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NILPOTENCY INDICES OF THE RADICALS OF p-GROUP ALGEBRAS

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Dedicated to Professor Kazuo Kishimoto on his 60th birthday

Let k be a field of characteristic $p>0$. We classify all finite p-groups G satisfying the inequality $p^{-2}|G|\leqq t(G)<p^{-1}|G|$, where $t(G)$ is the nilpotency index of the Jacobson radical of $k[G]$.

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1. Introduction

Let p be a prime number, G a finite p-group of order p^m and k a field of characteristic p. Denote by $t(G)$ the nilpotency index of the Jacobson radical of $k[G]$, the group algebra of G over k.

It is well known (see [**8**], for example) that

(A) $t(G)\leqq p^m$. Moreover $t(G)=p^m$ if and only if G is cyclic.

Let us assume that G is noncyclic and denote by $\exp G$ the exponent of G. Then $t(G)<p^m$ and it is known, by a result of Koshitani [**6**], that

(B) $p^{m-1}<t(G)<p^m$ if and only if $\exp G=p^{m-1}$. Moreover in this case $t(G)=p^{m-1}+p-1$.

We know also, by a result of Motose [**7**], that

(C) $t(G)=p^{m-1}$ if and only if G is either an elementary abelian 2-group of order 2^3 or $M(3)$, the nonabelian 3-group of order 3^3 and exponent 3.

The purpose of this note is to classify all finite p-groups G satisfying the inequality $p^{m-2}\leqq t(G)<p^{m-1}$. In a recent paper [**10**], Shalev showed that if $p\geqq 7$, then $t(G)\geqq p^{m-2}$ if and only if $\exp G\geqq p^{m-2}$. Therefore:

(D) If $p\geqq 7$, then $p^{m-2}<t(G)<p^{m-1}$ if and only if $\exp G=p^{m-2}$.

Now we are interested in the case when $p<7$. We know that in this case there exist five non-isomorphic groups G with $\exp G < p^{m-2}$ and $p^{m-2} < t(G) < p^{m-1}$. One of them is an elementary abelian 2-group of order 2^4. All the nonabelian p-groups of order p^4 are given by Burnside [1]. We know, by his result, that if $p \neq 2$ then two of them, which we denote by P and Q, are of exponent p:

$P = M(p) \times C_p$ (where $M(p)$ is the nonabelian p-group of order p^3 and exponent p, and C_p is a cyclic group of order p);

$Q = \langle a, b, c, d \rangle$, $p \geqq 5$, where $a^p = b^p = c^p = d^p = 1$, $[a,b] = [a,c] = [a,d] = [b,c] = 1$, $[c,d] = b$, $[b,d] = a$.

Then $t(P) = 5(p-1)+1$, $t(Q) = 7(p-1)+1$, and we see that

$$3^2 < t\langle P \rangle = 11 < 3^3 \text{ (for } p = 3),$$

$$5^2 < t(Q) = 29 < 5^3 \text{ (for } p = 5).$$

All the non-isomorphic 2-groups of order 2^5 are given by Hall and Senior [2]. We know, by their description, that all such groups of exponent 4 generated by two elements satisfy our condition. We have only two such groups, which we denote by R and S:

$R = \langle a, b, c \rangle$, $a^2 = b^4 = c^4 = 1$, $[a,b] = [a,c] = 1$, $[b,c] = a$;

$S = \langle a, b, c, d, e \rangle$, $a^2 = b^2 = c^2 = d^2 = 1$, $e^2 = c$, $[a,b] = [a,c] = [a,d] = [a,e] = [b,c] = [b,d] = 1$, $[b,e] = [c,d] = a$, $[d,e] = b$.

It is not difficult to see that $t(R) = 9$ and $t(S) = 10$. In this note, we shall show that the above five groups are all the p-groups such that $\exp G < p^{m-2}$ and $p^{m-2} < t(G) < p^{m-1}$. More precisely, we shall prove the following theorem.

Theorem 1. *Let G be a finite p-group of order p^m, where $m \geqq 3$. Then $p^{m-2} < t(G) < p^{m-1}$ if and only if one of the following seven cases holds*:

(1) $p \neq 2$, $\exp G = p^{m-2}$, $G \not\simeq M(3)$;

(2) $p = 3$, $m = 4$, $G \simeq M(3) \times C_3$;

(3) $p = 5$, $m = 4$, $G \simeq Q$;

(4) $p = 2$, $m \geqq 4$, $\exp G = 2^{m-2}$;

(5) $p=2$, $m=4$, $G \simeq C_2 \times C_2 \times C_2 \times C_2$;

(6) $p=2$, $m=5$, $G \simeq R$;

(7) $p=2$, $m=5$, $G \simeq S$.

Moreover, we shall give all the finite p-groups G with $t(G)=p^{m-2}$. Denote by $\Phi(G)$ the Frattini subgroup of G.

Theorem 2. *Let G be a finite p-group of order p^m, where $m \geqq 3$. Then $t(G)=p^{m-2}$ if and only if*

(1) $p=3$, $m=4$, $G \simeq C_3 \times C_3 \times C_3 \times C_3$; *or*

(2) $p=2$, $m=5$, $\exp G=2^2$ *and* $|G/\Phi(G)|=2^3$.

Let us note that if a group G satisfies condition (2) of Theorem 2, then G is either an abelian 2-group of type (2,2,1) or one of the fourteen nonabelian groups described in Section 3 (See Remark).

2. Preliminaries

To compute the nilpotency index of the Jacobson radical of the modular p-group algebra, Jennings' formula given in [4, Theorem 3.7] (see also [5]) is very useful. Let us recall this formula.

Let $\{\gamma_i(G)\}$ be the lower central series of G, that is, $\gamma_i(G)$ is defined inductively by

$$\gamma_1(G)=G,\ \gamma_{i+1}(G)=[\gamma_i(G),G] \quad \text{for} \quad i \geqq 1.$$

Denote by G^{p^i} the subgroup of G generated by $\{g^{p^i} \mid g \in G\}$, and let $\{\kappa_n(G)\}$ be the sequence defined by

$$\kappa_n(G)=\prod_{ip^j \geqq n} \gamma_i(G)^{p^j}.$$

Moreover, let $l(G)$ be the smallest integer such that $\kappa_{l(G)+1}(G)=\{1\}$ and put $|\kappa_n(G)/\kappa_{n+1}(G)|=p^{e_n}$, $1 \leqq n \leqq l(G)$. Jennings' formula for $t(G)$ is as follows:

$$t(G)=1+(p-1)\sum_{n=1}^{l(G)} ne_n.$$

Now let G be a finite p-group of order p^m. Suppose that $\exp G=p^{m-2}$. If G is abelian then G is either of type $(m-2,\ 2)$ or of type $(m-2,\ 1,\ 1)$, and correspondingly $t(G)=p^{m-2}+p^2-1$ or $p^{m-2}+2(p-1)$. Now we assume that G is nonabelian. If G is metacyclic then $t(G)=p^{m-2}+p^2-1$ (see [6,7]). In [9], we classified all the finite p-groups of order p^m and exponent p^{m-2}. This result implies that $|G/\Phi(G)|=p^2$ or p^3. Applying the above Jennings' formula to each group listed in [9, Theorems 1 and 2], we have $t(G)=p^{m-2}+3(p-1)$ if $|G/\Phi(G)|=p^2$ and G is nonmetacyclic; and $t(G)=p^{m-2}+2(p-1)$ if $|G/\Phi(G)|=p^3$. So we have the following:

Proposition 1. *Suppose that G is a finite p-group of order p^m and exponent p^{m-2}, where $m \geqq 3$.*

(1) *If G is metacyclic, then $t(G)=p^{m-2}+p^2-1$.*

(2) *If G is not metacyclic, but $|G/\Phi(G)|=p^2$, then $t(G)=p^{m-2}+3(p-1)$.*

(3) *If $|G/\Phi(G)|=p^3$, then $t(G)=p^{m-2}+2(p-1)$.*

3. Proofs of Theorems 1 and 2

We see, by Shalev's result (*D*), that if $p \geqq 7$ then Theorem 1 holds. In our proof of Theorem 1, we shall not use this fact. First we prove Theorems 1 and 2 in the case where G is abelian. For this aim we need the following lemma.

Lemma 1. *Let G be an abelian p-group of order p^m and exponent p^{m-3}, where $m \geqq 5$ provided $p=2$. Then $t(G) \leqq p^{m-2}$. Moreover $t(G)=p^{m-2}$ if and only if G is either an elementary abelian 3-group of order 3^4 or an abelian 2-group of type $(2,2,1)$.*

Proof. Assume that $p \neq 2$ and let A be a cyclic subgroup of G of order p^{m-3}. If G/A is cyclic then $t(G)=p^{m-3}+p^3-1<p^{m-2}$. If G/A is of type $(2,1)$ then

$$t(G)=p^{m-3}+p^2+p-2<p^{m-2}.$$

If G/A is elementary abelian then $t(G)=p^{m-3}+3(p-1) \leqq p^{m-2}$. In the last case, we see that if $t(G)=p^{m-2}$ then $p=3$ and $m=4$. Therefore our lemma is proved for the case $p \neq 2$. When $p=2$ we use a similar argument.

Now we are able to prove Theorem 1 in the case where G is abelian.

Proposition 2. *Let G be an abelian p-group of order p^m, where $m \geqq 3$. Then the following properties are equivalent:*

(1) $p^{m-2}<t(G)<p^{m-1}$.

(2) (i) $\exp G=p^{m-2}$ *and* $m \geqq 4$ *if* $p=2$; *or*

(ii) $G \simeq C_2 \times C_2 \times C_2 \times C_2$.

Proof. The implication $(2) \Rightarrow (1)$ is obvious. Assume now that (1) holds. Then by (A) and (B), $\exp G \leqq p^{m-2}$. Let $p=2$. If $m=3$ then G is elementary abelian and $t(G)=4$, but it is not our case. If $m=4$, then $\exp G \leqq 2^2$, so we see that (2) holds. Therefore we must show that if either $p \neq 2$, $m \geqq 3$, or $p=2$, $m \geqq 5$, then $\exp G=p^{m-2}$. For this aim, we use induction on m. If $p \neq 2$ and $m=3$ then $\exp G=p$. So assume $p=2$ and $m=5$. If exp $G \leqq 2^2$ then it is easy to see that $t(G) \leqq 2^3$. This shows that $\exp G=2^3$. Suppose that

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$m \geqq 4$ if $p \neq 2$, and $m \geqq 6$ if $p=2$. Let z be an element of G of order p. Then by [11, Theorem 2.4], $p \cdot t(G/\langle z \rangle) \geqq t(G) > p^{m-2}$, and consequently $t(G/(z\rangle) > p^{m-3}$. Moreover, $t(G/\langle z \rangle) < t(G) < p^{m-1}$. So we have $p^{m-3} < t(G/\langle z \rangle) < p^{m-1}$. Assume now that $t(G/\langle z \rangle) > p^{m-2}$. Then (because $|G/\langle z \rangle| = p^{m-1}$), $\exp G/\langle z \rangle = p^{m-2}$ by (B), and so $\exp G = p^{m-2}$. Since $t(G/\langle z \rangle) \neq p^{m-2}$ by (C), it is enough to prove that if $p^{m-3} < t(G/\langle z \rangle) < p^{m-2}$ then $\exp G = p^{m-2}$. In this case, by the induction hypothesis, we have $\exp G/\langle z \rangle = p^{m-3}$, which implies $\exp G \geqq p^{m-3}$. Hence, by Lemma 1, we obtain exp $G = p^{m-2}$. This completes the proof.

Corollary 1. *Let G be an abelian p-group of order p^m, where $m \geqq 3$. Then the following properties are equivalent:*

(1) $t(G) = p^{m-2}$.

(2) (i) $G \simeq C_3 \times C_3 \times C_3 \times C_3$; *or*

(ii) $G \simeq C_4 \times C_4 \times C_2$.

Proof. It suffices to prove that (1) implies (2). Suppose first $p \neq 2$. Then we have $\exp G < t(G) = p^{m-2}$, and so $m \geqq 4$. If $m=4$, then $p^2 = t(G) = 4(p-1)+1$, because G is elementary abelian, which forces p to be 3, and (i) follows. Therefore we must prove that if $m \geqq 5$ then $t(G) \neq p^{m-2}$. We proceed by induction on m. If $m=5$, then $\exp G \leqq p^2$ and so

$$t(G) = 2p^2 + p - 2 \quad \text{or} \quad p^2 + 3(p-1) \quad \text{or} \quad 5(p-1)+1.$$

Hence $t(G) \neq p^3$. Now let $m > 5$ and assume that $t(H) \neq p^{m-3}$ for any abelian group H of order p^{m-1}. Suppose by way of contradiction that there exists an abelian group G of order p^m such that $t(G) = p^{m-2}$. Choose an element z of order p in G. Then we have $p^{m-3} \leqq t(G/\langle z \rangle) < p^{m-2}$ and by the induction hypothesis $t(G/\langle z \rangle) \neq p^{m-3}$. Hence by Proposition 2, $\exp G/\langle z \rangle = p^{m-3}$, which yields $\exp G = p^{m-3}$ because $\exp G \leqq p^{m-3}$, and so $t(G) < p^{m-2}$ by Lemma 1, a contradiction. Thus the corollary is proved for the case p odd. When $p=2$, we use a similar argument.

In the rest of the paper, we denote by $cl(G)$ the class of G, and by $Z(G)$ the centre of G. Now, using the classification of finite p-groups of order $\leqq p^6$ (Hall and Senior [2] and James [3]), we shall prove the following three lemmas.

Lemma 2. *Let G be a 2-group of order 2^5 and exponent at most 2^2. Then the following hold:*

(1) *If* $\exp G = 2$ *then* $t(G) = 6$.

(2) *If* $\exp G = 2^2$ *then*

$$t(G) = \begin{cases} 7 & \text{if} \quad |G/\Phi(G)| = 2^4, \\ 8 & \text{if} \quad |G/\Phi(G)| = 2^3, \quad \text{and} \end{cases}$$

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$$t(G)=\begin{cases}9 & \text{if } |G/\Phi(G)|=2^2, \quad cl(G)=2,\\ 10 & \text{if } |G/\Phi(G)|=2^2, \quad cl(G)=3.\end{cases}$$

Lemma 3. *Let G be a 2-group of order 2^6 and exponent at most 2^3. Then $t(G)<2^4$.*

Lemma 4. *Let $p\neq 2$, and G a p-group of order p^5 and exponent at most p^2. Then $t(G)<p^3$.*

Proof of Lemma 2. If $\exp G=2$, G is elementary abelian, and so $t(G)=6$. Assume that $\exp G=2^2$. Since $\kappa_2(G)=\Phi(G)$, we have $2^{e_1}=|G/\Phi(G)|$, and so if G is abelian then $e_1=4$ or 3, and correspondingly $t(G)=7$ or 8. Hence the lemma holds for abelian groups. Now let G be nonabelian. Then G belongs to one of the families: Γ_2, Γ_4, Γ_5, Γ_7 (see [**2**]). If G belongs to Γ_2, Γ_4 or Γ_5 then $cl(G)=2$ and $\gamma_2(G)^2=\{1\}$. Therefore $l(G)=2$ and $(e_1,e_2)=(4,1)$ or $(3,2)$ or $(2,3)$; and correspondingly $t(G)=7$ or 8 or 9. On the other hand, in the family Γ_7, there is only one group G of order 2^5 and exponent 2^2. For this group, $e_1=2$, and $t(G)=10$, and we know that $cl(G)=3$. This completes the proof of Lemma 2.

Remark. There are twenty-one types of nonabelian 2-groups of order 2^5 and exponent 2^2. Five of them given below satisfy $t(G)=7$:

$$32\Gamma_2a_1, 32\Gamma_2a_2, 32\Gamma_2b, 32\Gamma_5a_1, 32\Gamma_5a_2.$$

The following fourteen groups satisfy $t(G)=8$:

$$32\Gamma_2c_1, 32\Gamma_2c_2, 32\Gamma_2e_1, 32\Gamma_2e_2, 32\Gamma_2f, 32\Gamma_4a_1, 32\Gamma_4a_2,$$

$$32\Gamma_4a_3, 32\Gamma_4b_1, 32\Gamma_4b_2, 32\Gamma_4c_1, 32\Gamma_4c_2, 32\Gamma_4c_3, 32\Gamma_4d.$$

The last two are the groups we presented in Section 1: $R=32\Gamma_2h$ with $t(R)=9$ and $S=32\Gamma_7a_1$ with $t(S)=10$.

Proof of Lemma 3. If $\exp G=2$, G is elementary abelian, and so $t(G)=7$. Suppose next that $\exp G=2^2$. If G is abelian then G is of type $(2,2,2)$ or $(2,2,1,1)$ or $(2,1,1,1,1)$, and we see that $t(G)\leqq 10$. Suppose G is nonabelian. Then G belongs to one of the families: Γ_2, Γ_4, Γ_5, Γ_7, Γ_9, Γ_{10}, Γ_{11}, Γ_{13}, Γ_{23}, Γ_{25}. If G belongs to a family other than Γ_{23}, then $cl(G)\leqq 3$, $\gamma_2(G)^2=\{1\}$ and $|\gamma_3(G)|\leqq 2$. Hence $\kappa_3(G)=\{1\}$ if $cl(G)=2$; and $\kappa_3(G)\simeq C_2$, $\kappa_4(G)=\{1\}$ if $cl(G)=3$. So it follows that $t(G)<2^4$. If G belongs to Γ_{23} then $|\kappa_2(G)|=2^4$ and $\kappa_3(G)=\gamma_3(G)\simeq C_2\times C_2$, $\kappa_4(G)=\gamma_4(G)\simeq C_2$, $\kappa_5(G)=\{1\}$, and so $t(G)=14$. Therefore the lemma holds if $\exp G\leqq 2^2$. Finally, consider the case $\exp G=2^3$. If G is abelian then G is of type (3,3) or (3,2,1) or (3,1,1,1), and so $t(G)\leqq 15$. So assume G is nonabelian. Then G belongs to one of the families: $\Gamma_2,\ldots,\Gamma_7$, Γ_{12}, $\Gamma_{14},\ldots,\Gamma_{18}$, Γ_{22}, Γ_{23}, Γ_{24}, Γ_{26}, and so $cl(G)\leqq 4$, $\gamma_2(G)^4=\gamma_3(G)^2=\{1\}$. This shows that $l(G)=4$. Because,

$e_2 \neq 0$, and $e_4 = 1$ if $e_2 = 1$ ([**10**, Corollary 1.5, Theorem 1.12(ii)]), noting that $e_1 = 2$, 3 or 4, we have the following possibilities:

$$(e_1, e_2, e_3. e_4) = (2,1,2,1) \text{ or } (2,2,1,1) \text{ or } (2,2,0,2) \text{ or } (3,1,1,1) \text{ or } (3,2,0,1) \text{ or } (4,1,0,1).$$

This implies that $t(G) < 2^4$ and Lemma 3 is proved.

Proof of Lemma 4. If G is abelian, then it is easy to see that

$$t(G) \leqq t(C_{p^2} \times C_{p^2} \times C_p) = 2p^2 + p - 2 < p^3.$$

Assume G is nonabelian. Then G belongs to one of the families: $\Phi_2, \ldots, \Phi_7$, Φ_9, Φ_{10} (see [3]). If G belongs to a family other than Φ_9 and Φ_{10} then $cl(G) \leqq 3$ and $\gamma_2(G)^p = \{1\}$. So we see that $l(G) \leqq p$, and by Jennings' formula, we conclude that

$$t(G) \leqq (1 \cdot 2 + p \cdot 3)(p-1) + 1 < p^3.$$

If G belongs to Φ_9 or Φ_{10} then $cl(G) = 4$ and $\gamma_2(G)^p = \{1\}$. Therefore, if $p > 3$ then $l(G) \leqq p$, so $t(G) < p^3$ again; while if $p = 3$ then $l(G) = 4$, $\kappa_4(G) \simeq C_3$, and we have

$$t(G) \leqq (1 \cdot 2 + 3 \cdot 2 + 4 \cdot 1)(3-1) + 1 = 25 < 3^3.$$

Thus Lemma 4 is proved.

Lemma 5. *Let G be a nonabelian p-group of order p^m, and let z be an element of order p lying in $Z(G) \cap \kappa_{l(G)}(G)$. Assume* $\exp G/\langle z \rangle = p^{m-3}$.

(1) *If $m \geqq 7$ then* $\exp G = p^{m-2}$.

(2) *If $m = 6$, $p \neq 2$ and $t(G) \geqq p^4$ then* $\exp G = p^4$.

Proof. (1) Suppose the result is false. Then there exists a p-group G of order p^m with $m \geqq 7$ such that $\exp G = \exp G/\langle z \rangle = p^{m-3}$. Since $G/\langle z \rangle$ is of order p^{m-1}, it is either an abelian group of type $(m-3, 2)$ or $(m-3, 1, 1)$, or isomorphic to one of the groups listed in [**9**, Theorems 1 and 2]. Because $m \geqq 7$, in either case, we have $\kappa_{p^2+1}(G) = G^{p^3} = \langle a^{p^3} \rangle$, where a is an element of G such that $a\langle z \rangle$ ($\in G/\langle z \rangle$) is of order p^{m-3}. But, because $\exp G = p^{m-3}$, $\langle a \rangle$ does not contain z, and so $\kappa_{p^2+1}(G) \not\ni z$, which contradicts the choice of z. Thus (1) is proved.

(2) Suppose the result is false. Then $\exp G = p^3$ and $G/\langle z \rangle$ is either an abelian group of type $(3,2)$ or $(3,1,1)$; or isomorphic to one of the groups: $G_1, G_2, \ldots, G_9$ given in [**9**, Theorem 1]. Hence it follows that $cl(G) \leqq 4$ and $\gamma_2(G)^{p^2} = \gamma_3(G)^p = \{1\}$, and so $l(G) = p^2$. Because $e_p \neq 0$, and $e_{p^2} = 1$ if $e_p = 1$ ([**10**]), noting that $e_1 = 2$ or 3, we have the following possibilities:

$$(e_1, e_p, e_{p^2}) = (2,1,1) \text{ or } (2,2,1) \text{ or } (2,2,2) \text{ or } (2,3,1) \text{ or } (3,1,1) \text{ or } (3,2,1).$$

This together with Jennings' formula implies that $t(G)$ does not exceed $(1\cdot 2+p\cdot 1+p^2\cdot 3)(p-1)+1<p^4$. This contradicts our assumption. Thus (2) is proved.

The next proposition is our Theorem 1 in the case when G is nonabelian.

Proposition 3. *Let G be a nonabelian p-group of order p^m, where $m\geqq 3$. Then the following properties are equivalent*:

(1) $p^{m-2}<t(G)<p^{m-1}$.

(2) *One of the following holds*:

(i) $\exp G=p^{m-2}$, *where* $(p,m)\neq(3,3)$;

(ii) $p=3$, $m=4$, $G\simeq M(3)\times C_3$;

(iii) $p=5$, $m=4$, $G\simeq Q$;

(iv) $p=2$, $m=5$, $G\simeq R$;

(v) $p=2$, $m=5$, $G\simeq S$.

Proof. Obviously (2) implies (1). Suppose (1) holds. Then $\exp G\leqq p^{m-2}$. Therefore, if $m=3$ then $\exp G=p$, and so, as G is nonabelian, p is odd and $G\simeq M(p)$. But then $t(G)=4p-3$. Hence the inequality $p^2>t(G)$ yields $p\neq 3$. Assume $m=4$. Then $\exp G\leqq p^2$ and we already know that (i), (ii) or (iii) holds in this case. Further if $p=2$ and $m=5$, by Proposition 1 and Lemma 2 (see also Remark), (i), (iv) or (v) holds. Therefore it suffices to prove that if either $p\neq 2$, $m\geqq 5$; or $p=2$, $m\geqq 6$, then $\exp G=p^{m-2}$. We proceed by induction on m. By Lemmas 3 and 4, the cases $p\neq 2$, $m=5$ and $p=2$, $m=6$ are done. Suppose $m>5$ if $p\neq 2$, and $m>6$ if $p=2$, and let z be an element of order p lying in $Z(G)\cap\kappa_{t(G)}(G)$. Then

$$p\cdot t(G/\langle z\rangle)\geqq t(G)>p^{m-2},\quad t(G/\langle z\rangle)<t(G)<p^{m-1},$$

and so $p^{m-3}<t(G/\langle z\rangle)<p^{m-1}$. If $t(G/\langle z\rangle)>p^{m-2}$, then $\exp G/\langle z\rangle=p^{m-2}$ by (B), and so $\exp G=p^{m-2}$ as desired. Since $t(G/\langle z\rangle)\neq p^{m-2}$ by (C), it remains only to show that if $p^{m-3}<t(G/\langle z\rangle)<p^{m-2}$ then $\exp G=p^{m-2}$. In this case, we have $\exp(G/\langle z\rangle)=p^{m-3}$; because if $G/\langle z\rangle$ is abelian, this follows from Proposition 2, and if $G/\langle z\rangle$ is nonabelian, this follows from the induction hypothesis. Therefore $\exp G=p^{m-2}$ by Lemma 5, and Proposition 3 is proved.

Corollary 2. *Let G be a nonabelian p-group of order p^m. Then the following properties are equivalent*:

(1) $t(G)=p^{m-2}$.

(2) $|G|=2^5$, $\exp G=2^2$, $|G/\Phi(G)|=2^3$.

Proof. The implication (2)$\Rightarrow$(1) follows from Lemma 2. Suppose (1) holds. Since $\exp G<t(G)=p^{m-2}$ and G is nonabelian, we see that $m\geqq 4$ if $p\neq 2$, and $m\geqq 5$ if $p=2$. Let $p=2$. If $m=5$, (2) follows from Lemma 2. Further, if $m=6$ then $t(G)\neq 2^4$ by Lemma 3. We next assume that $p\neq 2$. If $m=4$ then $\exp G=p$ and $G\simeq M(p)\times C_p$ or Q. We already

know that $t(G) \neq p^2$ in either case. If $m=5$ then $t(G) \neq p^3$ by Lemma 4. Therefore it suffices to prove that if either $p=2$ and $m \geqq 7$, or $p \neq 2$ and $m \geqq 6$, then $t(G) \neq p^{m-2}$. Suppose that it is false and let G be a nonabelian p-group of minimal order satisfying $t(G)=p^{m-2}$. Let z be an element of order p lying in $Z(G) \cap \kappa_{l(G)}(G)$. Then by $p^{m-2}=t(G) \leqq p \cdot t(G/\langle z \rangle)$, we get $p^{m-3} \leqq t(G/\langle z \rangle)$. Suppose now $t(G/\langle z \rangle)=p^{m-3}$. Then by Lemmas 3 and 4, we have $p=2$, $m>7$ or $p \neq 2$, $m>6$, and $G/\langle z \rangle$ is abelian by the minimality of G. But, by Corollary 1, this is impossible. Hence $p^{m-3}<t(G/\langle z \rangle)$. Now the inequality $t(G/\langle z \rangle)<t(G)=p^{m-2}$ implies $p^{m-3}<t(G/\langle z \rangle)<p^{m-2}$. Therefore by Propositions 2 and 3, $\exp G/\langle z \rangle=p^{m-3}$, and so $\exp G=p^{m-2}$ by Lemma 5, a contradiction. Thus the corollary is proved.

Theorem 1 now follows from Propositions 2 and 3; and Theorem 2 follows from Corollaries 1 and 2.

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