \frac{1}{2^2}R^{[ab}_{[ab}R^{cd]}_{cd]} = \frac{1}{2^2}R^{ab}_{[ab}R^{cd}_{cd]}\\ &= \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{R^{ab}_{ab}R^{cd}_{cd} + R^{ab}_{ac}R^{cd}_{db} + R^{ab}_{ad}R^{cd}_{bc} + R^{ab}_{bc}R^{cd}_{ad} + R^{ab}_{bd}R^{cd}_{ca} + R^{ab}_{cd}R^{cd}_{ab}\right\}\end{aligned}$$

where [ ] is a skew symmetrization, and $\{e_k\}_{k=1}^{8}$ is an orthonormal frame. We analyze the terms of this sum individually:

(i) $R^{ab}_{ab}R^{cd}_{cd} = 1/2S^2 - 4\,\mathrm{ric}^c_{0c}\,\mathrm{ric}^c_{0c} + 2R^{cd}_{cd}R^{cd}_{cd}$;
(ii) $R^{ab}_{ac}R^{cd}_{db} = -\mathrm{ric}^b_{0c}\,\mathrm{ric}^c_{0b} + R^{db}_{dc}R^{dc}_{db}$.

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Thus we obtain

$$\operatorname{trace} R_4 = \frac{1}{2^2}\left(\frac{1}{6}\right)\left\{\frac{1}{2}S^2 - 4\operatorname{ric}^c_{0c}\operatorname{ric}^c_{0c} - 4\operatorname{ric}^b_{0c}\operatorname{ric}^c_{0b} + 2R^{cd}_{cd}R^{cd}_{cd} + 4R^{db}_{dc}R^{dc}_{db} + R^{ab}_{cd}R^{cd}_{ab}\right\}$$

and this completes the proof. □

Now we are ready to prove the main result.

THEOREM 3.1. *Suppose that (M^8, g) is a compact orientable Einstein-Thorpe manifold and that*

$$\chi = \left(\frac{2!2!}{4!}\right)|P_2|.$$

Then (M^8, g) must be a flat manifold.

PROOF. By Theorem 2.1, we see that the above topological condition together with the Thorpe condition can be expressed as

$$\operatorname{trace} R_4 R_4 = |\operatorname{trace} R_4 * R_4|.$$

We consider any orthonormal basis $\{A_i\}_{i=1}^{14}$ in $\bigwedge^+(M^8)$, and any orthonormal basis $\{B_i\}_{i=1}^{14}$ in $\bigwedge^-(M^8)$, where $\bigwedge^+(M^8)$ and $\bigwedge^-(M^8)$ denote the self dual space and the anti-self-dual space with respect to the Hodge $*$ operator, respectively. Then we have

$$\langle R_4(A_i), R_4(A_i)\rangle = \sum_{j=1}^{14}|R_4(A_i, A_j)|^2 + \sum_{j=1}^{14}|R_4(A_i, B_j)|^2,$$

$$\langle R_4(B_i), R_4(B_i)\rangle = \sum_{j=1}^{14}|R_4(B_i, B_j)|^2 + \sum_{j=1}^{14}|R_4(B_i, A_j)|^2,$$

$$\langle *R_4(A_i), R_4(A_i)\rangle = \sum_{j=1}^{14}|R_4(A_i, A_j)|^2 - \sum_{j=1}^{14}|R_4(A_i, B_j)|^2,$$

$$\langle *R_4(B_i), R_4(B_i)\rangle = \sum_{j=1}^{14}|R_4(B_i, A_j)|^2 - \sum_{j=1}^{14}|R_4(B_i, B_j)|^2,$$

for each $i = 1, 2, \ldots, 14$.

If we assume $\operatorname{trace} R_4 * R_4 \geq 0$, then by the given condition

$$R_4(A_i, B_j) = R_4(B_i, B_j) = 0 \quad \text{for } i, j = 1, 2, \ldots, 14$$

and this means that

$$R_4^- = \frac{R_4 - *R_4}{2} \equiv 0.$$

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Furthermore, by the Bianchi identity,

$$\operatorname{trace} *R_4 \equiv 0,$$

and so we obtain

$$\operatorname{trace} R_4 \equiv 0.$$

However, by Lemma 3.1 the Einstein condition (that means $\mathrm{ric}_0 = 0$) implies $\operatorname{trace} R_4 \geq 0$ and equality holds when its metric is flat and so we conclude that the given metric is flat.

On the other hand, if we assume

$$\operatorname{trace} R_4 * R_4 \leq 0,$$

then we can repeat the above argument with a different choice of sign and this completes the proof. □

COROLLARY 3.1. (i) *The product manifold of* T^4 *with any compact orientable hyperbolic manifold of dimension* 4 *does not admit an Einstein-Thorpe metric.*

(ii) *The product manifold of* T^4 *with any compact complex hyperbolic manifold of complex dimension* 2 *does not admit an Einstein-Thorpe metric.*
The manifolds described in (i) *and* (ii) *satisfy* $\chi = 0$ *and* $P_2 = 0$.

PROOF. It is easy to see that the manifolds described in part (i) and (ii) satisfy $\chi = 0$ and $P_2 = 0$. This implies that any Einstein-Thorpe metric on the manifolds described in (i) and (ii) must be flat by Theorem 3.1, hence a contradiction. □

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