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HARDY'S THEOREM FOR GABOR TRANSFORM

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Abstract

Hardy's uncertainty principle for the Gabor transform is proved for locally compact abelian groups having noncompact identity component and groups of the form $\mathbb{R}^n \times K$, where K is a compact group having irreducible representations of bounded dimension. We also show that Hardy's theorem fails for a connected nilpotent Lie group G which admits a square integrable irreducible representation. Further, a similar conclusion is made for groups of the form $G \times D$, where D is a discrete group.

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1. Introduction

Hardy's uncertainty principle states that a nonzero integrable function f on $\mathbb{R}$ and its Fourier transform $\widehat{f}$ cannot both be compactly supported. For $f \in L^2(\mathbb{R})$, the Fourier transform $\widehat{f}$ is given by

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i \xi x}\, dx.$$

The following theorem of Hardy (see [7] for the proof) makes the above statement more precise.

THEOREM 1.1. *Let f be a measurable function on $\mathbb{R}$ such that:*

(1) $|f(x)| \le Ce^{-a\pi x^2}$ *for all* $x \in \mathbb{R}$;
(2) $|\widehat{f}(\xi)| \le Ce^{-b\pi \xi^2}$ *for all* $\xi \in \mathbb{R}$,

where a, b and C are positive constants. If $ab > 1$, then $f = 0$ a.e. (almost everywhere).

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Hardy's theorem has been proved for the Fourier transform in the setting of $\mathbb{R}^n$ and the Heisenberg group $\mathbb{H}_n$ (see [16]), locally compact abelian groups and some classes of solvable Lie groups (see [2]), the Euclidean motion group (see [15, 17]), nilpotent Lie groups (see [1, 9, 12, 13, 18]) and noncompact connected semisimple Lie groups with finite centre (see [14]). For a detailed survey of the uncertainty principles for the Fourier transform, refer to [5].

In recent decades, the Fourier transform was an indispensable tool in applied mathematics, especially in signal processing. It has also been recognised that the global Fourier transform is of little practical value in analysing the frequency spectrum of a long signal. So, there is a necessity of the notion of frequency analysis that is local in time, in other words, a *joint time–frequency analysis*. In recent times, the *Gabor transform* is one of the tools that has established itself in this direction. The approach used in this technique is cutting the signal into segments using a smooth window function (usually a square integrable function) and then computing the Fourier transform separately on each smaller segment. It results in a two-dimensional representation of the signal.

Let $\psi \in L^2(\mathbb{R})$ be a fixed function usually called a *window function*. The Gabor transform of a function $f \in L^2(\mathbb{R})$ with respect to the window function ψ is defined by $G_\psi f : \mathbb{R} \times \widehat{\mathbb{R}} \to \mathbb{C}$ as

$$G_\psi f(t, \xi) = \int_{\mathbb{R}} f(x)\overline{\psi(x - t)}e^{-2\pi i \xi x}\, dx$$

for all $(t, \xi) \in \mathbb{R} \times \widehat{\mathbb{R}}$.

Hardy's theorem for the Gabor transform on $\mathbb{R}^n$ has been established in [6]. In Section 3, Hardy's theorem for the Gabor transform on a second countable, locally compact, abelian group having noncompact identity component is established. In the next section, we show that Hardy's theorem holds for groups of the form $\mathbb{R}^n \times K$, where K is a compact group having irreducible unitary representations of bounded dimension. Section 5 deals with connected nilpotent Lie groups for which Hardy's theorem does not hold. Finally, some auxiliary results regarding groups having discrete part and quotient group are proved, thereby showing that Hardy's theorem fails for groups of the form $G \times D$, where G is a connected nilpotent Lie group with a square integrable irreducible representation and D is a discrete group.

2. Continuous Gabor transform

Let G be a second countable, unimodular group of type I. Let dx denote the Haar measure on G and $d\pi$ the Plancherel measure on $\widehat{G}$. For each $(x, \pi) \in G \times \widehat{G}$, we define

$$\mathcal{H}_{(x,\pi)} = \pi(x)\mathrm{HS}(\mathcal{H}_\pi),$$

where $\pi(x)\mathrm{HS}(\mathcal{H}_\pi) = \{\pi(x)T : T \in \mathrm{HS}(\mathcal{H}_\pi)\}$. It can be easily seen that $\mathcal{H}_{(x,\pi)}$ forms a Hilbert space with the inner product given by

$$\langle \pi(x)T, \pi(x)S \rangle_{\mathcal{H}_{(x,\pi)}} = \mathrm{tr}\,(S^*T) = \langle T, S \rangle_{\mathrm{HS}(\mathcal{H}_\pi)}.$$

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Also, $\mathcal{H}_{(x,\pi)} = \mathrm{HS}(\mathcal{H}_\pi)$ for all $(x,\pi) \in G \times \widehat{G}$. The family $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ of Hilbert spaces indexed by $G \times \widehat{G}$ is a field of Hilbert spaces over $G \times \widehat{G}$. Let $\mathcal{H}^2(G \times \widehat{G})$ denote the direct integral of $\{\mathcal{H}_{(x,\pi)}\}_{(x,\pi)\in G\times\widehat{G}}$ with respect to the product measure $dx\,d\pi$, that is, the space of all measurable vector fields F on $G \times \widehat{G}$ such that

$$\|F\|^2_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \|F(x,\pi)\|^2_{(x,\pi)}\,dx\,d\pi < \infty.$$

One can observe that $\mathcal{H}^2(G \times \widehat{G})$ forms a Hilbert space with the inner product given by

$$\langle F, K\rangle_{\mathcal{H}^2(G\times\widehat{G})} = \int_{G\times\widehat{G}} \mathrm{tr}\,[F(x,\pi)K(x,\pi)^*]\,dx\,d\pi.$$

Let $f \in C_c(G)$, the set of all continuous complex-valued functions on G with compact supports, and let ψ be a fixed function in $L^2(G)$. For $(x,\pi) \in G \times \widehat{G}$, the continuous *Gabor transform* of f with respect to the window function ψ can be defined as a measurable field of operators on $G \times \widehat{G}$ by

$$G_\psi f(x,\pi) := \int_G f(y)\overline{\psi(x^{-1}y)}\pi(y)^*\,dy. \tag{2.1}$$

The operator-valued integral (2.1) is considered in the weak sense, that is, for each $(x,\pi) \in G \times \widehat{G}$ and $\xi, \eta \in \mathcal{H}_\pi$,

$$\langle G_\psi f(x,\pi)\xi, \eta\rangle = \int_G f(y)\overline{\psi(x^{-1}y)}\langle \pi(y)^*\xi, \eta\rangle\,dy.$$

One can verify that $G_\psi f(x,\pi)$ is a Hilbert–Schmidt operator for all $x \in G$ and for almost all $\pi \in \widehat{G}$. We can extend G_ψ uniquely to a bounded linear operator from $L^2(G)$ into a closed subspace H of $\mathcal{H}^2(G \times \widehat{G})$, which we still denote by G_ψ. As in [4], for $f_1, f_2 \in L^2(G)$ and window functions ψ_1 and ψ_2,

$$\langle G_{\psi_1} f_1, G_{\psi_2} f_2\rangle = \langle \psi_2, \psi_1\rangle\langle f_1, f_2\rangle. \tag{2.2}$$

3. Locally compact abelian groups

Throughout this section, G will be a second countable, locally compact, abelian group and $\widehat{G}$ the dual group of G. For $z \in G$ and $\omega \in \widehat{G}$, we define the *translation operator* T_z on $L^2(G)$ as

$$(T_z f)(y) = f(z^{-1}y)$$

and the *modulation operator* M_ω on $L^2(G)$ as

$$(M_\omega f)(y) = f(y)\omega(y),$$

where $f \in L^2(G)$ and $y \in G$. In the next lemma, we list some properties of the Gabor transform which can be verified easily, so we omit the proofs.

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Lemma 3.1. *Let $f, \psi \in L^2(G)$. For $x, z \in G$ and $\gamma, \omega \in \widehat{G}$, we have:*

(i) $G_\psi f(x,\gamma) = \gamma(x^{-1})\overline{G_f\psi(x^{-1},\gamma^{-1})}$;
(ii) $G_\psi(M_\omega T_z f)(x,\gamma) = (\omega^{-1}\gamma)(z^{-1})G_\psi f(z^{-1}x, \omega^{-1}\gamma)$;
(iii) $G_{(M_\omega T_z\psi)}(M_\omega T_z f)(x,\gamma) = \omega(x)\gamma(z^{-1})G_\psi f(x,\gamma)$.

Lemma 3.2. *Let $f, \psi \in L^2(G)$ and $F : G \times \widehat{G} \to \mathbb{C}$ be defined as*

$$F(x,\gamma) = \gamma(x)G_\psi f(x,\gamma)G_\psi f(x^{-1},\gamma^{-1}),$$

where $(x,\gamma) \in G \times \widehat{G}$. For $(\omega, z) \in \widehat{G} \times G$, the Fourier transform $\widehat{F}$ of F is given by $\widehat{F}(\omega, z) = F(z^{-1}, \omega)$, where z in $\widehat{F}(\omega, z)$ is identified with τ_z via the isomorphism $z \to \tau_z$ of G onto $\widehat{\widehat{G}}$ (see [8, Theorem 24.2]).

Proof. For $z \in G$ and $\omega \in \widehat{G}$,

$$\begin{aligned}
\widehat{F}(\omega, z) &= \int_G \int_{\widehat{G}} F(x,\gamma)\omega(x^{-1})\gamma(z^{-1})\,dx\,d\gamma \\
&= \int_G \int_{\widehat{G}} \gamma(x)G_\psi f(x,\gamma)\gamma(x^{-1})\overline{G_f\psi(x,\gamma)}\omega(x^{-1})\gamma(z^{-1})\,dx\,d\gamma \\
&\qquad\qquad \text{[by Lemma 3.1(i)]} \\
&= \int_G \int_{\widehat{G}} G_\psi f(x,\gamma)\overline{G_{(M_\omega T_{z^{-1}} f)}(M_\omega T_{z^{-1}}\psi)(x,\gamma)}\,dx\,d\gamma \\
&\qquad\qquad \text{[by Lemma 3.1(iii)]} \\
&= \langle G_\psi f, G_{(M_\omega T_{z^{-1}} f)}(M_\omega T_{z^{-1}}\psi)\rangle \\
&= \langle f, M_\omega T_{z^{-1}}\psi\rangle\overline{\langle \psi, M_\omega T_{z^{-1}} f\rangle} \qquad \text{[using (2.2)]} \\
&= \int_G f(x)\overline{M_\omega T_{z^{-1}}\psi(x)}\,dx \int_G \overline{\psi(y)}M_\omega T_{z^{-1}} f(y)\,dy \\
&= \int_G f(x)\overline{\psi(zx)\omega(x)}\,dx \int_G \overline{\psi(y)}f(zy)\omega(y)\,dy \\
&= G_\psi f(z^{-1},\omega) \int_G \overline{\psi(z^{-1}y)}f(y)\omega(z^{-1}y)\,dy \\
&= G_\psi f(z^{-1},\omega)\omega(z^{-1}) \int_G f(y)\overline{\psi(z^{-1}y)}\omega(y)\,dy \\
&= \omega(z^{-1})G_\psi f(z^{-1},\omega)G_\psi f(z,\omega^{-1}) \\
&= F(z^{-1},\omega).
\end{aligned}$$

□

The structure theory of locally compact abelian groups asserts that G decomposes into a direct product $G = \mathbb{R}^n \times H$, where $n \geq 0$ and H contains a compact open subgroup. So, the connected component of the identity of G is noncompact if and only if $n \geq 1$ (see [8]). Let G be a locally compact abelian group such that the connected component of the identity is noncompact. We can write $G = \mathbb{R} \times K$, where

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$K = \mathbb{R}^{n-1} \times H$, and $\widehat{G} = \widehat{\mathbb{R}} \times \widehat{K}$. We now prove the following analogue of Hardy's theorem for the Gabor transform.

THEOREM 3.3. *Let G be a locally compact abelian group such that the connected component of the identity is noncompact. Let $f, \psi \in L^2(G)$ be such that*

$$|G_\psi f(x,k,\xi,\gamma)| \leq e^{-\pi(ax^2+b\xi^2)/2}\varphi(k)\eta(\gamma)$$

for all $(x,k) \in G = \mathbb{R} \times K$, $(\xi,\gamma) \in \widehat{G} = \widehat{\mathbb{R}} \times \widehat{K}$, where a, b are positive real numbers; φ and η are bounded functions in $L^2(K)$ and $L^2(\widehat{K})$, respectively. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

PROOF. For $(x,k),(z,t) \in \mathbb{R} \times K$ and $(\xi,\gamma),(\zeta,\chi) \in \widehat{\mathbb{R}} \times \widehat{K}$, we define

$$\begin{aligned} F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma) = e^{2\pi i\xi x}&\gamma(k)G_\psi(M_{\zeta,\chi}T_{z,t}f)(x,k,\xi,\gamma) \\ &\times G_\psi(M_{\zeta,\chi}T_{z,t}f)(-x,k^{-1},-\xi,\gamma^{-1}). \end{aligned}$$

The function $F_{(z,t,\zeta,\chi)}$ is continuous and belongs to $L^1(\mathbb{R} \times K \times \widehat{\mathbb{R}} \times \widehat{K})$. By Lemma 3.2,

$$\widehat{F_{(z,t,\zeta,\chi)}}(\omega,\delta,y,v) = F_{(z,t,\zeta,\chi)}(-y,v^{-1},\omega,\delta).$$

Using Lemma 3.1(ii),

$$\begin{aligned} F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma) = e^{2\pi i\xi x}&\gamma(k)e^{-2\pi i(\xi-\zeta)z}(\chi^{-1}\gamma)(t^{-1})G_\psi f(x-z,t^{-1}k,\xi-\zeta,\chi^{-1}\gamma) \\ &\times e^{-2\pi i(-\xi-\zeta)z}(\chi^{-1}\gamma^{-1})(t^{-1})G_\psi f(-x-z,t^{-1}k^{-1},-\xi-\zeta,\chi^{-1}\gamma^{-1}), \end{aligned}$$

which implies that

$$\begin{aligned} &|F_{(z,t,\zeta,\chi)}(x,k,\xi,\gamma)| \\ &\quad = |G_\psi f(x-z,t^{-1}k,\xi-\zeta,\chi^{-1}\gamma)|\,|G_\psi f(-x-z,t^{-1}k^{-1},-\xi-\zeta,\chi^{-1}\gamma^{-1})| \\ &\quad \leq e^{-\pi[a(x-z)^2+b(\xi-\zeta)^2]/2}\varphi(t^{-1}k)\eta(\chi^{-1}\gamma)e^{-\pi[a(-x-z)^2+b(-\xi-\zeta)^2]/2}\varphi(t^{-1}k^{-1})\eta(\chi^{-1}\gamma^{-1}) \\ &\quad = e^{-\pi a(x^2+z^2)}e^{-\pi b(\xi^2+\zeta^2)}\varphi(t^{-1}k)\eta(\chi^{-1}\gamma)\varphi(t^{-1}k^{-1})\eta(\chi^{-1}\gamma^{-1}) \\ &\quad = e^{-\pi ax^2}\beta_{(z,t,\zeta,\chi)}(k,\xi,\gamma), \end{aligned}$$

where $\beta_{(z,t,\zeta,\chi)}$ is the function on $K \times \widehat{\mathbb{R}} \times \widehat{K} = S$ (say) given by

$$\beta_{(z,t,\zeta,\chi)}(k,\xi,\gamma) = e^{-\pi az^2}e^{-\pi b(\xi^2+\zeta^2)}\varphi(t^{-1}k)\eta(\chi^{-1}\gamma)\varphi(t^{-1}k^{-1})\eta(\chi^{-1}\gamma^{-1}).$$

Using the Cauchy–Schwarz inequality, $\beta_{(z,t,\zeta,\chi)} \in L^1(S) \cap L^2(S)$. Also,

$$\begin{aligned} |\widehat{F_{(z,t,\zeta,\chi)}}(\omega,\delta,y,v)| &= |F_{(z,t,\zeta,\chi)}(-y,v^{-1},\omega,\delta)| \\ &\leq e^{-\pi a(y^2+z^2)}e^{-\pi b(\omega^2+\zeta^2)}\varphi(t^{-1}v^{-1})\eta(\chi^{-1}\delta)\varphi(t^{-1}v)\eta(\chi^{-1}\delta^{-1}) \\ &= e^{-\pi b\omega^2}\rho_{(z,t,\zeta,\chi)}(\delta,y,v), \end{aligned}$$

where $\rho_{(z,t,\zeta,\chi)}$ is the function on $\widehat{S} = \widehat{K} \times \mathbb{R} \times K$ given by

$$\rho_{(z,t,\zeta,\chi)}(\delta,y,v) = e^{-\pi a(y^2+z^2)}e^{-\pi b\zeta^2}\varphi(t^{-1}v^{-1})\eta(\chi^{-1}\delta)\varphi(t^{-1}v)\eta(\chi^{-1}\delta^{-1}).$$

Again, using the Cauchy–Schwarz inequality, we have $\rho_{(z,t,\zeta,\chi)} \in L^1(\widehat{S}) \cap L^2(\widehat{S})$ and it is also bounded.

Thus, by [2, Theorem 1.1], we have $F_{(z,t,\zeta,\chi)} \equiv 0$ for all (z,t,ζ,χ) whenever $ab > 1$.

Since $F_{(-z,t^{-1},-\zeta,\chi^{-1})}(0,e,0,I) = e^{4\pi i\zeta z}\,\chi(t)^2(G_\psi f(z,t,\zeta,\chi))^2$, it follows that $G_\psi f \equiv 0$ whenever $ab > 1$.

By (2.2), we have $\|\psi\|_2\|f\|_2 = 0$, which implies that either $f = 0$ a.e. or $\psi = 0$ a.e. whenever $ab > 1$. □

4. $\mathbb{R}^n \times K$, K a compact group

In this section, we shall prove Hardy's theorem for the Gabor transform for the groups of the form $\mathbb{R}^n \times K$ when K is a compact group with irreducible unitary representations of bounded dimension, that is, there exists a positive integer M such that $d_\sigma \leq M$ for every $\sigma \in \widehat{K}$.

A well-known theorem of Moore [10] implies that compact groups with all irreducible representations of bounded dimension are precisely the compact groups with an abelian subgroup of finite index. Thus, groups of the form $N \rtimes F$, where N is a compact abelian group and F is a finite group, are examples of such groups. In particular, $\mathbb{T}^2 \rtimes \{1,-1\}$ with multiplication given by

$$(e^{it_1}, e^{it_2}, \alpha_1)\cdot(e^{it'_1}, e^{it'_2}, \alpha_2) = (e^{i(t_1+\alpha_1 t'_1)}, e^{i(t_2+\alpha_1 t'_2)}, \alpha_1\alpha_2).$$

Theorem 4.1. *Let $f,\psi \in L^2(\mathbb{R}^n \times K)$, where K is a compact group with all irreducible unitary representations of bounded dimension such that*

$$\|G_\psi f(x,k,\xi,\sigma)\|_{HS} \leq Ce^{-\pi(a\|x\|^2+b\|\xi\|^2)/2}\varphi(\sigma) \tag{4.1}$$

for all $(x,k) \in \mathbb{R}^n \times K$, $(\xi,\sigma) \in \widehat{\mathbb{R}^n} \times \widehat{K}$, where a, b and C are positive real numbers and φ is a function in $L^2(\widehat{K})$. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

Proof. Assume that $\psi \neq 0$. For $\omega,\gamma \in \widehat{K}$, let $\mathcal{H}_\omega$ and $\mathcal{H}_\gamma$ be the Hilbert spaces of dimensions d_ω and d_γ with orthonormal bases $\{e_i^\omega\}_{i=1}^{d_\omega}$ and $\{e_i^\gamma\}_{i=1}^{d_\gamma}$, respectively.

For fixed e_r^γ, e_s^γ, we define $\tau : \mathbb{R}^n \to \mathbb{C}$ by

$$\tau(x) = \int_K \psi(x,k)\overline{\langle\gamma(k)^* e_r^\gamma, e_s^\gamma\rangle}\,dk.$$

Using Hölder's inequality, it follows that $\tau \in L^2(\mathbb{R}^n)$. We fix $\gamma \in \widehat{K}$ for which $\tau \neq 0$. For $\sigma \in \widehat{K}$, we can write

$$\gamma(k)e_r^\gamma = \sum_{j=1}^{d_\gamma} C_{j,r}^k e_j^\gamma$$

and

$$\gamma \otimes \sigma = \sum_{\delta \in K_\sigma} m_\delta \delta, \tag{4.2}$$

where K_σ is a finite subset of $\widehat{K}$ and the $C_{j,r}^k$, m_δ are scalars (see [8]).

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For fixed e_p^ω, e_q^ω, we define $g : \mathbb{R}^n \to \mathbb{C}$ such that

$$g(x) = \int_K f(x,k)\overline{\langle \omega(k)^* e_p^\omega, e_q^\omega\rangle}\, dk.$$

Clearly, $g \in L^2(\mathbb{R}^n)$. Consider a function $\varphi : \mathbb{R}^n \times K \to \mathbb{C}$ defined by

$$\varphi(x,k) = \psi(x,k)\overline{\langle \gamma(k)^* e_r^\gamma, e_s^\gamma\rangle}.$$

Then $\varphi \in L^2(\mathbb{R}^n \times K)$ and $G_\varphi f(x,k,\xi,\sigma)$ is a Hilbert–Schmidt operator for all $(x,k) \in \mathbb{R}^n \times K$ and for almost all $(\xi,\sigma) \in \widehat{\mathbb{R}^n} \times \widehat{K}$.

For $\sigma \in \widehat{K}$ and fixed e_l^σ, e_m^σ,

$$\begin{aligned}
&\langle G_\varphi f(x,k,\xi,\sigma)e_l^\sigma, e_m^\sigma\rangle \\
&\quad = \int_{\mathbb{R}^n\times K} f(y,v)\overline{\psi(y-x,k^{-1}v)}\langle \gamma(k^{-1}v)^* e_r^\gamma, e_s^\gamma\rangle e^{-2\pi i\xi y}\langle \sigma(v)^* e_l^\sigma, e_m^\sigma\rangle\, dy\, dv \\
&\quad = \int_{\mathbb{R}^n\times K} f(y,v)\overline{\psi(y-x,k^{-1}v)} e^{-2\pi i\xi y}\langle (\gamma\otimes\sigma)(v)^*(\gamma(k)e_r^\gamma)\otimes e_l^\sigma, e_s^\gamma\otimes e_m^\sigma\rangle\, dy\, dv \\
&\quad = \sum_{j=1}^{d_\gamma} C_{j,r}^k \int_{\mathbb{R}^n\times K} f(y,v)\overline{\psi(y-x,k^{-1}v)} e^{-2\pi i\xi y}\langle (\gamma\otimes\sigma)(v)^* e_j^\gamma\otimes e_l^\sigma, e_s^\gamma\otimes e_m^\sigma\rangle\, dy\, dv \\
&\quad = \sum_{j=1}^{d_\gamma} C_{j,r}^k \int_{\mathbb{R}^n\times K} f(y,v)\overline{\psi(y-x,k^{-1}v)} e^{-2\pi i\xi y} \sum_{\delta\in K_\sigma} m_\delta \langle \delta^*(v) e_{l,j}^\delta, e_{m,s}^\delta\rangle\, dy\, dv \\
&\quad = \sum_{j=1}^{d_\gamma}\sum_{\delta\in K_\sigma} C_{j,r}^k m_\delta \langle G_\psi f(x,k,\xi,\delta)e_{l,j}^\delta, e_{m,s}^\delta\rangle.
\end{aligned}$$

Let $M_\sigma = \max\{m_\delta : \delta \in K_\sigma\}$; then, using ([4.1]) and the Cauchy–Schwarz inequality, we can write

$$\begin{aligned}
\|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}}^2 &= \sum_{l,m=1}^{d_\sigma} |\langle G_\varphi f(x,k,\xi,\sigma)e_l^\sigma, e_m^\sigma\rangle|^2 \\
&= \sum_{l,m=1}^{d_\sigma} \Big|\sum_{j=1}^{d_\gamma}\sum_{\delta\in K_\sigma} C_{j,r}^k m_\delta \langle G_\psi f(x,k,\xi,\delta)e_{l,j}^\delta, e_{m,s}^\delta\rangle\Big|^2 \\
&\le \sum_{l,m=1}^{d_\sigma} \Big(\sum_{j=1}^{d_\gamma}\sum_{\delta\in K_\sigma} |C_{j,r}^k m_\delta \langle G_\psi f(x,k,\xi,\delta)e_{l,j}^\delta, e_{m,s}^\delta\rangle|\Big)^2 \\
&\le \sum_{l,m=1}^{d_\sigma} M_\sigma^2 |K_\sigma| d_\gamma \Big(\sum_{j=1}^{d_\gamma}\sum_{\delta\in K_\sigma} |\langle G_\psi f(x,k,\xi,\delta)e_{l,j}^\delta, e_{m,s}^\delta\rangle|^2\Big)
\end{aligned}$$

$$\leq \sum_{l,m=1}^{d_\sigma} M_\sigma^2 |K_\sigma| d_\gamma \sum_{j=1}^{d_\gamma} \sum_{\delta \in K_\sigma} \|G_\psi f(x,k,\xi,\delta)\|_{\mathrm{HS}}^2$$
$$\leq d_\sigma^2 M_\sigma^2 |K_\sigma| d_\gamma^2 C^2 e^{-\pi(a\|x\|^2+b\|\xi\|^2)} \sum_{\delta \in K_\sigma} |\varphi(\delta)|^2.$$

So,

$$\|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}}^2 \leq d_\sigma^2 M_\sigma^2 |K_\sigma| d_\gamma^2 C^2 \|\varphi\|_2^2 e^{-\pi(a\|x\|^2+b\|\xi\|^2)}. \tag{4.3}$$

Now, from (4.2), we have $\dim(\gamma \otimes \sigma) = d_\gamma d_\sigma$, so $|K_\sigma| \leq d_\gamma d_\sigma \leq M^2$.

Also, $M_\sigma = \max\{m_\delta : \delta \in K_\sigma\} \leq M$ (see [8, Corollary 27.31]).

It follows that (4.3) can be written as

$$\begin{aligned}\|G_\varphi f(x,k,\xi,\sigma)\|_{\mathrm{HS}} &\leq M^4 C \|\varphi\|_2 e^{-\pi(a\|x\|^2+b\|\xi\|^2)/2} \\ &= C_1 e^{-\pi(a\|x\|^2+b\|\xi\|^2)/2},\end{aligned} \tag{4.4}$$

where $C_1 = M^4 C \|\varphi\|_2$.

Also,

$$\begin{aligned}G_\tau g(x,\xi) &= \int_{\mathbb{R}^n}\int_K\int_K f(y,v)\overline{\psi(y-x,k)} e^{-2\pi i\xi y} \langle \gamma(k)^* e_r^\gamma, e_s^\gamma\rangle \langle \omega(v)^* e_p^\omega, e_q^\omega\rangle \, dy\, dk\, dv \\ &= \int_{\mathbb{R}^n}\int_K\int_K f(x,v)\overline{\psi(y-x,k^{-1}v)} e^{-2\pi i\xi y} \langle \gamma(k^{-1}v)^* e_r^\gamma, e_s^\gamma\rangle \langle \omega(v)^* e_p^\omega, e_q^\omega\rangle \, dy\, dk\, dv \\ &= \int_K \langle G_\varphi f(x,k,\xi,\omega) e_p^\omega, e_q^\omega\rangle \, dk.\end{aligned}$$

Using (4.4),

$$\begin{aligned}|G_\tau g(x,\xi)|^2 &= \left|\int_K \langle G_\varphi f(x,k,\xi,\omega) e_p^\omega, e_q^\omega\rangle \, dk\right|^2 \\ &\leq \int_K |\langle G_\varphi f(x,k,\xi,\omega) e_p^\omega, e_q^\omega\rangle|^2 \, dk \\ &\leq \|G_\varphi f(x,k,\xi,\omega)\|_{\mathrm{HS}}^2 \\ &\leq C_1 e^{-\pi(a\|x\|^2+b\|\xi\|^2)}.\end{aligned}$$

By Hardy's theorem for the Gabor transform on $\mathbb{R}^n$, we have $g = 0$ a.e.

Since $\omega \in \widehat{K}$ is arbitrary, $f = 0$ a.e. □

5. Connected nilpotent Lie groups

In this section, we shall prove that Hardy's theorem fails for a connected nilpotent Lie group G having a square integrable irreducible representation. We shall use the notation of [1].

Let G be a connected nilpotent Lie group and $\widetilde{G}$ be its simply connected covering group. Let Γ be a discrete subgroup of $\widetilde{G}$ such that $G = \widetilde{G}/\Gamma$. Let $\mathfrak{g}$ be the Lie algebra

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of G and $\widetilde{G}$. The exponential maps of G and $\widetilde{G}$ are denoted by $\exp_G : \mathfrak{g} \to G$ and $\exp_{\widetilde{G}} : \mathfrak{g} \to \widetilde{G}$, respectively.

Let $\mathcal{B} = \{X_1, X_2, \ldots, X_n\}$ be a strong Malcev basis of $\mathfrak{g}$ through the ascending central series of $\mathfrak{g}$. The norm function on $\mathfrak{g}$ is defined as the Euclidean norm of X with respect to the basis $\mathcal{B}$, that is, for $X = \sum_{j=1}^{n} x_j X_j \in \mathfrak{g}$, $x_j \in \mathbb{R}$,

$$\|X\| = \Big(\sum_{j=1}^{n} x_j^2\Big)^{1/2}.$$

Define a 'norm function' on G by setting

$$\|x\| = \inf\{\|X\| : X \in \mathfrak{g} \text{ such that } \exp_G X = x\}.$$

The composed map

$$\mathbb{R}^n \to \mathfrak{g} \to \widetilde{G},$$

given as

$$(x_1, \ldots, x_n) \to \sum_{j=1}^{n} x_j X_j \to \exp_{\widetilde{G}}\Big(\sum_{j=1}^{n} x_j X_j\Big),$$

is a diffeomorphism and maps the Lebesgue measure on $\mathbb{R}^n$ to the Haar measure on $\widetilde{G}$. In this manner, we shall always identify $\mathfrak{g}$, and sometimes $\widetilde{G}$, as sets with $\mathbb{R}^n$. Thus, measurable (integrable) functions on $\widetilde{G}$ can be viewed as such functions on $\mathbb{R}^n$.

Let $\mathfrak{g}^*$ denote the vector space dual of $\mathfrak{g}$ and $\{X_1^*, \ldots, X_n^*\}$ the basis of $\mathfrak{g}^*$ which is dual to $\{X_1, \ldots, X_n\}$. Then $\{X_1^*, \ldots, X_n^*\}$ is a Jordan–Hölder basis for the coadjoint action of G on $\mathfrak{g}^*$. We shall identify $\mathfrak{g}^*$ with $\mathbb{R}^n$ via the map

$$\xi = (\xi_1, \ldots, \xi_n) \to \sum_{j=1}^{n} \xi_j X_j^*$$

and on $\mathfrak{g}^*$ we introduce the Euclidean norm relative to the basis $\{X_1^*, \ldots, X_n^*\}$, that is,

$$\Big\|\sum_{j=1}^{n} \xi_j X_j^*\Big\| = \Big(\sum_{j=1}^{n} \xi_j^2\Big)^{1/2} = \|\xi\|.$$

Let $\mathcal{U}$ denote the Zariski open subset of $\mathfrak{g}^*$ of generic elements under the coadjoint action of $\widetilde{G}$ with respect to the basis $\{X_1^*, \ldots, X_n^*\}$. Let S be the set of jump indices, $T = \{1, \ldots, n\} \backslash S$ and $V_T = \mathbb{R}\text{-span}\{X_i^* : i \in T\}$. Then $\mathcal{W} = \mathcal{U} \cap V_T$ is a cross-section for the generic orbits and $\mathcal{W}$ supports the Plancherel measure on $\widetilde{G}$.

We now state Hardy's theorem for the Gabor transform on connected nilpotent Lie groups.

Hardy's theorem (Conjecture). Let $f, \psi \in L^2(G)$ be such that

$$\|G_\psi f(x, \tau)\|_{\mathrm{HS}} \le C e^{-\pi(a\|x\|^2 + b\|\tau\|^2)/2}$$

for all $(x, \tau) \in G \times \mathcal{W}$, where a, b and C are positive real numbers. If $ab > 1$, then either $f = 0$ a.e. or $\psi = 0$ a.e.

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The following theorem shows that Hardy's theorem for the Gabor transform fails for a connected nilpotent Lie group G having a square integrable irreducible representation.

Theorem 5.1. *Let G be a connected nilpotent Lie group having a square integrable irreducible representation σ. Then there exist nonzero functions $f, \psi \in L^2(G)$ such that*

$$\|G_\psi f(x,\tau)\|_{HS} \leq Ce^{-\pi(a\|x\|^2+b\|\tau\|^2)/2}$$

for all $x \in G$, $\tau \in \widehat{G}$, where a, b and C are positive real numbers with $ab > 1$.

Proof. Given that G has a square integrable irreducible representation σ, the centre $Z(G)$ of G is compact. It implies that $Z(G) = \mathbb{T}^d$ for some $d \in \mathbb{N}$. Let χ_0 be the character of $Z(G)$ such that $\sigma|_{Z(G)}$ is a multiple of χ_0. Suppose that χ_0 is given by

$$\chi_0(z_1, z_2, \ldots, z_d) = \prod_{j=1}^{d} z_j^{-m_j},$$

where $(m_1, m_2, \ldots, m_d) \in \mathbb{Z}^d$. Given $a > 0$, we define functions f and ψ on G as

$$f(\exp_G X) = \exp\Big(-a\pi \sum_{j=d+1}^{n} x_j^2\Big)\exp\Big(-2\pi i \sum_{j=1}^{d} x_j m_j\Big)$$

and

$$\psi(\exp_G X) = \exp\Big(-a\pi \sum_{j=d+1}^{n} x_j^2\Big),$$

where $X = \sum_{j=1}^{n} x_j X_j \in \mathfrak{g}$. For $x = \exp_G\,(\sum_{j=1}^{n} x_j X_j)$,

$$\begin{aligned}\|x\|^2 &= \Big(\sum_{j=d+1}^{n} x_j^2\Big) + \inf\Big\{\sum_{j=1}^{d} y_j^2 : y_j - x_j \in \mathbb{Z} \text{ for } 1 \leq j \leq d\Big\} \\ &\leq d + \sum_{j=d+1}^{n} x_j^2. \end{aligned} \tag{5.1}$$

It implies that

$$\exp\,(-a\pi\|x\|^2) \geq \exp\Big(-a\pi \sum_{j=d+1}^{n} x_j^2\Big)\exp\,(-a\pi d).$$

For $C' = \exp\,(a\pi d)$, we can write

$$|f(x)| = \exp\Big(-a\pi \sum_{j=d+1}^{n} x_j^2\Big) \leq C' \exp\,(-a\pi\|x\|^2)$$

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and

$$|\psi(x)| = \exp\Big(-a\pi \sum_{j=d+1}^{n} x_j^2\Big) \le C' \exp(-a\pi\|x\|^2).$$

So, $f, \psi \in L^2(G)$ and are both nonzero functions.

Let $q : \widetilde{G} \to G$ be the quotient homomorphism; then the irreducible representation $\pi = \sigma \circ q$ of $\widetilde{G}$ is square integrable modulo $Z(\widetilde{G})$.

By [3, Corollary 4.5.4 and Theorem 3.2.3], it follows that the induced representation $\mathrm{ind}_{Z(\widetilde{G})}^{\widetilde{G}}(\pi|_{Z(\widetilde{G})})$ is a multiple of π. Since $Z(G) = q(Z(\widetilde{G}))$, it follows that $\mathrm{ind}_{Z(G)}^{G}(\sigma|_{Z(G)})$ is a multiple of σ.

Let τ be an irreducible representation of G and $\chi \in \widehat{Z(G)}$ be such that $\tau|_{Z(G)}$ is a multiple of χ. We normalise the Haar measures on G, $Z(G)$ and $G/Z(G)$ so that $Z(G)$ has measure one and Weil's formula holds. Let $\mathcal{H}_\tau$ denote the Hilbert space of τ. Then, for any $\zeta, \eta \in \mathcal{H}_\tau$,

$$\begin{aligned}
\langle G_\psi f(x,\tau)\zeta, \eta\rangle &= \int_G f(y)\overline{\psi(x^{-1}y)}\langle \tau(y)^*\zeta, \eta\rangle\, dy \\
&= \int_{G/Z(G)} \int_{Z(G)} f(yz)\psi(x^{-1}yz)\langle \tau(z)^*\tau(y)^*\zeta, \eta\rangle\, d\dot{y}\, dz \\
&= \int_{G/Z(G)} \Big(\int_{Z(G)} f(yz)\psi(x^{-1}yz)\overline{\chi(z)}\, dz\Big)\langle \tau(y)^*\zeta, \eta\rangle\, d\dot{y}. \qquad (5.2)
\end{aligned}$$

If $y = \exp_G(\sum_{j=1}^n y_j X_j) \in G$ and $z = \exp_G(\sum_{j=1}^d z_j X_j) \in Z(G)$, then

$$\begin{aligned}
f(yz) &= f\Big(\exp_G\Big(\sum_{j=1}^{d}(y_j + z_j)X_j + \sum_{j=d+1}^{n} y_j X_j\Big)\Big) \\
&= \exp\Big(-a\pi \sum_{j=d+1}^{n} y_j^2\Big)\exp\Big(-2\pi i \sum_{j=1}^{d}(y_j + z_j)m_j\Big) \\
&= f(y)\exp\Big(-2\pi i \sum_{j=1}^{d} z_j m_j\Big)
\end{aligned}$$

and

$$\begin{aligned}
\psi(x^{-1}yz) &= \psi\Big(\exp_G\Big(\sum_{j=1}^{d}(-x_j + y_j + z_j)X_j + \sum_{j=d+1}^{n}(-x_j + y_j)X_j\Big)\Big) \\
&= \exp\Big(-a\pi \sum_{j=d+1}^{n}(-x_j + y_j)^2\Big) \\
&= \psi(x^{-1}y).
\end{aligned}$$

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This implies that

$$\int_{Z(G)} f(yz)\psi(x^{-1}yz)\overline{\chi(z)}\,dz = \int_{Z(G)} f(y)\exp\Big(-2\pi i\sum_{j=1}^{d} z_j m_j\Big)\psi(x^{-1}y)\overline{\chi(z)}\,dz$$

$$= f(y)\psi(x^{-1}y)\int_{Z(G)} \chi_0(z)\overline{\chi(z)}\,dz$$

$$= \begin{cases} f(y)\psi(x^{-1}y) & \text{if } \chi=\chi_0,\\ 0 & \text{otherwise.}\end{cases} \tag{5.3}$$

On combining (5.2) and (5.3),

$$\langle G_\psi f(x,\tau)\zeta,\eta\rangle = \begin{cases} \displaystyle\int_{G/Z(G)} f(y)\psi(x^{-1}y)\langle\tau(y)^*\zeta,\eta\rangle\,d\dot{y} & \text{if } \tau|_{Z(G)} \text{ is a multiple of } \chi_0,\\ 0 & \text{otherwise,}\end{cases}$$

which implies that $\|G_\psi f(x,\tau)\|_{\mathrm{HS}} = 0$ for all $\tau \neq \sigma$.

Let $\{e_i\}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_\sigma$. Then

$$\langle G_\psi f(x,\sigma)e_r,e_s\rangle = \int_{G/Z(G)} f(y)\psi(x^{-1}y)\langle\sigma(y)^*e_r,e_s\rangle\,d\dot{y}$$

$$= \int_{G/Z(G)} \exp\Big(-a\pi\sum_{j=d+1}^{n} y_j^2\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)$$

$$\times \exp\Big(-a\pi\sum_{j=d+1}^{n}(y_j-x_j)^2\Big)\langle\sigma(y)^*e_r,e_s\rangle\,d\dot{y}$$

$$= \int_{G/Z(G)} \exp\Big(-a\pi\sum_{j=d+1}^{n}[y_j^2+(y_j-x_j)^2]\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle\sigma(y)^*e_r,e_s\rangle\,d\dot{y}$$

$$= \int_{G/Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}[x_j^2+(2y_j-x_j)^2]\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle\sigma(y)^*e_r,e_s\rangle\,d\dot{y}$$

$$= \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n} x_j^2\Big)\int_{G/Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j-x_j)^2\Big)$$

$$\times \exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle\sigma(y)^*e_r,e_s\rangle\,d\dot{y}. \tag{5.4}$$

Let

$$g(y) = \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j)^2\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)$$

and

$$t = \exp\Big(\sum_{j=d+1}^{n}\frac{x_j}{2}X_j\Big),$$

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so that

$$ {}_tg(y) = g(yt^{-1}) = \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big). $$

Therefore,

$$ \begin{aligned} \langle \sigma({}_tg(y))e_r, e_s\rangle &= \int_G {}_tg(y)\langle \sigma(y)^* e_r, e_s\rangle\, dy \\ &= \int_G \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big)\exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle \sigma(y)^* e_r, e_s\rangle\, dy \\ &= \int_{G/Z(G)}\int_{Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big) \\ &\qquad \times \exp\Big(-2\pi i\sum_{j=1}^{d} (y_j + u_j) m_j\Big)\langle \sigma(yu)^* e_r, e_s\rangle\, du\, d\dot{y} \\ &= \int_{G/Z(G)}\int_{Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big) \\ &\qquad \times \exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\chi_0(u)\overline{\chi_0(u)}\langle \sigma^*(y) e_r, e_s\rangle\, du\, d\dot{y} \\ &= \int_{G/Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big) \\ &\qquad \times \exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle \sigma(y)^* e_r, e_s\rangle\, d\dot{y}. \end{aligned} \tag{5.5} $$

Hence, (5.1), (5.4) and (5.5) imply that

$$ \begin{aligned} |\langle G_\psi f(x,\sigma)e_r, e_s\rangle| &\le \exp\Big(\frac{a\pi}{2}(d - \|x\|^2)\Big)\Big|\int_{G/Z(G)} \exp\Big(-\frac{a\pi}{2}\sum_{j=d+1}^{n}(2y_j - x_j)^2\Big) \\ &\qquad \times \exp\Big(-2\pi i\sum_{j=1}^{d} y_j m_j\Big)\langle \sigma(y)^* e_r, e_s\rangle\, d\dot{y}\Big| \\ &= C_1 e^{-a\pi\|x\|^2/2}|\langle \sigma({}_tg)e_r, e_s\rangle|. \end{aligned} $$

Thus,

$$ \begin{aligned} \|G_\psi f(x,\sigma)\|_{\mathrm{HS}} &= \Big(\sum_{r,s=1}^{n} |\langle G_\psi f(x,\sigma)e_r, e_s\rangle|^2\Big)^{1/2} \\ &\le C_1 e^{-a\pi\|x\|^2/2}\Big(\sum_{r,s=1}^{n} |\langle \sigma({}_tg)e_r, e_s\rangle|^2\Big)^{1/2} \end{aligned} $$

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$$= C_1 e^{-a\pi\|x\|^2/2}\|\sigma({}_t g)\|_{\mathrm{HS}}$$
$$\leq C_1 e^{-\pi(a\|x\|^2+b\|\sigma\|^2)/2} e^{\pi b\|\sigma\|^2/2}\|\sigma(t)\|\,\|\sigma(g)\|_{\mathrm{HS}}$$
$$= C e^{-\pi(a\|x\|^2+b\|\sigma\|^2)/2},$$

where $C = C_1 e^{\pi b\|\sigma\|^2/2}\|\sigma(g)\|_{\mathrm{HS}}$. □

Remark. Hardy's theorem for the Gabor transform does not hold for $G_{5,1}/\mathbb{Z}$, $G_{5,3}/\mathbb{Z}$ and $G_{5,6}/\mathbb{Z}$, since each of these groups admits a square integrable irreducible representation. See [11] for relevant data about these low-dimensional connected nilpotent Lie groups. The same conclusion can be made for the reduced Weyl–Heisenberg group.

6. Auxiliary results

In this section, we shall establish some auxiliary results related to Hardy's theorem for the Gabor transform.

Theorem 6.1. *Let H be a separable unimodular locally compact group of type I and D be a unimodular discrete group of type I. If Hardy's theorem holds for the Gabor transform on $\mathbb{R}^n \times H \times D$, then it also holds for the Gabor transform on $\mathbb{R}^n \times H$.*

Proof. Let $f, \psi \in L^2(\mathbb{R}^n \times H)$ be such that

$$\|G_\psi f(x, h, \xi, \sigma)\|_{\mathrm{HS}} \leq e^{-\pi(a\|x\|^2+b\|\xi\|^2)/2}\varphi_1(h)\varphi_2(\sigma) \tag{6.1}$$

for all $(x, h) \in \mathbb{R}^n \times H$, $(\xi, \sigma) \in \widehat{\mathbb{R}^n} \times \widehat{H}$, where a and b are positive real numbers; φ_1 and φ_2 are bounded functions in $L^2(H)$ and $L^2(\widehat{H})$, respectively.

We show that either $f = 0$ a.e. or $\psi = 0$ a.e. whenever $ab > 1$.

Define a function $g : \mathbb{R}^n \times H \times D \to \mathbb{C}$ by setting

$$g(x, h, t) = f(x, h)\chi_{\{e\}}(t)$$

and a function $\tau : \mathbb{R}^n \times H \times D \to \mathbb{C}$ by

$$\tau(x, h, t) = \psi(x, h)\chi_{\{e\}}(t)$$

for all $(x, h, t) \in \mathbb{R}^n \times H \times D$. Here $\chi_{\{e\}}$ denotes the characteristic function of $\{e\}$, e being the identity of D.

Since

$$\int_{\mathbb{R}^n\times H}\sum_{t\in D}|g(x, h, t)|^2\,dx\,dh = \int_{\mathbb{R}^n\times H}|f(x, h)|^2\,dx\,dh < \infty$$

and

$$\int_{\mathbb{R}^n\times H}\sum_{t\in D}|\tau(x, h, t)|^2\,dx\,dh = \int_{\mathbb{R}^n\times H}|\psi(x, h)|^2\,dx\,dh < \infty,$$

we have $g, \tau \in L^2(\mathbb{R}^n \times H \times D)$.

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For $(x, h, t) \in \mathbb{R}^n \times H \times D$ and $(\xi, \sigma, \delta) \in \mathbb{R}^n \times \widehat{H} \times \widehat{D}$,

$$\begin{aligned} G_\tau g(x, h, t, \xi, \sigma, \delta) &= \int_{\mathbb{R}^n \times H} \sum_{u \in D} g(y, k, u) \overline{\tau(y - x, h^{-1}k, t^{-1}u)} e^{-2\pi i y \xi} \sigma(k^{-1}) \delta(u^{-1})\, dy\, dk \\ &= \int_{\mathbb{R}^n \times H} f(y, k) \overline{\psi(y - x, h^{-1}k) \chi_{\{e\}}(t^{-1})} e^{-2\pi i y \xi} \sigma(k^{-1})\, dy\, dk \\ &= G_\psi f(x, h, \xi, \sigma) \overline{\chi_{\{e\}}(t^{-1})}. \end{aligned}$$

Using (6.1), we can write

$$\begin{aligned} \|G_\tau g(x, h, t, \xi, \sigma, \delta)\|_{\mathrm{HS}} &= \|G_\psi f(x, h, \xi, \sigma)\|_{\mathrm{HS}} \chi_{\{e\}}(t^{-1}) \\ &\le e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2} \varphi_1(h) \varphi_2(\sigma) \chi_{\{e\}}(t^{-1}) \\ &\le e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2} \rho_1(h, t) \rho_2(\sigma, \delta), \end{aligned} \tag{6.2}$$

where $\rho_1 : H \times D \to \mathbb{C}$ and $\rho_2 : \widehat{H} \times \widehat{D} \to \mathbb{C}$ are defined by

$$\rho_1(h, t) = \varphi_1(h) \chi_{\{e\}}(t^{-1})$$

and

$$\rho_2(\sigma, \delta) = \varphi_2(\sigma).$$

It is easy to see that $\rho_1 \in L^2(H \times D)$ and $\rho_2 \in L^2(\widehat{H} \times \widehat{D})$, where $\widehat{H} \times \widehat{D}$ is equipped with a product of Plancherel measures.

Also, ρ_1 and ρ_2 are bounded functions as φ_1 and φ_2 are bounded functions.

From (6.2), it is clear that g and τ satisfy the conditions of Hardy's theorem for the Gabor transform on $\mathbb{R}^n \times H \times D$. So, either $g = 0$ a.e. or $\tau = 0$ a.e. whenever $ab > 1$.

If $g = 0$ a.e., then there exists $M \subseteq \mathbb{R}^n \times H \times D = G$ such that $m(M) = 0$ and $g(x, h, t) = 0$ for all $(x, h, t) \in G \backslash M$.

Let $p(M)$ denote the projection of M on $\mathbb{R}^n \times H$. Then

$$(m_{\mathbb{R}^n} \times m_H)(p(M)) \le \int_{\mathbb{R}^n \times H} \sum_{t \in D} \chi_M(x, h, t)\, dx\, dh = m(M) = 0.$$

Also, $(x, k) \notin p(M)$ implies that $f(x, k) = 0$, so $f = 0$ a.e.

Similarly, if $\tau = 0$ a.e., then $\psi = 0$ a.e. □

REMARK. We can conclude from the proof of the above theorem that if G is a connected nilpotent Lie group and Hardy's theorem for the Gabor transform holds for $G \times D$, where D is a discrete group of type I, then Hardy's theorem for the Gabor transform holds for G. In particular, by Theorem 5.1, if G has a square integrable representation, then Hardy's theorem for the Gabor transform fails for $G \times D$ also. A similar conclusion can be made in case of groups of the form $G \times K$, where K is a compact group.

THEOREM 6.2. *Let H be a separable unimodular locally compact group of type I. If Hardy's theorem for the Gabor transform holds for $\mathbb{R}^n \times H$, then it also holds for $\mathbb{R}^n \times (H/K)$, where K is a compact normal subgroup of H.*

Proof. Let $f, \psi \in L^2(\mathbb{R}^n \times (H/K))$ be such that

$$\|G_\psi f(x, u, \xi, \sigma)\|_{\mathrm{HS}} \leq e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2} \varphi_1(u)\varphi_2(\sigma) \tag{6.3}$$

for all $(x, u) \in \mathbb{R}^n \times (H/K)$, $(\xi, \sigma) \in \widehat{\mathbb{R}^n} \times \widehat{H/K}$, where a and b are positive real numbers; φ_1 and φ_2 are bounded functions in $L^2(H/K)$ and $L^2(\widehat{H/K})$, respectively.

We show that either $f = 0$ a.e. or $\psi = 0$ a.e. whenever $ab > 1$.

Let $q : H \to H/K$ be the quotient map. Define $g : \mathbb{R}^n \times H \to \mathbb{C}$ by

$$g(x, k) = f(x, q(k))$$

and $\tau : \mathbb{R}^n \times H \to \mathbb{C}$ by

$$\tau(x, k) = \psi(x, q(k))$$

for all $(x, k) \in \mathbb{R}^n \times H$.

One can verify that $g, \tau \in L^2(\mathbb{R} \times H)$. For $(x, h) \in \mathbb{R}^n \times H$, $(\xi, \gamma) \in \mathbb{R}^n \times \widehat{H}$ and $\zeta, \eta \in \mathcal{H}_\sigma$,

$$\begin{aligned}
\langle G_\tau g(x, h, \xi, \gamma)\zeta, \eta\rangle &= \int_{\mathbb{R}^n}\int_{H/K}\int_K g(y, vk)\overline{\tau(y - x, h^{-1}vk)} e^{-2\pi i\xi y}\langle \gamma(vk)^*\zeta, \eta\rangle \, dy\, d\dot{v}\, dk \\
&= \int_{\mathbb{R}^n}\int_{H/K}\int_K f(y, q(v))\overline{\psi(y - x, q(h^{-1}v))} e^{-2\pi i\xi y}\langle \gamma(k)^*\gamma(v)^*\zeta, \eta\rangle \, dy\, d\dot{v}\, dk \\
&= \begin{cases} \langle G_\psi f(x, \dot{h}, \xi, \gamma)\zeta, \eta\rangle & \text{if } \gamma \in A(K, \widehat{H}), \\ 0 & \text{otherwise,} \end{cases}
\end{aligned}$$

where $A(K, \widehat{H}) = \{\gamma \in \widehat{H} : \gamma(k) = 1_{\mathcal{H}_\gamma}, \text{ for all } k \in K\}$. Using (6.3),

$$\begin{aligned}
\|G_\tau g(x, h, \xi, \gamma)\|_{HS} &= \|G_\psi f(x, \dot{h}, \xi, \gamma)\|_{\mathrm{HS}} \\
&\leq e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2}\varphi_1(\dot{h})\varphi_2(\gamma) \\
&= e^{-\pi(a\|x\|^2 + b\|\xi\|^2)/2}\varphi_1(q(h))\varphi_2(\gamma).
\end{aligned}$$

By Hardy's theorem for the Gabor transform on $\mathbb{R}^n \times H$, we have either $g = 0$ a.e. or $\psi = 0$ a.e. Hence, either $f = 0$ a.e. or $\psi = 0$ a.e. □

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