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AN ALGEBRAIC INTERPRETATION OF THE SUPER CATALAN NUMBERS

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Abstract

We extend the notion of polynomial integration over an arbitrary circle C in the Euclidean geometry over general fields $\mathbb{F}$ of characteristic zero as a normalised $\mathbb{F}$-linear functional on $\mathbb{F}[\alpha_1, \alpha_2]$ that maps polynomials that evaluate to zero on C to zero and is $\mathrm{SO}(2, \mathbb{F})$-invariant. This allows us to not only build a purely algebraic integration theory in an elementary way, but also give the super Catalan numbers

$$S(m,n) = \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}$$

an algebraic interpretation in terms of values of this algebraic integral over some circle applied to the monomials $\alpha_1^{2m}\alpha_2^{2n}$.

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1. Introduction

This is the second of our series of papers building an integration theory of polynomials over unit circles over a general field $\mathbb{F}$. The first paper [9] deals with the case where $\mathbb{F}$ is finite of odd characteristic; the family of integers $S(m, n)$ called the *super Catalan numbers* and their closely related family of rational numbers $\Omega(m, n)$ called the *circular super Catalan numbers* play a prominent role. These numbers are defined by

$$S(m,n) := \frac{(2m)!\,(2n)!}{m!\,n!\,(m+n)!}, \quad \Omega(m,n) := \frac{S(m,n)}{4^{m+n}}$$

and are indexed by two elements in $\mathbb{N}$ (the nonnegative integers including 0).

The super Catalan numbers were first introduced by Catalan [3] in 1874 and the first modern study of these numbers was initiated by Gessel [7] in 1992. They generalise the Catalan numbers c_n since $S(1, n) = 2c_n$. The integrality of $S(m, n)$ can be observed from the relation $4S(m, n) = S(m + 1, n) + S(m, n + 1)$ which yields the Pascal-like property $\Omega(m, n) = \Omega(m + 1, n) + \Omega(m, n + 1)$.

No combinatorial interpretation of $S(m,n)$ is known for general m and n, in contrast to over 200 interpretations of the Catalan numbers [12]. However, for $m = 2$, there are some interpretations in terms of cubic trees by Pippenger and Schleich [10] and blossom trees by Schaeffer [11], and when $m = 2, 3$, as pairs of Dyck paths with restricted heights by Gessel and Xin [8]. When $n = m + s$ for $0 \leq s \leq 3$, Chen and Wang showed that there is an interpretation in terms of restricted lattice paths [4]. There is also a weighted interpretation of $S(m,n)$ as a certain value of Krawtchouk polynomials by the work of Georgiadis *et al.* [6], and another in terms of positive and negative 2-Motzkin paths by Allen and Gheorghiciuc [1].

The aim of this paper is twofold. The first is to build, in a rather elementary way, a polynomial integration theory over circles in the Euclidean geometry over general fields of characteristic zero without recourse to the usual Riemann integral and limiting processes. We shall see that this allows us to give the super Catalan numbers a purely algebraic interpretation, which is our second objective.

Here and throughout, $\mathbb{F}$ is a general field of characteristic zero with multiplicative identity $1_{\mathbb{F}}$ or sometimes just 1 if the context is clear. We denote by $\mathbb{F}[\alpha_1, \alpha_2]$ the algebra of polynomials in α_1 and α_2 over $\mathbb{F}$ with multiplicative identity $\mathbf{1}$. Our algebraic integral over a circle C is a linear functional ϕ on $\mathbb{F}[\alpha_1, \alpha_2]$, called a circular integral functional with respect to C, which satisfies three conditions: $\phi(\mathbf{1}) = 1_{\mathbb{F}}$ (*Normalisation*), $\phi(\pi) = 0$ whenever π evaluates to the zero function on C (*Locality*) and ϕ is rotationally invariant (*Invariance*).

When $\mathbb{F} = \mathbb{R}$, there is a well-known formula for the integral of polynomials on the unit sphere S^{n-1} (see [2] or [5]).

THEOREM 1.1. *Let $n \geq 2$ and S^{n-1} denote the $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$. If μ is the usual rotationally invariant measure on S^{n-1}, then by writing $b_i = \frac{1}{2}(d_i + 1)$,*

$$\int_{S^{n-1}} x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}\, d\mu = \begin{cases} \dfrac{2}{\Gamma(b_1 + b_2 + \cdots + b_n)} \displaystyle\prod_{i=1}^{n} \Gamma(b_i) & \text{if each } d_i \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

We may obtain from Theorem 1.1, when $n = 2$, $d_1 = 2m$ and $d_2 = 2n$, that

$$\frac{2\Gamma(b_1)\Gamma(b_2)}{\Gamma(b_1 + b_2)} = \frac{2\Gamma(m + \frac{1}{2})\Gamma(n + \frac{1}{2})}{\Gamma(m + n + 1)} = 2\pi \frac{(2m)!\,(2n)!}{4^{m+n} m!\, n!\,(m+n)!} = 2\pi\Omega(m,n).$$

So if the integral is normalised, we get just the circular super Catalan numbers.

In [9], we showed that the polynomial integration theory over a finite field of odd characteristic is analogous to the $\mathbb{F} = \mathbb{R}$ case, which we summarise in the next theorem.

THEOREM 1.2. *Let $p > 2$ be a prime and $q = p^r$ for some $r \in \mathbb{N}$. In the Euclidean geometry over $\mathbb{F}_q$ with multiplicative identity 1_q, the unit circle is defined by $S^1 = \{(x_1, x_2) \in \mathbb{F}_q^2 : x_1^2 + x_2^2 = 1_q\}$. Let k and ℓ be any natural numbers for which $0 \leq k + \ell < q - 1$. Then the functional $\psi_{b,q}$ on $\mathbb{F}_q[\alpha_1, \alpha_2]$ given by*

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$$\psi_{b,q}(\alpha_1^k\alpha_2^\ell) = -\left(\frac{-1}{p}\right)^r \sum_{(x_1,x_2)\in S^1} x_1^k x_2^\ell = \begin{cases} \Omega(m,n) \bmod p & \textit{if } k = 2m \textit{ and } \ell = 2n, \\ 0 & \textit{otherwise,} \end{cases}$$

is the unique circular integral functional with respect to S^1*. Here* $(\frac{-1}{p})$ *is the usual Legendre symbol.*

Now we present our main result. For $a \in \mathbb{Q}$, we denote by $a1_{\mathbb{F}}$ the embedding of a in $\mathbb{F}$. The unit circle S^1 in this setting will be defined in the next section.

THEOREM 1.3. *For any* $k, l \in \mathbb{N}$*, the linear functional* ψ *on* $\mathbb{F}[\alpha_1, \alpha_2]$ *defined by*

$$\psi(\alpha_1^k\alpha_2^\ell) = \begin{cases} \Omega(m,n)1_{\mathbb{F}} & \textit{if } k = 2m \textit{ and } \ell = 2n, \\ 0 & \textit{otherwise,} \end{cases}$$

is the unique circular integral functional with respect to S^1.

2. Circular integral functional

Denote by $\mathbb{A} = \mathbb{A}(\mathbb{F})$ the two-dimensional *affine plane* $\{(x_1, x_2) : x, y \in \mathbb{F}\}$, with the objects (x_1, x_2) called *points*. The space $\mathbb{F}^{\mathbb{A}}$ consists of functions from $\mathbb{A}$ to $\mathbb{F}$, and is an $\mathbb{F}$-algebra under pointwise addition and multiplication and with the evaluation map $\varepsilon\colon \mathbb{F}[\alpha_1, \alpha_2] \to \mathbb{F}^{\mathbb{A}}$ which is an algebra homomorphism. Clearly we may regard $\mathbb{F}[\alpha_1]$ as a subalgebra of $\mathbb{F}[\alpha_1, \alpha_2]$. Recall that any nonzero polynomial in $\mathbb{F}[\alpha_1]$ of degree d has at most d roots.

The group $\mathrm{GL}(2, \mathbb{F})$ of invertible 2×2 matrices with entries in $\mathbb{F}$ left-acts on $\mathbb{F}[\alpha_1, \alpha_2]$ as follows: if $\pi = \pi(\alpha_1, \alpha_2)$, then

$$h \cdot \pi = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \cdot \pi := \pi(h_{11}\alpha_1 + h_{21}\alpha_2, h_{12}\alpha_1 + h_{22}\alpha_2). \tag{2.1}$$

Additionally, $\mathrm{GL}(2, \mathbb{F})$ right-acts on $\mathbb{A}$ and left-acts on $\mathbb{F}^{\mathbb{A}}$ as follows:

$$(x_1, x_2) \cdot h := (h_{11}x_1 + h_{21}x_2, h_{12}x_1 + h_{22}x_2),$$
$$(h \cdot f)(x_1, x_2) := f((x_1, x_2) \cdot h) = f(h_{11}x_1 + h_{21}x_2, h_{12}x_1 + h_{22}x_2).$$

The group $\mathrm{SO}(2, \mathbb{F})$ of matrices h that satisfy $h^{-1} = h^T$ of determinant $1_{\mathbb{F}}$ is a subgroup of $\mathrm{GL}(2, \mathbb{F})$ and is called the rotation group. The action of $\mathrm{SO}(2, \mathbb{F})$ on $\mathbb{F}[\alpha_1, \alpha_2]$ is induced as the restriction of the action of $\mathrm{GL}(2, \mathbb{F})$ on $\mathbb{F}[\alpha_1, \alpha_2]$. This action respects evaluation: for any $h \in \mathrm{SO}(2, \mathbb{F})$ and $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$,

$$\varepsilon(h \cdot \pi) = h \cdot \varepsilon(\pi). \tag{2.2}$$

In a similar manner, we also get an action of $\mathrm{SO}(2, \mathbb{F})$ on $\mathbb{A}$ and $\mathbb{F}^{\mathbb{A}}$.

We define a symmetric bilinear form on $\mathbb{A}$, given by $(x_1, x_2) \cdot (y_1, y_2) := x_1y_1 + x_2y_2$. The associated quadratic form gives rise to the (Euclidean) unit circle

$$S^1 = S^1(\mathbb{F}) := \{(x_1, x_2) \in \mathbb{A} : x_1^2 + x_2^2 = 1_{\mathbb{F}}\}.$$

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LEMMA 2.1. *Each point on* S^1 *except* $(-1, 0)$ *can be written as*

$$\left(\frac{1-u^2}{1+u^2}, \frac{2u}{1+u^2}\right)$$

for some $u \in \mathbb{F}$ *such that* $1 + u^2 \neq 0$. *Consequently,* S^1 *is an infinite set.*

PROOF. The identity $((1 - u^2)/(1 + u^2))^2 + (2u/(1 + u^2))^2 = 1_{\mathbb{F}}$ holds for all $u \in \mathbb{F}$ for which $u^2 \neq -1$. The line $x_2 = ux_1 + u$ through the points $(-1, 0)$ and $(0, u)$ intersects S^1 at exactly two points, $(-1, 0)$ and $((1 - u^2)/(1 + u^2), 2u/(1 + u^2))$. Hence, every point on S^1 except $(-1, 0)$ corresponds to exactly one $u \in \mathbb{F}$ for which $u^2 \neq -1$. Since there are infinitely many $u \in \mathbb{F}$ for which $u^2 \neq -1$, S^1 is an infinite set. □

COROLLARY 2.2. *The rotation group* $\mathrm{SO}(2, \mathbb{F})$ *admits a parametrisation*

$$\mathrm{SO}(2, \mathbb{F}) = \left\{ h_u = \frac{1}{1+u^2}\begin{pmatrix} 1-u^2 & -2u \\ 2u & 1-u^2 \end{pmatrix} :\ u \in \mathbb{F}, u^2 \neq -1 \right\} \cup \{-I\}.$$

We now introduce the central object of this paper: a linear functional on $\mathbb{F}[\alpha_1, \alpha_2]$ that generalises normalised integration over the Euclidean unit circle over $\mathbb{R}$. We say that a linear functional $\phi\colon \mathbb{F}[\alpha_1, \alpha_2] \to \mathbb{F}$ is a *circular integral functional* on $\mathbb{F}[\alpha_1, \alpha_2]$ with respect to S^1 precisely when it satisfies the following three conditions.

(Normalisation) For the multiplicative identity $\mathbf{1}$ of $\mathbb{F}[\alpha_1, \alpha_2]$, we have $\phi(\mathbf{1}) = 1_{\mathbb{F}}$.

(Locality) If $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$ such that $\varepsilon(\pi)$ is the zero function on S^1, then $\phi(\pi) = 0$.

(Invariance) The functional ϕ is $\mathrm{SO}(2, \mathbb{F})$-invariant: $\phi(h \cdot \pi) = \phi(\pi)$ holds for any $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$ and $h \in \mathrm{SO}(2, \mathbb{F})$.

3. Existence and uniqueness

Our strategy to prove Theorem 1.3 is divided into two main steps. First, we show that ψ satisfies the Normalisation, Locality and Invariance conditions. Next, we demonstrate that if such a circular integral functional $\mathbb{F}[\alpha_1, \alpha_2]$ with respect to S^1 exists, it is uniquely determined.

It is easy to see that the Normalisation condition holds. The next two lemmas are needed to prove the Locality of ψ.

LEMMA 3.1. *Both* $S^1_{x_1} = \{x_2 \in \mathbb{F} : (x_1, x_2) \in S^1\}$ *and* $S^1_{x_2} = \{x_1 \in \mathbb{F} : (x_1, x_2) \in S^1\}$ *have infinitely many elements.*

PROOF. For any $(x_1, x_2) \in S^1$, we have that $(x_2, x_1) \in S^1$, so $S^1_{x_1} = S^1_{x_2}$. If $S^1_{x_1} = S^1_{x_2}$ is finite, then so is $S^1_{x_1} \times S^1_{x_2}$ and consequently S^1. This contradicts the fact that S^1 is infinite from Lemma 2.1. □

The crucial property of polynomials in $\mathbb{F}[\alpha_1, \alpha_2]$ that evaluate to the zero function on S^1 is that they must lie in $\langle \alpha_1^2 + \alpha_2^2 - 1 \rangle$, the ideal generated by $\alpha_1^2 + \alpha_2^2 - 1$. We offer an elementary proof below by using the multivariate polynomial division which

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requires a choice of monomial ordering. This has a flavour of Hilbert's Nullstellensatz which usually works over algebraically closed fields, although our argument does not assume that $\mathbb{F}$ is an algebraically closed field.

LEMMA 3.2. *If $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$ satisfies $\varepsilon(\pi) = 0$ on S^1, then $\pi \in \langle \alpha_1^2 + \alpha_2^2 - 1 \rangle$.*

PROOF. Fix a monomial ordering $\preccurlyeq$ such that $\alpha_1^{k_1}\alpha_2^{\ell_1} \preccurlyeq \alpha_1^{k_2}\alpha_2^{\ell_2}$ if either $k_1 < k_2$ or $k_1 = k_2$ and $\ell_1 < \ell_2$. With respect to this ordering, any $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$ can be written as $\pi = (\alpha_1^2 + \alpha_2^2 - 1)\pi_0 + \alpha_2\omega + \rho$ for some $\pi_0 \in \mathbb{F}[\alpha_1, \alpha_2]$ and $\omega, \rho \in \mathbb{F}[\alpha_1]$.

Since π evaluates to the zero function on S^1,

$$0 = \varepsilon(\pi)(x_1, x_2) = x_2\varepsilon(\omega)(x_1, x_2) + \varepsilon(\rho)(x_1, x_2) = x_2\varepsilon(\omega)(x_1, 0) + \varepsilon(\rho)(x_1, 0) \tag{3.1}$$

for all $(x_1, x_2) \in S^1$, where the last equation holds because $\omega, \rho \in \mathbb{F}[\alpha_1]$. Now consider the set $S^1_* = \{(x_1, x_2) \in S^1 : x_2 \neq 0\}$ which is nonempty since $(0, 1) \in S^1_*$. For any $(x_1, x_2) \in S^1_*$, the point $(x_1, -x_2) \in S^1_*$ is different from (x_1, x_2) so (3.1) forces $\varepsilon(\omega)(x_1, 0) = \varepsilon(\rho)(x_1, 0) = 0$ for all $(x_1, x_2) \in S^1$. By Lemma 3.1, ω and ρ have infinitely many roots, so they must be the zero polynomial. □

THEOREM 3.3 (Locality of ψ). *The linear functional ψ satisfies the Locality condition.*

PROOF. If $\pi \in \mathbb{F}[\alpha_1, \alpha_2]$ evaluates to the zero function on S^1, then $\pi \in \langle \alpha_1^2 + \alpha_2^2 - 1 \rangle$ from Lemma 3.2. By linearity, it suffices to show that

$$\psi((\alpha_1^2 + \alpha_2^2 - 1)\alpha_1^k\alpha_2^\ell) = \psi(\alpha_1^{k+2}\alpha_2^\ell) + \psi(\alpha_1^k\alpha_2^{\ell+2}) - \psi(\alpha_1^k\alpha_2^\ell)$$

vanishes. Clearly it does if either k or ℓ is odd, and if $k = 2m$ and $\ell = 2n$, the right-hand side simplifies to $\Omega(m + 1, n) + \Omega(m, n + 1) - \Omega(m, n) = 0$ by the Pascal-like property. □

Finally, to prove the Invariance of ψ, we need the following three lemmas.

LEMMA 3.4. *For any $h \in \mathrm{SO}(2, \mathbb{F})$ and natural numbers k, ℓ, each term of $h \cdot \alpha_1^k\alpha_2^\ell$ has degree $k + \ell$.*

PROOF. By using (2.1), $h \cdot \alpha_1^k\alpha_2^\ell = (h_{11}\alpha_1 + h_{21}\alpha_2)^k(h_{12}\alpha_1 + h_{22}\alpha_2)^\ell$. Expanding this using the binomial theorem, we see that the degree of each term is always $k + \ell$. □

LEMMA 3.5. *For any natural numbers m and $h \in \mathrm{SO}(2, \mathbb{F})$, we have $\psi(h \cdot \alpha_1^{2m}) = \psi(\alpha_1^{2m})$ and $\psi(h \cdot \alpha_1^{2m-1}\alpha_2) = \psi(\alpha^{2m-1}\alpha_2)$.*

PROOF. The statement is obviously true for $h = -I$. Now for $h = h_u$ defined in Corollary 2.2,

$$\psi(h_u \cdot \alpha_1^{2m}) = \sum_{s=0}^{2m} \binom{2m}{s} \left(\frac{1 - u^2}{1 + u^2}\right)^s \left(\frac{2u}{1 + u^2}\right)^{2m-s} \psi(\alpha_1^s\alpha_2^{2m-s}).$$

However, since the odd indices do not contribute to the sum, we just need to consider the even indices:

$$\begin{aligned}\psi(h_u\cdot\alpha_1^{2m}) &= \sum_{s=0}^{m}\binom{2m}{2s}\Big(\frac{1-u^2}{1+u^2}\Big)^{2s}\Big(\frac{2u}{1+u^2}\Big)^{2m-2s}\psi(\alpha_1^{2s}\alpha_2^{2m-2s})\\ &= \sum_{s=0}^{m}\binom{2m}{2s}\Omega(s,m-s)\Big(\frac{1-u^2}{1+u^2}\Big)^{2s}\Big(\frac{2u}{1+u^2}\Big)^{2m-2s}1_{\mathbb{F}}\\ &= \frac{(2m)!}{4^m m!\,m!}\sum_{s=0}^{m}\binom{m}{s}\Big(\frac{1-u^2}{1+u^2}\Big)^{2s}\Big(\frac{2u}{1+u^2}\Big)^{2m-2s}1_{\mathbb{F}}. \end{aligned}\tag{3.2}$$

Using the binomial theorem, (3.2) simplifies to

$$\psi(\alpha_1^{2m})\Big(\Big(\frac{1-u^2}{1+u^2}\Big)^2+\Big(\frac{2u}{1+u^2}\Big)^2\Big)^m = \psi(\alpha_1^{2m})1_{\mathbb{F}}.$$

The proof that $\psi(h_u\cdot\alpha_1^{2m-1}\alpha_2)=\psi(\alpha_1^{2m-1}\alpha_2)$ is more involved but done similarly. □

LEMMA 3.6. *If $\pi\in\mathbb{F}[\alpha_1,\alpha_2]$ evaluates to the zero function on S^1, then so does $h\cdot\pi$ for any $h\in\mathrm{SO}(2,\mathbb{F})$.*

PROOF. For an arbitrary $h\in\mathrm{SO}(2,\mathbb{F})$, the map $(x_1,x_2)\mapsto(x_1,x_2)\cdot h$ is a bijection on S^1. Choose any $(x_1,x_2)\in S^1$. Then $(x_1,x_2)=(u_1,u_2)\cdot h^{-1}$ for some $(u_1,u_2)\in S^1$. Using (2.2),

$$\varepsilon(h\cdot\pi)(x_1,x_2)=(h\cdot\varepsilon(\pi))(x_1,x_2)=\varepsilon(\pi)((x_1,x_2)\cdot h)=\varepsilon(\pi)(u_1,u_2)=0,$$

where the last equality follows from the assumption on π. Consequently, $h\cdot\pi$ evaluates to the zero function on S^1. □

THEOREM 3.7 (Invariance of ψ). *The linear functional ψ satisfies the Invariance condition.*

PROOF. It is sufficient to show that $\psi(h\cdot\alpha_1^k\alpha_2^\ell)=\psi(\alpha_1^k\alpha_2^\ell)$ for any $k,\ell\in\mathbb{N}$ and $h\in\mathrm{SO}(2,\mathbb{F})$. As before, the statement is obviously true for $h=-I$, so we will only show that $\psi(h_u\cdot\alpha_1^k\alpha_2^\ell)=\psi(\alpha_1^k\alpha_2^\ell)$. If $k+\ell$ is odd, then by Lemma 3.4, each term of $h_u\cdot\alpha_1^k\alpha_2^\ell$ has an odd degree and therefore $\psi(h_u\cdot\alpha_1^k\alpha_2^\ell)=0=\psi(\alpha_1^k\alpha_2^\ell)$.

The polynomial $\pi=\alpha_1^{2m}\alpha_2^{2n}-\alpha_1^{2m}(1-\alpha_1^2)^n$ evaluates to the zero function on S^1 and therefore by Lemma 3.6,

$$\psi(h_u\cdot\alpha_1^{2m}\alpha_2^{2n}) =\psi(h_u\cdot\alpha_1^{2m}(1-\alpha_1^2)^n)=\sum_{s=0}^{n}(-1)^s\binom{n}{s}\psi(h_u\cdot\alpha_1^{2m+2s}).$$

Now by Lemma 3.5, $\psi(h_u\cdot\alpha_1^{2m+2s})=\psi(\alpha_1^{2m+2s})$ and therefore this lets us retrace the steps:

$$\psi(h_u\cdot\alpha_1^{2m}\alpha_2^{2n})=\sum_{s=0}^{n}(-1)^s\binom{n}{s}\psi(\alpha_1^{2m+2s})=\psi(\alpha_1^{2m}(1-\alpha_1^2)^n)=\psi(\alpha_1^{2m}\alpha_2^{2n}),$$

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where in the last equality, we used the Locality of π again. The case when k and ℓ are both odd is treated similarly. The conclusion thus follows by the linearity of ψ. $\square$

Next, we proceed to show that ψ is the only circular integral functional on $\mathbb{F}[\alpha_1, \alpha_2]$ with respect to S^1.

THEOREM 3.8 (Existence implies uniqueness). *If ϕ is any circular integral functional on $\mathbb{F}[\alpha_1, \alpha_2]$ with respect to S^1, then ϕ is uniquely determined.*

PROOF. By linearity, it suffices to show that $\phi(\alpha_1^k \alpha_2^\ell)$ is uniquely determined for any $k, \ell \in \mathbb{N}$. Using the Invariance property with $h = -I$, we obtain $\phi(\alpha_1^k \alpha_2^\ell) = \phi(-I \cdot \alpha_1^k \alpha_2^\ell) = (-1)^{k+\ell} \phi(\alpha_1^k \alpha_2^\ell)$, so $\phi(\alpha_1^k \alpha_2^\ell) = 0$ whenever $k + \ell$ is odd.

For $m \geq 1$, another application of the Invariance property with $h = h_u$ from Corollary 2.2 gives

$$\phi(\alpha_1^{2m}) = \phi(h_u \cdot \alpha_1^{2m}) = \frac{1}{(1+u^2)^{2m}} \sum_{s=0}^{2m} \binom{2m}{s} (1-u^2)^s (2u)^{2m-s} \phi(\alpha_1^s \alpha_2^{2m-s}).$$

We multiply both sides by $(1+u^2)^{2m}$ and split the summation depending on the parity of s to obtain

$$\begin{aligned}(1+u^2)^{2m}\phi(\alpha_1^{2m}) &= \sum_{s=0}^{m} \binom{2m}{2s} (1-u^2)^{2s} (2u)^{2m-2s} \phi(\alpha_1^{2s} \alpha_2^{2m-2s}) \\ &\quad + \sum_{s=1}^{m} \binom{2m}{2s-1} (1-u^2)^{2s-1} (2u)^{2m-2s+1} \phi(\alpha_1^{2s-1} \alpha_2^{2m-2s+1}). \qquad (3.3)\end{aligned}$$

The two polynomials $\pi_1 = \alpha_1^{2s}\alpha_2^{2m-2s} - \alpha_1^{2s}(1-\alpha_1^2)^{m-s}$ and $\pi_2 = \alpha_1^{2s-1}\alpha_2^{2m-2s+1} - \alpha_1^{2s-1}(1-\alpha_1^2)^{m-s}\alpha_2$ both evaluate to the zero function on S^1 so by Locality,

$$\phi(\alpha_1^{2s}\alpha_2^{2m-2s}) = \phi(\alpha_1^{2s}(1-\alpha_1^2)^{m-s}) = \sum_{t=0}^{m-s} (-1)^t \binom{m-s}{t} \phi(\alpha_1^{2s+2t}), \qquad (3.4)$$

$$\phi(\alpha_1^{2s-1}\alpha_2^{2m-2s+1}) = \phi(\alpha_1^{2s-1}(1-\alpha_1^2)^{m-s}\alpha_2) = \sum_{t=0}^{m-s} (-1)^t \binom{m-s}{t} \phi(\alpha_1^{2s+2t-1}\alpha_2), \qquad (3.5)$$

respectively. By (3.4) and (3.5), (3.3) becomes

$$\begin{aligned}&(1+u^2)^{2m}\phi(\alpha_1^{2m}) \\ &\quad = \sum_{s=0}^{m}\sum_{t=0}^{m-s} (-1)^t \binom{2m}{2s}\binom{m-s}{t} (1-u^2)^{2s}(2u)^{2m-2s}\phi(\alpha_1^{2s+2t}) \\ &\qquad + \sum_{s=0}^{m}\sum_{t=0}^{m-s} (-1)^t \binom{2m}{2s-1}\binom{m-s}{t} (1-u^2)^{2s-1}(2u)^{2m-2s+1}\phi(\alpha_1^{2s+2t-1}\alpha_2).\end{aligned}$$

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Now the following polynomial of degree at most $4m$ in $\mathbb{F}[\beta]$, namely

$$\pi = (1+\beta^2)^{2m}\phi(\alpha_1^{2m}) - \sum_{s=0}^{m}\sum_{t=0}^{m-s}(-1)^t\binom{2m}{2s}\binom{m-s}{t}(1-\beta^2)^{2s}(2\beta)^{2m-2s}\phi(\alpha_1^{2s+2t})$$
$$-\sum_{s=0}^{m}\sum_{t=0}^{m-s}(-1)^t\binom{2m}{2s-1}\binom{m-s}{t}(1-\beta^2)^{2s-1}(2\beta)^{2m-2s+1}\phi(\alpha_1^{2s+2t-1}\alpha_2),$$

has infinitely many roots, so π is identically zero. By extracting the coefficient of β and β^2, respectively we get $4m\phi(\alpha_1^{2m-1}\alpha_2) = 0$ and $8m^2\phi(\alpha_1^{2m}) - 4m(2m-1)\phi(\alpha_1^{2m-2}) = 0$.

Since m is arbitrary, for any $m \geq 1$, we must have $\phi(\alpha_1^{2m-1}\alpha_2) = 0$ and the first-order recurrence relation $2m\phi(\alpha_1^{2m}) = (2m-1)\phi(\alpha_1^{2m-2})$ with the initial condition $\phi(\mathbf{1}) = 1_{\mathbb{F}}$. Thus, we see that $\phi(\alpha_1^{2m})$ and $\phi(\alpha_1^{2m-1}\alpha_2)$ are uniquely determined for all $m \geq 1$.

Finally, by using the Locality condition again, both $\phi(\alpha_1^{2m+1}\alpha_2^{2n+1})$ and $\phi(\alpha_1^{2m}\alpha_2^{2n})$ are uniquely determined since

$$\phi(\alpha_1^{2m+1}\alpha_2^{2n+1}) = \phi(\alpha_1^{2m+1}(1-\alpha_1^2)^n\alpha_2) = \sum_{s=0}^{n}(-1)^n\binom{n}{s}\phi(\alpha_1^{2m+2s+1}\alpha_2),$$
$$\phi(\alpha_1^{2m}\alpha_2^{2n}) = \phi(\alpha_1^{2m}(1-\alpha_1^2)^n) = \sum_{s=0}^{n}(-1)^n\binom{n}{s}\phi(\alpha_1^{2m+2s}).$$

This concludes the proof. □

4. Generalisation to arbitrary circles

Fix a point $(a,b) \in \mathbb{A}$ and a nonzero $r \in \mathbb{F}$. We define $S^1_{r,(a,b)}$ to be the collection of points $(x_1,x_2) \in \mathbb{A}$ such that $(x-a)^2 + (y-b)^2 = r^2$. A linear functional $\phi_{r,(a,b)}\colon \mathbb{F}[\alpha_1,\alpha_2] \to \mathbb{F}$ is a circular integral functional on $\mathbb{F}[\alpha_1,\alpha_2]$ with respect to $S^1_{r,(a,b)}$ if the following conditions are satisfied.

(Normalisation) For the multiplicative identity $\mathbf{1}$ of $\mathbb{F}[\alpha_1,\alpha_2]$, we have $\phi_{r,(a,b)}(\mathbf{1}) = r$.

(Locality) If $\pi \in \mathbb{F}[\alpha_1,\alpha_2]$ such that $\varepsilon(\pi) = 0$ on $S^1_{r,(a,b)}$, we have $\phi_{r,(a,b)}(\pi) = 0$.

(Invariance) For any $\pi \in \mathbb{F}[\alpha_1,\alpha_2]$ and $h \in \mathrm{SO}(2,\mathbb{F})$, $\phi_{r,(a,b)}(h\cdot\pi) = \phi_{r,(a,b)}(\pi)$.

By employing the same analysis, the existence and uniqueness of a circular integral functional on $\mathbb{F}[\alpha_1,\alpha_2]$ with respect to $S^1_{r,(a,b)}$ can be derived from that of ψ.

THEOREM 4.1. *There is one and only one circular integral functional on* $\mathbb{F}[\alpha_1,\alpha_2]$ *with respect to* $S^1_{r,(a,b)}$, *given by*

$$\psi_{r,(a,b)}(\alpha_1^k\alpha_2^\ell) = r\psi((a+r\alpha_1)^k(b+r\alpha_2)^\ell),$$

where ψ *is the circular integral functional on* $\mathbb{F}[\alpha_1,\alpha_2]$ *with respect to* S^1.

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Now we are finally able to give an algebraic interpretation of the super Catalan numbers $S(m,n)$.

THEOREM 4.2 (An algebraic interpretation of $S(m,n)$). *Over $\mathbb{Q}$, for any $m,n \in \mathbb{N}$, we have $2S(m,n) = \psi_{2,(0,0)}(\alpha_1^{2m}\alpha_2^{2n})$.*

PROOF. The result follows immediately from Theorem 4.1 since

$$\psi_{2,(0,0)}(\alpha_1^{2m}\alpha_2^{2n}) = 2\psi((2\alpha_1)^{2m}(2\alpha_2)^{2n}) = \frac{2^{2m+2n+1}}{4^{m+n}}S(m,n)1_{\mathbb{Q}} = 2S(m,n). \qquad \square$$

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