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TRAVELLING WAVE SOLUTIONS IN NONLOCAL REACTION–DIFFUSION SYSTEMS WITH DELAYS AND APPLICATIONS

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Abstract

This paper deals with two-species convolution diffusion-competition models of Lotka–Volterra type with delays which describe more accurate information than the Laplacian diffusion-competition models. We first investigate the existence of travelling wave solutions of a class of nonlocal convolution diffusion systems with weak quasimonotonicity or weak exponential quasimonotonicity by a cross-iteration technique and Schauder's fixed point theorem. When the results are applied to the convolution diffusion-competition models with delays, we establish the existence of travelling wave solutions as well as asymptotic behaviour.

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1. Introduction

The theory of travelling wave solutions for Laplacian diffusion equations has attracted much attention due to its significant nature in biology, chemistry, epidemiology and physics (see [7, 13, 22, 24, 26]). Recently, many researchers have focused on the existence of travelling wave solutions for Laplacian diffusion equations with discrete time delays (see [2, 10, 11, 18, 19]) and spatiotemporal or nonlocal delays (see [6, 15, 23, 25] and the references therein). Despite the popularity of Laplacian diffusion models, they have some drawbacks. One important shortcoming for ecological and epidemiological models is that Laplacian diffusion is a local operator where individuals in the population can only influence their immediate neighbours. With diffusion models there is some disconnection between the experimentally collected data and the limited number of parameters that are available to fit that data. One

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method in overcoming these problems with the Laplacian operator is to describe these models concerning spatial migration by integral equations, such as the diffusion system with the convolution operator

$$\frac{\partial u(x,t)}{\partial t} = D[(J * u)(x,t) - u(x,t)] + f(u(x,t)), \tag{1.1}$$

where $*$ is the spatial convolution operator and

$$(J * u)(x,t) = \int_{\mathbb{R}} J(y) u(x-y,t)\, dy.$$

Lee *et al.* [9] argue that, for processes where the spatial scale for movement is large in comparison with its temporal scale, nonlocal models using integro-differential equations may allow for a better estimation of the parameters from the data and provide more insight into the biological system. In (1.1) if the diffusion kernel $J(x) = \delta(x) + \delta''(x)$, where δ is the Dirac delta (see Medlock and Kot [12]), then (1.1) reduces to the traditional reaction-diffusion model

$$\frac{\partial}{\partial t} u(x,t) = D \frac{\partial u^2(x,t)}{\partial x^2} + f(u(x,t)). \tag{1.2}$$

For the nonlocal convolution diffusion model (1.1) without delays, one can refer to [1, 3–5] and the references cited therein. Delays are incorporated into (1.1) by the authors in [17] and [16] with the quasimonotone condition and the exponential quasimonotone condition, respectively. Unfortunately, it is quite common for the reaction terms in some systems arising from a practical problem to not satisfy the above conditions; for example, this is true for the following nonlocal convolution diffusion competitive models with delays:

$$\begin{cases} \dfrac{\partial}{\partial t} u_1(x,t) = d_1[(J_1 * u_1)(x,t) - u_1(x,t)] \\ \qquad\qquad + r_1 u_1(x,t)[1 - a_1 u_1(x,t) - b_1 u_2(x,t-\tau_1)], \\ \dfrac{\partial}{\partial t} u_2(x,t) = d_2[(J_2 * u_2)(x,t) - u_2(x,t)] \\ \qquad\qquad + r_2 u_2(x,t)[1 - b_2 u_1(x,t-\tau_2) - a_2 u_2(x,t)] \end{cases} \tag{1.3}$$

and

$$\begin{cases} \dfrac{\partial}{\partial t} u_1(x,t) = d_1[(J_1 * u_1)(x,t) - u_1(x,t)] \\ \qquad\qquad + r_1 u_1(x,t)[1 - a_1 u_1(x,t-\tau_1) - b_1 u_2(x,t-\tau_2)], \\ \dfrac{\partial}{\partial t} u_2(x,t) = d_2[(J_2 * u_2)(x,t) - u_2(x,t)] \\ \qquad\qquad + r_2 u_2(x,t)[1 - b_2 u_1(x,t-\tau_3) - a_2 u_2(x,t-\tau_4)]. \end{cases} \tag{1.4}$$

Here $u_i(t,x)$, $i = 1, 2$, are the densities of the populations of two species at location x and time t. With a diffusion rate d_i, $i = 1, 2$, individuals move from their current location x and instantaneously arrive at some new location y. This process is represented by integro-differential equations (1.3) and (1.4): the second terms on the right-hand sides, $-d_i u_i$, $i = 1, 2$, represent individuals leaving location x, while the first terms, the convolutions $d_i J_i * u_i$, $i = 1, 2$, represent the total number of individuals arriving at x from all possible locations y. Here r_i, $i = 1, 2$, are net birth rates, $1/b_i$, $i = 1, 2$, are carrying capacities, a_i, $i = 1, 2$, are competition coefficients and the delays τ_i, $i = 1, \ldots, 4$, are nonnegative constants. In particular, letting $J_1(x) = J_2(x) = \delta(x) + \delta''(x)$, (1.3) and (1.4) reduce to the corresponding Laplacian diffusion competition systems which are investigated in [10, 18, 24]. In addition, if $\tau_i = 0$, $i = 1, \ldots, 4$, (1.3) and (1.4) reduce to competition systems used to describe the local interaction between the externally introduced gray squirrel and the indigenous red squirrel in Britain [8, 14] (see also [20, 21]).

In order to focus on the mathematical ideas, we will discuss the following general convolution diffusion systems with delays:

$$\begin{cases} \dfrac{\partial}{\partial t} u_1(x, t) = d_1[(J_1 * u_1)(x, t) - u_1(x, t)] + f_1(u_{1,t}(x), u_{2,t}(x)), \\ \dfrac{\partial}{\partial t} u_2(x, t) = d_2[(J_2 * u_2)(x, t) - u_2(x, t)] + f_2(u_{1,t}(x), u_{2,t}(x)), \end{cases} \tag{1.5}$$

where $d_i > 0$, $u_{i,t}(x)(\theta) = u_i(x, t + \theta)$, $-\tau \le \theta \le 0$, τ denotes the maximal time delay [24], $f_i : \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and $J_i : \mathbb{R} \to \mathbb{R}$ are nonnegative functions with $\int_{\mathbb{R}} J_i(y)\, dy = 1$, for $i = 1, 2$. Applying the method developed in [10], we prove the existence of travelling wave solutions of (1.5) with nonlocal dispersal. If we take $J_1(x) = J_2(x) = \delta(x) + \delta''(x)$ in (1.5), then (1.5) reduces to the Laplacian reaction diffusion equations with weak quasimonotonicity or weak exponential quasimonotonicity, which were investigated in [10]. Thus, our results contain those of [10].

This paper is organized as follows. Section 2 is devoted to preliminary and abstract results. More precisely, we investigate the existence of travelling wave solutions of a class of nonlocal convolution diffusion systems with weak quasimonotonicity (WQM) or weak exponential quasimonotonicity (WQM*) by applying the method developed in [10]. In Section 3, we apply our main results to the convolution diffusion-competition systems (1.3) and (1.4) and prove the existence of travelling wave solutions by constructing a pair of suitable upper–lower solutions. Because of the introduction of nonlocal diffusion, it is more difficult to construct and verify upper–lower solutions. Thus we choose a pair of suitable upper–lower solutions which make a slight difference to our calculations from those in [10] (see Remark 1).

2. Preliminary and abstract results

We first introduce the usual notation for the standard ordering in $\mathbb{R}^2$. That is, for $u = (u_1, u_2)$ and $v = (v_1, v_2)$, we denote $u \le v$ if $u_i \le v_i$, $i = 1, 2$, and $u < v$ if $u \le v$ but $u \ne v$. In particular, we denote $u \ll v$ if $u \le v$ but $u_i \ne v_i$, $i = 1, 2$.

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If $u \le v$, we also denote $(u, v] = \{w \in \mathbb{R}^2, u < w \le v\}$, $[u, v) = \{w \in \mathbb{R}^2, u \le w < v\}$ and $[u, v] = \{w \in \mathbb{R}^2, u \le w \le v\}$. Let $|\cdot|$ denote the Euclidean norm in $\mathbb{R}^2$ and $\|\cdot\|$ denote the supremum norm in $C([-\tau, 0], \mathbb{R}^2)$. A travelling wave solution of (1.5) is a special translation invariant solution of the form $u_1(x, t) = \phi(x + ct)$, $u_2(x, t) = \psi(x + ct)$, where $(\phi, \psi) \in C^1(\mathbb{R}, \mathbb{R}^2)$ are the profiles of the wave that propagates through the one-dimensional spatial domain at a constant velocity $c > 0$. Substituting $u_1(x, t) = \phi(x + ct)$, $u_2(x, t) = \psi(x + ct)$ into (1.5), then (ϕ, ψ) satisfies

$$\begin{cases} c\phi'(t) = d_1(J_1 * \phi)(t) - d_1\phi(t) + f_1^c(\phi_t, \psi_t), \\ c\psi'(t) = d_2(J_2 * \psi)(t) - d_2\psi(t) + f_2^c(\phi_t, \psi_t), \end{cases} \tag{2.1}$$

where $f_i^c(\phi, \psi) : C([-\tau, 0], \mathbb{R}^2) \to \mathbb{R}$ is defined by $f_i(\phi^c, \psi^c)$, $i = 1, 2$, and $\phi^c(s) = \phi(cs)$ and $\psi^c(s) = \psi(cs)$, for $s \in [-\tau, 0]$.

Motivated by the background of travelling wave solutions, we also require that (ϕ, ψ) satisfies asymptotic boundary conditions

$$\lim_{t\to-\infty} (\phi(t), \psi(t)) = (0, 0), \qquad \lim_{t\to\infty} (\phi(t), \psi(t)) = (k_1, k_2), \tag{2.2}$$

where $(0, 0)$ and (k_1, k_2) are two equilibria of (2.1).

In this paper we are interested in travelling wave solutions of (2.1) and (2.2) when the reaction terms satisfy the following WQM or WQM* conditions.

DEFINITION 2.1. Suppose that for any $\phi_1(s), \phi_2(s), \psi_1(s), \psi_2(s) \in C([-\tau, 0], \mathbb{R})$ such that

$$0 \le \phi_2(s) \le \phi_1(s) \le M_1, \quad 0 \le \psi_2(s) \le \psi_1(s) \le M_2 \quad \text{for } s \in [-\tau, 0], \tag{2.3}$$

there exist two positive numbers β_1 and β_2 such that

$$\begin{gathered} f_1(\phi_1(s), \psi_1(s)) - f_1(\phi_2(s), \psi_1(s)) + (\beta_1 - d_1)(\phi_1(0) - \phi_2(0)) \ge 0, \\ f_1(\phi_1(s), \psi_1(s)) - f_1(\phi_1(s), \psi_2(s)) \le 0, \\ f_2(\phi_1(s), \psi_1(s)) - f_2(\phi_1(s), \psi_2(s)) + (\beta_2 - d_2)(\psi_1(0) - \psi_2(0)) \ge 0, \\ f_2(\phi_1(s), \psi_1(s)) - f_2(\phi_2(s), \psi_1(s)) \le 0. \end{gathered} \tag{2.4}$$

Then the reaction terms have the property of WQM.

Suppose that for any $\phi_1(s), \phi_2(s), \psi_1(s), \psi_2(s) \in C([-\tau, 0], \mathbb{R})$ satisfying (2.3) there exist two positive numbers β_1 and β_2 such that (2.4) holds and $e^{\beta_1 s}[\phi_1(s) - \phi_2(s)]$ and $e^{\beta_2 s}[\psi_1(s) - \psi_2(s)]$ are nondecreasing in $s \in [-\tau, 0]$. In this case, the reaction terms have the property of WQM*.

For $\mathbf{0} = (0, 0)$ and $\mathbf{M} = (M_1, M_2)$, let

$$C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) = \big\{(\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) \mid 0 \le \phi(s) \le M_1, 0 \le \psi(s) \le M_2, s \in \mathbb{R}\big\}.$$

Define the operator $H = (H_1, H_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\begin{aligned} H_1(\phi, \psi)(t) &= f_1^c(\phi_t, \psi_t) + d_1(J_1 * \phi)(t) + (\beta_1 - d_1)\phi(t), \\ H_2(\phi, \psi)(t) &= f_2^c(\phi_t, \psi_t) + d_2(J_2 * \psi)(t) + (\beta_2 - d_2)\psi(t), \end{aligned} \tag{2.5}$$

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and the operator $\mathbf{F} = (F_1, F_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$ by

$$\begin{cases} F_1(\phi, \psi)(t) = \dfrac{1}{c} e^{-\beta_1 t/c} \displaystyle\int_{-\infty}^{t} e^{\beta_1 s/c} H_1(\phi, \psi)(s)\, ds, \\ F_2(\phi, \psi)(t) = \dfrac{1}{c} e^{-\beta_2 t/c} \displaystyle\int_{-\infty}^{t} e^{\beta_2 s/c} H_2(\phi, \psi)(s)\, ds. \end{cases} \tag{2.6}$$

Then the problem is changed into investigating whether the fixed point of $\mathbf{F}$, which is a travelling wave solution of (1.5) connecting $\mathbf{0} = (0, 0)$ and $\mathbf{K} = (k_1, k_2)$, satisfies (2.2).

In the following we introduce the exponential decay norm. For $0 < \mu < \min\{\beta_1/c, \beta_2/c\}$, define

$$B_\mu(\mathbb{R}, \mathbb{R}^2) = \left\{\Phi \mid \Phi(t) \in C(\mathbb{R}, \mathbb{R}^2) \text{ and } \sup_{t\in\mathbb{R}} |\Phi(t)| e^{-\mu|t|} < \infty\right\}.$$

It is easy to check that $B_\mu(\mathbb{R}, \mathbb{R}^2)$ is a Banach space equipped with the norm $|\cdot|_\mu$ defined by $|\Phi|_\mu = \sup_{t\in\mathbb{R}} |\Phi(t)| e^{-\mu|t|}$ for $\Phi \in B_\mu(\mathbb{R}, \mathbb{R}^2)$.

We make the following assumptions concerning (1.5) throughout this paper:

(A1) $f_i(0, 0) = f_i(k_1, k_2) = 0$ for $i = 1, 2$;

(A2) there exist two positive constants L_1 and L_2 such that

$$|f_i(\phi_1, \psi_1) - f_i(\phi_2, \psi_2)| \le L_i \|\Phi - \Psi\|, \quad i = 1, 2,$$

for $\Phi = (\phi_1, \psi_1), \Psi = (\phi_2, \psi_2) \in C([-\tau, 0], \mathbb{R}^2)$ where $0 \le \phi_i(s), \psi_i(s) \le M_i$, $s \in [-\tau, 0]$ and $M_i > k_i$ is positive constant, $i = 1, 2$;

(A3) $J_i : \mathbb{R} \to \mathbb{R}$ is a nonnegative function with $\int_{\mathbb{R}} J_i(y)\, dy = 1$, $i = 1, 2$;

(A4) $\int_{\mathbb{R}} J_i(y) e^{\lambda y}\, dy < \infty$, $i = 1, 2$ for any $\lambda \in \mathbb{R}$.

2.1. The WQM case In this subsection, we consider the existence of travelling wave solutions of (2.1) when the delayed reaction terms f_1 and f_2 are WQM.

DEFINITION 2.2. The two functions $\overline{\Phi} = (\overline{\phi}, \overline{\psi})$ and $\underline{\Phi} = (\underline{\phi}, \underline{\psi}) \in C(\mathbb{R}, \mathbb{R}^2)$ are called the upper and lower solutions of (2.1), respectively, if there exists $\mathbb{T} = \{T_i \mid i = 1, \ldots, m\}$ such that $\overline{\Phi}$ and $\underline{\Phi}$ are once continuously differentiable in $\mathbb{R}\backslash\mathbb{T}$ and satisfy

$$\begin{cases} c\overline{\phi}'(t) \ge d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) + f_1^c(\overline{\phi}_t, \underline{\psi}_t) \\ c\overline{\psi}'(t) \ge d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) + f_2^c(\underline{\phi}_t, \overline{\psi}_t) \end{cases} \quad \text{in } \mathbb{R}\backslash\mathbb{T} \tag{2.7}$$

and

$$\begin{cases} c\underline{\phi}'(t) \le d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) + f_1^c(\underline{\phi}_t, \overline{\psi}_t) \\ c\underline{\psi}'(t) \le d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) + f_2^c(\underline{\phi}_t, \overline{\psi}_t) \end{cases} \quad \text{in } \mathbb{R}\backslash\mathbb{T}. \tag{2.8}$$

We assume that (2.1) has an upper solution $\overline{\Phi}$ and a lower solution $\underline{\Phi}$ satisfying the following hypotheses:

(P1) $(0, 0) \le (\underline{\phi}(t), \underline{\psi}(t)) \le (\overline{\phi}(t), \overline{\psi}(t)) \le (M_1, M_2)$;

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(P2)
$$\lim_{t\to-\infty}(\overline{\phi}(t), \overline{\psi}(t)) = (0, 0),$$
$$\lim_{t\to+\infty}(\underline{\phi}(t), \underline{\psi}(t)) = \lim_{t\to+\infty}(\overline{\phi}(t), \overline{\psi}(t)) = (k_1, k_2).$$

Now we formulate our main result as follows.

THEOREM 2.1. *Assume that* (A1)–(A4) *and WQM hold. If* (2.1) *has an upper solution* $(\overline{\phi}(t), \overline{\psi}(t)) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *and a lower solution* $(\underline{\phi}(t), \underline{\psi}(t)) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *such that* (P1) *and* (P2) *hold, then* (2.1) *has a travelling wave solution satisfying* (2.2).

PROOF. Assume that $\overline{\Phi}$ and $\underline{\Phi} \in C(\mathbb{R}, \mathbb{R}^2)$ are a pair of upper–lower solutions of (2.1) satisfying (P1) and (P2). Define the wave profile set Γ by

$$\Gamma\left([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}]\right) = \left\{(\phi, \psi) \in C(\mathbb{R}, \mathbb{R}^2) \;\middle|\; \underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t),\ \underline{\psi}(t) \le \psi(t) \le \overline{\psi}(t),\ t \in \mathbb{R}\right\}.$$

Obviously, $\Gamma([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$ is a nonempty, closed and bounded convex set. Similarly to the proof in [10, 16, 17], it is easily seen that

$$\mathbf{F} = (F_1, F_2) : C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \to C(\mathbb{R}, \mathbb{R}^2)$$

is continuous with respect to the norm $|\cdot|_\mu$ in $B_\mu(\mathbb{R}, \mathbb{R}^2)$ and

$$\underline{\phi} \le F_1(\underline{\phi}, \overline{\psi}) \le F_1(\overline{\phi}, \underline{\psi}) \le \overline{\phi}, \quad \underline{\psi} \le F_2(\overline{\phi}, \underline{\psi}) \le F_2(\underline{\phi}, \overline{\psi}) \le \overline{\psi}.$$

Moreover, $\mathbf{F}\big(\Gamma([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])\big) \subset \Gamma([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$ and the map $\mathbf{F}$ is compact with respect to the decay norm $|\cdot|_\mu$. By Schauder's fixed point theorem, there exists a fixed point $(\phi^*, \psi^*) \in \Gamma([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$, so it is a solution of (2.1).

Next, we need to verify the asymptotic boundary condition (2.2). By (P2) and

$$\mathbf{0} \le (\underline{\phi}(t), \underline{\psi}(t)) \le (\phi^*(t), \psi^*(t)) \le (\overline{\phi}(t), \overline{\psi}(t)) \le (M_1, M_2),$$

we see that $\lim_{t\to-\infty}(\phi^*(t), \psi^*(t)) = (0, 0)$ and $\lim_{t\to+\infty}(\phi^*(t), \psi^*(t)) = (k_1, k_2)$. This completes the proof. □

2.2. The WQM* case

We assume that (2.1) has an upper solution $\overline{\Phi} = (\overline{\phi}, \overline{\psi})$ and a lower solution $\underline{\Phi} = (\underline{\phi}, \underline{\psi})$ not only satisfying the hypotheses (P1) and (P2) but also:

(P3) $e^{\beta_1 t}[\overline{\phi}(t) - \underline{\phi}(t)]$ and $e^{\beta_2 t}[\overline{\psi}(t) - \underline{\psi}(t)]$ are nondecreasing for $t \in \mathbb{R}$.

Define the set Γ^* consisting of wave profiles by

$$\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}]) = \left\{(\phi, \psi) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2) \;\middle|\; \begin{array}{l} \text{(i)}\ \underline{\phi}(t) \le \phi(t) \le \overline{\phi}(t), \quad \underline{\psi}(t) \le \psi(t) \le \overline{\psi}(t), \\ \quad t \in \mathbb{R}, \\ \text{(ii)}\ e^{\beta_1 t}[\phi(t) - \underline{\phi}(t)], \quad e^{\beta_1 t}[\overline{\phi}(t) - \phi(t)], \\ \quad e^{\beta_2 t}[\psi(t) - \underline{\psi}(t)], \quad e^{\beta_2 t}[\overline{\psi}(t) - \psi(t)] \\ \quad \text{are nondecreasing for } t \in \mathbb{R} \end{array}\right\}.$$

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Obviously, $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$ is nonempty. In fact, by (P3), $(\underline{\phi}, \underline{\psi})$, $(\overline{\phi}, \overline{\psi})$ satisfy (i) and (ii) of $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$. Moreover, $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$ is a closed and bounded convex set. Thus, we have the following result.

THEOREM 2.2. *Assume that* (A1)–(A4) *and WQM* hold. Suppose that* (2.1) *has an upper solution* $(\overline{\phi}(t), \overline{\psi}(t)) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *and a lower solution* $(\underline{\phi}(t), \underline{\psi}(t)) \in C_{[\mathbf{0},\mathbf{M}]}(\mathbb{R}, \mathbb{R}^2)$ *such that* (P1), (P2) *and* (P3) *hold. Then, for any* $c \geq 1$, (2.1) *has a travelling wave solution in* $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$ *satisfying* (2.2).

PROOF. Letting $(\phi, \psi) \in \Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$, it is easy to see that

$$\underline{\phi} \leq F_1(\underline{\phi}, \overline{\psi}) \leq F_1(\phi, \psi) \leq F_1(\overline{\phi}, \underline{\psi}) \leq \overline{\phi} \quad \text{and}$$
$$\underline{\psi} \leq F_2(\overline{\phi}, \underline{\psi}) \leq F_2(\phi, \psi) \leq F_2(\underline{\phi}, \overline{\psi}) \leq \overline{\psi}.$$

This implies that $\mathbf{F}(\phi, \psi) = (F_1(\phi, \psi), F_2(\phi, \psi))$ satisfies statement (i) of $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$.

Now we need to prove that $(F_1(\phi, \psi), F_2(\phi, \psi))$ satisfies statement (ii) of $\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$. Letting $F_1(\phi, \psi) = \phi_1$ for $(\phi, \psi) \in \Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$, we then have

$$\begin{aligned}
e^{\beta_1 t}[\overline{\phi}(t) - \phi_1(t)] &= \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^{t} e^{\beta_1 s/c} \big\{[c\overline{\phi}'(s) + \beta_1 \overline{\phi}(s)] \\
&\quad - [c\phi_1'(s) + \beta_1 \phi_1(s)]\big\}\, ds \\
&= \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^{t} e^{\beta_1 s/c} \big\{[c\overline{\phi}'(s) + \beta_1 \overline{\phi}(s) - H_1(\phi, \psi)(s)] \\
&\quad - [c\phi_1'(t) + \beta_1 \phi_1(t) - H_1(\phi, \psi)(s)]\big\}\, ds \\
&= \frac{1}{c} e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^{t} e^{\beta_1 s/c} [c\overline{\phi}'(s) + \beta_1 \overline{\phi}(s) - H_1(\phi, \psi)(s)]\, ds.
\end{aligned}$$

Hence, for $c \geq 1$, we have, for $t \in \mathbb{R}$,

$$\begin{aligned}
&\frac{d}{dt}\{e^{\beta_1 t}[\overline{\phi}(t) - \phi_1(t)]\} \\
&\quad = \frac{1}{c}\left(\beta_1 - \frac{\beta_1}{c}\right) e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^{t} e^{\beta_1 s/c} [c\overline{\phi}'(s) + \beta_1 \overline{\phi}(s) - H_1(\phi, \psi)(s)]\, ds \\
&\qquad + \frac{1}{c} e^{\beta_1 t}[c\overline{\phi}'(t) + \beta_1 \overline{\phi}(t) - H_1(\phi, \psi)(t)] \\
&\quad \geq \frac{1}{c}\left(\beta_1 - \frac{\beta_1}{c}\right) e^{(\beta_1 - \beta_1/c)t} \int_{-\infty}^{t} e^{\beta_1 s/c} [c\overline{\phi}'(s) + \beta_1 \overline{\phi}(s) - H_1(\overline{\phi}, \underline{\psi})(s)]\, ds \\
&\qquad + \frac{1}{c} e^{\beta_1 t}[c\overline{\phi}'(t) + \beta_1 \overline{\phi}(t) - H_1(\overline{\phi}, \underline{\psi})(t)] \geq 0.
\end{aligned}$$

Thus, $e^{\beta_1 t}[\overline{\phi}(t) - F_1(\phi, \psi)(t)]$ is nondecreasing for $t \in \mathbb{R}$.

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Similarly, we can prove that $e^{\beta_2 t}[\overline{\psi}(t) - F_2(\phi, \psi)(t)]$, $e^{\beta_2 t}[F_2(\phi, \psi)(t) - \underline{\psi}(t)]$ and $e^{\beta_1 t}[F_1(\phi, \psi)(t) - \underline{\phi}(t)]$ are nondecreasing in $t \in \mathbb{R}$. Thus

$$\mathbf{F}\Big(\Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])\Big) \subset \Gamma^*([\underline{\phi}, \underline{\psi}], [\overline{\phi}, \overline{\psi}])$$

and, similar to the proof in [10, 16, 17], $\mathbf{F}$ is compact with respect to the decay norm $|\cdot|_\mu$. Therefore, there exists $(\phi^*(t), \psi^*(t))$ satisfying the asymptotic boundary condition (2.2). This completes the proof. □

3. Convolution diffusion-competition models

In this section, we apply Theorems 2.1 and 2.2 to establish the existence of travelling wave solutions for Systems (1.3) and (1.4).

3.1. Model (1.3) We consider the existence of travelling wave solutions for the nonlocal diffusion-competition system (1.3), where $d_i > 0$, $a_i > 0$, $b_i > 0$, $\tau_i \geq 0$, $i = 1, 2$ are constants. Suppose that J_1 and J_2 are even functions which satisfy (A3) and (A4).

Let $c > 0$. A travelling wave solution $(\phi(x + ct), \psi(x + ct))$ of (1.3) must satisfy

$$\begin{cases} c\phi'(t) = d_1(J_1 * \phi)(t) - d_1\phi(t) + r_1\phi(t)(1 - a_1\phi(t) - b_1\psi(t - c\tau_1)), \\ c\psi'(t) = d_2(J_1 * \psi)(t) - d_2\psi(t) + r_2\psi(t)(1 - b_2\phi(t - c\tau_2) - a_2\psi(t)). \end{cases} \tag{3.1}$$

As with the theory established in Section 2, we are interested in the solution of (3.1) with asymptotic boundary conditions

$$\lim_{t\to-\infty}(\phi(t), \psi(t)) = (0, 0), \qquad \lim_{t\to+\infty}(\phi(t), \psi(t)) = (k_1, k_2),$$

where

$$k_1 = \frac{a_2 - b_1}{a_1a_2 - b_1b_2} > 0, \quad k_2 = \frac{a_1 - b_2}{a_1a_2 - b_1b_2} > 0,$$

provided that

$$a_1 > b_2, \quad a_2 > b_1. \tag{3.2}$$

For $\phi, \psi \in C([-\tau, 0], \mathbb{R})$ where $\tau = \max\{\tau_1, \tau_2\}$, denote

$$\begin{aligned} f_1(\phi, \psi) &= r_1\phi(0)[1 - a_1\phi(0) - b_1\psi(-\tau_1)], \\ f_2(\phi, \psi) &= r_2\psi(0)[1 - b_2\phi(-\tau_2) - a_2\psi(0)]. \end{aligned}$$

Obviously, (A1) and (A2) hold. We now verify that $\mathbf{F} = (f_1, f_2)$ is WQM.

LEMMA 3.1. *The function $\mathbf{F} = (f_1, f_2)$ is WQM.*

PROOF. Take $(\phi_1(s), \phi_2(s)), (\psi_1(s), \psi_2(s)) \in C([-\tau, 0], \mathbb{R}^2)$ where

$$0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1, \quad 0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2 \quad \text{for } s \in [-\tau, 0].$$

According to $M_1 > k_1$ and $M_2 > k_2$, we have $\beta_1 := r_1(2a_1 M_1 + b_1 M_2 - 1) + d_1 > 0$ and

$$2a_1 M_1 + b_1 M_2 - 1 > 2a_1 k_1 + b_1 k_2 - 1 = a_1 k_1 > 0.$$

Then, defining $\Upsilon = [\phi_1(0) - \phi_2(0)]$, we have

$$\begin{aligned} &f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) \\ &\quad = r_1\phi_1(0)[1 - a_1\phi_1(0) - b_1\psi_1(-\tau_1)] - r_1\phi_2(0)[1 - a_1\phi_2(0) - b_1\psi_1(-\tau_1)] \\ &\quad = r_1\Upsilon - r_1 a_1(\phi_1^2(0) - \phi_2^2(0)) - r_1 b_1\psi_1(-\tau_1)\Upsilon \\ &\quad \geq r_1\Upsilon(1 - 2a_1 M_1 - b_1 M_2) \\ &\quad = -r_1(2a_1 M_1 + b_1 M_2 - 1)\Upsilon - d_1\Upsilon + d_1\Upsilon = -\beta_1\Upsilon + d_1\Upsilon. \end{aligned}$$

So, $f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) + (\beta_1 - d_1)[\phi_1(0) - \phi_2(0)] \geq 0$ and

$$\begin{aligned} &f_1(\phi_1, \psi_1) - f_1(\phi_1, \psi_2) \\ &\quad = r_1\phi_1(0)[1 - a_1\phi_1(0) - b_1\psi_1(-\tau_1)] - r_1\phi_1(0)[1 - a_1\phi_1(0) - b_1\psi_2(-\tau_1)] \\ &\quad = -r_1 b_1\phi(0)[\psi_1(-\tau_1) - \psi_2(-\tau_1)] \leq 0. \end{aligned}$$

In a similar way, we can prove that f_2 is WQM. This completes the proof. □

Define

$$\Delta_1(\lambda, c) = d_1 J_1 * e^{\lambda\cdot} - d_1 - c\lambda + r_1, \tag{3.3}$$

$$\Delta_2(\lambda, c) = d_2 J_2 * e^{\lambda\cdot} - d_2 - c\lambda + r_2. \tag{3.4}$$

We can easily obtain the following conclusions.

LEMMA 3.2.

(i) *There exists a* $c_1^* > 0$ *such that* (3.3) *has two distinct positive roots,* $\lambda_1(c)$ *and* $\lambda_2(c)$*, with* $\lambda_1(c) < \lambda_2(c)$ *for any* $c > c_1^*$.

(ii) *There exists a* $c_2^* > 0$ *such that* (3.4) *has two distinct positive roots,* $\lambda_3(c)$ *and* $\lambda_4(c)$*, with* $\lambda_3(c) < \lambda_4(c)$ *for any* $c > c_2^*$.

(iii) *There exists a* $c' \leq \min\{c_1^*, c_2^*\}$ *such that* (3.3) *and* (3.4) *have no real root if* $c < c'$.

For fixed

$$\eta \in \left(1, \min\left\{2, \frac{\lambda_2(c)}{\lambda_1(c)}, \frac{\lambda_4(c)}{\lambda_3(c)}, \frac{\lambda_1(c) + \lambda_3(c)}{\lambda_1(c)}, \frac{\lambda_2(c) + \lambda_4(c)}{\lambda_2(c)}\right\}\right), \tag{3.5}$$

where $c > c^* = \max\{c_1^*, c_2^*\}$ and we have a large constant $q > 0$, we consider the functions $l_1(t) = e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}$ and $l_3(t) = e^{\lambda_3(c)t} - qe^{\eta\lambda_3(c)t}$. It is easy to see that $l_1(t)$ and $l_3(t)$ have global maxima $m_1 > 0$ and $m_3 > 0$, respectively. Then there exist t_1 and t_3 such that $l_1(t_1) = m_1$ and $l_3(t_3) = m_3$, where $t_i = [1/(\eta\lambda_i - \lambda_i)]\ln(1/(q\eta)) < 0$, $i = 1, 3$, and t_i $(i = 1, 3)$ is large and negative enough for large enough q. Furthermore, $l_i'(t) \geq 0$ for $t \leq t_i$ and $l_i'(t) \leq 0$ for $t \geq t_i$ $(i = 1, 3)$.

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Then, for any given $\lambda > 0$, there exist $\varepsilon_2 > 0$ and $\varepsilon_4 > 0$ such that

$$k_1 - \varepsilon_2 e^{-\lambda t_1} = l_1(t_1) = m_1 \quad \text{and} \quad k_2 - \varepsilon_4 e^{-\lambda t_3} = l_3(t_3) = m_3.$$

According to (3.2), there exist ε_0, ε_1 and ε_3 such that

$$\begin{cases} a_1\varepsilon_1 - b_1\varepsilon_4 > \varepsilon_0, & a_2\varepsilon_3 - b_2\varepsilon_2 > \varepsilon_0, \\ a_1\varepsilon_2 - b_1\varepsilon_3 > \varepsilon_0, & a_2\varepsilon_4 - b_2\varepsilon_1 > \varepsilon_0. \end{cases} \tag{3.6}$$

For the above constants and for suitable constants t_2 and t_4, we can define continuous functions as follows:

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_1(c)t}, & t \le t_2, \\ k_1 + \varepsilon_1 e^{-\lambda t}, & t \ge t_2, \end{cases} \qquad \overline{\psi}(t) = \begin{cases} e^{\lambda_3(c)t}, & t \le t_4, \\ k_2 + \varepsilon_3 e^{-\lambda t}, & t \ge t_4, \end{cases}$$

and

$$\underline{\phi}(t) = \begin{cases} e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}, & t \le t_1, \\ k_1 - \varepsilon_2 e^{-\lambda t}, & t \ge t_1, \end{cases} \qquad \underline{\psi}(t) = \begin{cases} e^{\lambda_3(c)t} - qe^{\eta\lambda_1(c)t}, & t \le t_3, \\ k_2 - \varepsilon_4 e^{-\lambda t}, & t \ge t_3, \end{cases}$$

where $q > 0$ is large enough and $\lambda > 0$ is small enough. It is easy to see that $M_1 = \sup_{t\in\mathbb{R}} \overline{\phi}(t) > k_1$, $M_2 = \sup_{t\in\mathbb{R}} \overline{\psi}(t) > k_2$, $\overline{\phi}(t)$, $\overline{\psi}(t)$, $\underline{\phi}(t)$ and $\underline{\psi}(t)$ satisfy (P1) and (P2), since $t_i = [1/(\eta\lambda_i - \lambda_i)] \ln(1/(q\eta)) < 0$, $i = 1, 3$, are small enough and

$$\min\{t_2, t_4\} - c\max\{\tau_1, \tau_2\} \ge \max\{t_1, t_3\}$$

for sufficiently large $q > 0$ and sufficiently small $\lambda > 0$. We now prove that the continuous functions $(\overline{\phi}(t), \overline{\psi}(t))$ and $(\underline{\phi}(t), \underline{\psi}(t))$ are an upper solution and a lower solution of (3.1), respectively.

REMARK 1. The t_i, $i = 1, 3$, used here are different from $t_i = \max\{t \mid l_i(t) = m_i/2\}$, $i = 1, 3$, in [10]. In the current paper $l_i(t)$ $i = 1, 3$ are nondecreasing for $t \le t_i$, $i = 1, 3$, and $l_i(t)$ $i = 1, 3$, are nonincreasing for $t \ge t_i$, $i = 1, 3$, where t_i, $i = 1, 3$, are negative and $-t_i$, $i = 1, 3$, are large enough. In the current paper the lower solutions are increasing functions but they are not monotone in [10].

LEMMA 3.3. *Assume that* (A3), (A4) *and* (3.2) *hold and that* J_1 *and* J_2 *are even functions. Then* $(\overline{\phi}(t), \overline{\psi}(t))$ *is an upper solution of* (3.1).

PROOF. For $(\overline{\phi}(t), \overline{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2)$, if $t \le t_2$, then $\underline{\psi}(t - c\tau_1) \ge 0$, $\overline{\phi}(t) = e^{\lambda_1(c)t}$ and

$$\begin{aligned} &d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)] \\ &\quad = d_1\left(\int_{-\infty}^{t_2} J_1(t-s)\overline{\phi}(s)\,ds + \int_{t_2}^{\infty} J_1(t-s)\overline{\phi}(s)\,ds\right) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) \\ &\qquad + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)] \\ &\quad = d_1(J_1 * (e^{\lambda_1(c)\cdot}))(t) + d_1\int_{t_2}^{\infty} J_1(t-s)[k_1 + \varepsilon_1 e^{-\lambda s} - e^{\lambda_1 s}]\,ds \\ &\qquad - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)]. \end{aligned} \tag{3.7}$$

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The expression (3.7) is less than or equal to

$$\begin{aligned}
&d_1(J_1 * (e^{\lambda_1(c)\cdot}))(t) + d_1 \int_{t_2}^{\infty} J_1(t-s)[k_1 + \varepsilon_1 e^{-\lambda t_2} - e^{\lambda_1 t_2}]\, ds \\
&\quad - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)] \\
&\qquad \le d_1(J_1 * (e^{\lambda_1(c)\cdot}))(t) - d_1 e^{\lambda_1(c)t} - c(e^{\lambda_1(c)t})' + r_1 e^{\lambda_1(c)t} \\
&\qquad = [d_1(J_1 * (e^{\lambda_1(c)\cdot})) - d_1 - c\lambda_1(c) + r_1]e^{\lambda_1(c)t} = 0.
\end{aligned}$$

Similarly,

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)[1 - b_2\underline{\phi}(t - c\tau_2) - a_2\overline{\psi}(t)] \le 0.$$

If $t \ge t_2$, then $t - c\tau_1 \ge t_2 - c\tau_1 \ge t_3$ and $\underline{\psi}(t - c\tau_1) = k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_1)}$. For any $t \in \mathbb{R}$, we have that $0 < \overline{\phi}(t) < k_1 + \varepsilon_1$, and it follows that

$$\begin{aligned}
&d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)] \\
&\quad \le d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) - c((k_1 + \varepsilon_1 e^{-\lambda t}))' \\
&\qquad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_1)})] \\
&\quad = d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t} \\
&\qquad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda t}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_1)})] =: I(\lambda).
\end{aligned}$$

Obviously,

$$\begin{aligned}
I(0) &= d_1 J_1 * (k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) \\
&\quad + r_1(k_1 + \varepsilon_1)(1 - a_1 k_1 - b_1 k_2 - a_1\varepsilon_1 + b_1\varepsilon_4) \\
&= d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) + r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1) \\
&= r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1).
\end{aligned}$$

Since $a_1\varepsilon_1 - b_1\varepsilon_4 > \varepsilon_0$, we have $I(0) < -r_1(k_1 + \varepsilon_1)\varepsilon_0 < 0$, and there exists a $\lambda_1^* > 0$ such that $I(\lambda) < 0$ for $\lambda \in (0, \lambda_1^*)$. Thus,

$$d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)(1 - a_1\overline{\phi}(t) - b_1\underline{\psi}(t - c\tau_1)) \le 0.$$

Similarly, there exists a $\lambda_2^* > 0$ such that, for $\lambda \in (0, \lambda_2^*)$,

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)[1 - b_2\underline{\phi}(t - c\tau_2) - a_2\overline{\psi}(t)] \le 0.$$

This completes the proof. □

LEMMA 3.4. *Assume that* (A3), (A4) *and* (3.2) *hold and, in addition,* J_1, J_2 *are even functions. Then* $(\underline{\phi}, \underline{\psi})$ *is a lower solution of* (3.1).

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PROOF. Equation (3.5) implies that $\lambda_1(c) < \eta\lambda_1(c) < \lambda_2(c)$, $2\lambda_1(c) > \eta\lambda_1(c)$, $\lambda_3(c) + \lambda_1(c) > \eta\lambda_1(c)$ and $\Delta_1(\eta\lambda_1(c), c) < 0$.

For $t \le t_1$,

$$\begin{aligned}
&d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t - c\tau_1))\\
&\quad = d_1\left(\int_{t_1}^{\infty} J_1(t-s)\underline{\phi}(s)\,ds + \int_{-\infty}^{t_1} J_1(t-s)\underline{\phi}(s)\,ds\right) - d_1\underline{\phi}(t) - c\underline{\phi}'(t)\\
&\qquad + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t - c\tau_1))\\
&\quad = d_1\bigg\{J_1 * [e^{\lambda_1(c)\cdot} - qe^{\eta\lambda_1(c)\cdot}](t)\\
&\qquad + \int_{t_1}^{\infty} J_1(t-s)[k_1 - \varepsilon_2 e^{-\lambda s} - (e^{\lambda_1(c)s} - qe^{\eta\lambda_1(c)s})]\,ds\bigg\}\\
&\qquad - d_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}] - c[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}]'\\
&\qquad + r_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}]\left\{1 - a_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}] - b_1e^{\lambda_3(c)(t-c\tau_1)}\right\}\\
&\quad \ge d_1\left\{J_1 * [e^{\lambda_1(c)\cdot}]\right\}(t) - d_1[e^{\lambda_1(c)t}] - c\lambda_1(c)e^{\lambda_1(c)t} + r_1e^{\lambda_1(c)t}\\
&\qquad - \left\{d_1J_1 * [qe^{\eta\lambda_1(c)\cdot}](t) - d_1qe^{\eta\lambda_1(c)t} - cq\eta\lambda_1(c)e^{\eta\lambda_1(c)t} + r_1qe^{\eta\lambda_1(c)t}\right\}\\
&\qquad + d_1\int_{t_1}^{\infty} J_1(t-s)[k_1 - \varepsilon_2 e^{-\lambda t_1} - (e^{\lambda_1(c)t_1} - qe^{\eta\lambda_1(c)t_1})]\,ds\\
&\qquad - r_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}]\left\{a_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}] + b_1e^{\lambda_3(c)(t-c\tau_1)}\right\}\\
&\quad = \Delta_1(\lambda_1(c), c)e^{\lambda_1(c)t} - \Delta_1(\eta\lambda_1(c), c)qe^{\eta\lambda_1(c)t}\\
&\qquad - r_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}]\left\{a_1[e^{\lambda_1(c)t} - qe^{\eta\lambda_1(c)t}] + b_1e^{\lambda_3(c)(t-c\tau_1)}\right\}\\
&\quad \ge -\Delta_1(\eta\lambda_1(c), c)qe^{\eta\lambda_1(c)t} - r_1e^{\lambda_1(c)t}[a_1e^{\lambda_1(c)t} + b_1e^{\lambda_3(c)t}]\\
&\quad = -qe^{\eta\lambda_1(c)t}\left\{\Delta_1(\eta\lambda_1(c), c) + \frac{r_1}{q}[a_1e^{(2\lambda_1(c)-\eta\lambda_1(c))t} + b_1e^{(\lambda_3(c)+\lambda_1(c)-\eta\lambda_1(c))t}]\right\}\\
&\quad \ge -qe^{\eta\lambda_1(c)t}\left\{\Delta_1(\eta\lambda_1(c), c) + \frac{r_1}{q}[a_1e^{(2\lambda_1(c)-\eta\lambda_1(c))t_1} + b_1e^{(\lambda_3(c)+\lambda_1(c)-\eta\lambda_1(c))t_1}]\right\}\\
&\quad \ge -qe^{\eta\lambda_1(c)t}\left\{\Delta_1(\eta\lambda_1(c), c) + \frac{r_1}{q}[a_1e^{(2\lambda_1(c)-\eta\lambda_1(c))\alpha} + b_1e^{(\lambda_3(c)+\lambda_1(c)-\eta\lambda_1(c))\alpha}]\right\}\\
&\quad \ge 0 \quad \text{for large enough } q,
\end{aligned}$$

where $\alpha = (\ln(1/\eta q))/(\lambda_1(c)(\eta - 1))$.

By a similar argument,

$$d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) + r_2\underline{\psi}(t)\left(1 - b_2\overline{\phi}(t - c\tau_2) - a_2\underline{\psi}(t)\right) \ge 0.$$

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If $t > t_1$ and $-t_1$ is large enough, then for any $\varepsilon'(0 < \varepsilon' < r_1\varepsilon_0/d_1)$ there exists a large enough $T(q) > 0$ such that $t_1 < -T(q)$ and

$$\int_{t_1}^{\infty} J_1(s)\,ds > \int_{-\infty}^{\infty} J_1(s)\,ds - \varepsilon' = 1 - \varepsilon'.$$

By the definition of $\overline{\psi}(t)$, it is easy to verify that $0 < \overline{\psi}(t) \le k_2 + \varepsilon_3$ for any $t \in \mathbb{R}$.

Then,

$$\begin{aligned}
&d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t - c\tau_1))\\
&\quad \ge d_1(J_1 * \underline{\phi})(t) - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c\lambda e^{-\lambda t}\\
&\qquad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)]\\
&\quad = d_1 \int_{-\infty}^{t_1} J_1(t-s)[e^{\lambda_1(c)s} - q e^{\eta\lambda_1(c)s}]\,ds + d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda s})\,ds\\
&\qquad - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c\lambda e^{-\lambda t}\\
&\qquad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)]\\
&\quad \ge d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda s})\,ds - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + c\lambda e^{-\lambda t}\\
&\qquad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})[1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t}) - b_1(k_2 + \varepsilon_3)] =: \tilde{I}(\lambda).
\end{aligned}$$

Thus,

$$\begin{aligned}
\tilde{I}(0) &= d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2)\,ds - d_1(k_1 - \varepsilon_2) + r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3)\\
&\ge d_1(k_1 - \varepsilon_2)(1 - \varepsilon') - d_1(k_1 - \varepsilon_2) + r_1(k_1 - \varepsilon_2)\varepsilon_0\\
&= -d_1(k_1 - \varepsilon_2)\varepsilon' + r_1(k_1 - \varepsilon_2)\varepsilon_0 > 0,
\end{aligned}$$

which implies that there exists a $\lambda_3^* > 0$ such that $\tilde{I}(\lambda) > 0$ for $\lambda \in (0, \lambda_3^*)$. Thus,

$$d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t) - b_1\overline{\psi}(t - c\tau_1)) \ge 0.$$

Similarly, there exists a $\lambda_4^* > 0$ such that, for $\lambda \in (0, \lambda_4^*)$,

$$d_2(J_2 * \underline{\psi})(t) - d_2\underline{\psi}(t) - c\underline{\psi}'(t) + r_2\underline{\psi}(t)(1 - b_2\overline{\phi}(t - c\tau_2) - a_2\underline{\psi}(t)) \ge 0.$$

This completes the proof. □

By Theorem 2.1 and Lemmas 3.3 and 3.4, the following result holds.

THEOREM 3.5. *Assume that* (A3), (A4) *and* (3.2) *hold and, in addition,* J_1, J_2 *are even functions. Then, for every* $c > c^*$, *Model* (3.1) *has a travelling wave solution* $(\phi(x + ct), \psi(x + ct))$, *which connects* $(0, 0)$ *and* (k_1, k_2). *Furthermore,* $\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = \lim_{\xi\to-\infty} \psi(\xi)e^{-\lambda_3\xi} = 1$.

3.2. Model (1.4) Next we consider the existence of travelling wave solutions for the nonlocal diffusion-competition model (1.4) with delays, where all constants and

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assumptions are the same as the above arguments in Model (1.3) and $\tau_i \geq 0$, $i = 1, 2, 3, 4$.

Letting $c > 0$, the corresponding wave profile equations of (1.4) are

$$\begin{cases} c\phi'(t) = d_1(J_1 * \phi)(t) - d_1\phi(t) + r_1\phi(t)(1 - a_1\phi(t - c\tau_1) - b_1\psi(t - c\tau_2)), \\ c\psi'(t) = d_2(J_2 * \psi)(t) - d_2\psi(t) + r_2\psi(t)(1 - b_2\phi(t - c\tau_3) - a_2\psi(t - c\tau_4)). \end{cases} \tag{3.8}$$

For $\phi, \psi \in C([-\tau, 0], \mathbb{R})$, where $\tau = \max\{\tau_1, \tau_2, \tau_3, \tau_4\}$, denote

$$f_1(\phi, \psi) = r_1\phi(0)(1 - a_1\phi(-\tau_1) - b_1\psi(-\tau_2)),$$
$$f_2(\phi, \psi) = r_2\psi(0)(1 - b_2\phi(-\tau_3) - a_2\psi(-\tau_4)).$$

It is obvious that (A1) and (A2) hold.

LEMMA 3.6. *For τ_1 and τ_4 small enough, the function $\mathbf{F} = (f_1, f_2)$ is WQM*.*

PROOF. Take $(\phi_1(s), \phi_2(s)), (\psi_1(s), \psi_2(s)) \in C([-\tau, 0], \mathbb{R}^2)$ where:

(i) $0 \leq \phi_2(s) \leq \phi_1(s) \leq M_1$, $0 \leq \psi_2(s) \leq \psi_1(s) \leq M_2$, for $s \in [-\tau, 0]$; and
(ii) $e^{\beta_1 s}[\phi_1(s) - \phi_2(s)]$ and $e^{\beta_1 s}[\psi_1(s) - \psi_2(s)]$ are nondecreasing in $s \in [-\tau, 0]$.

If τ_1 is small enough, then we can choose $\beta_1 > 0$ such that

$$r_1(a_1M_1 + b_1M_2 + a_1M_1e^{\beta_1\tau_1} - 1) + d_1 < \beta_1.$$

Thus, we have

$$\begin{aligned} f_1(\phi_1, \psi_1) &- f_1(\phi_2, \psi_1) \\ &= r_1\phi_1(0)(1 - a_1\phi_1(-\tau_1) - b_1\psi_1(-\tau_2)) \\ &\quad - r_1\phi_2(0)[1 - a_1\phi_2(-\tau_1) - b_1\psi_1(-\tau_2)] \\ &= r_1\Upsilon - r_1a_1(\phi_1(0)\phi_1(-\tau_1) - \phi_2(0)\phi_2(-\tau_1)) - r_1b_1\psi_1(-\tau_2)\Upsilon \\ &\geq (r_1 - r_1b_1M_2)\Upsilon - r_1a_1\phi_1(0)[\phi_1(-\tau_1) - \phi_2(-\tau_1)] - r_1a_1\phi_2(-\tau_1)\Upsilon \\ &\geq -r_1(a_1M_1 + b_1M_2 - 1)\Upsilon - r_1a_1\phi_1(0)e^{\beta_1\tau_1}e^{-\beta_1\tau_1}[\phi_1(-\tau_1) - \phi_2(-\tau_1)] \\ &\geq -r_1(a_1M_1 + b_1M_2 + a_1M_1e^{\beta_1\tau_1} - 1)\Upsilon \geq -\beta_1\Upsilon + d_1\Upsilon. \end{aligned}$$

So, $f_1(\phi_1, \psi_1) - f_1(\phi_2, \psi_1) + (\beta_1 - d_1)[\phi_1(0) - \phi_2(0)] \geq 0$ and

$$\begin{aligned} f_1(\phi_1, \psi_1) - f_1(\phi_1, \psi_2) &= r_1\phi_1(0)[1 - a_1\phi_1(-\tau_1) - b_1\psi_1(-\tau_2)] \\ &\quad - r_1\phi_1(0)[1 - a_1\phi_1(-\tau_1) - b_1\psi_2(-\tau_2)] \\ &= -r_1b_1\phi(0)[\psi_1(-\tau_2) - \psi_2(-\tau_2)] \leq 0. \end{aligned}$$

Similarly, f_2 is WQM* if τ_4 is small enough. This completes the proof. □

Now we define $\overline{\phi}(t)$, $\overline{\psi}(t)$, $\underline{\phi}(t)$ and $\underline{\phi}(t)$ as in Model (1.3), where $c > c^* = \max\{c_1^*, c_2^*, 1\}$ and

$$\min\{t_2, t_4\} - c\max\{\tau_1, \tau_2, \tau_3, \tau_4\} \geq \max\{t_1, t_3\}.$$

It is easy to see that $\overline{\phi}(t)$, $\overline{\psi}(t)$, $\underline{\phi}(t)$ and $\underline{\phi}(t)$ satisfy (P1)–(P3). In fact, (P3) holds for λ small enough and $\beta > \lambda$.

LEMMA 3.7. *Assume that* (A3), (A4) *and* (3.2) *hold and, in addition,* J_1 *and* J_2 *are even functions. If* τ_1 *and* τ_4 *are small enough, then* $(\overline{\phi}(t), \overline{\psi}(t))$ *is an upper solution and* $(\underline{\phi}(t), \underline{\psi}(t))$ *is a lower solution of* (3.8).

PROOF. First, we prove that $(\overline{\phi}(t), \overline{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2)$ is an upper solution. For $t \geq t_2 + c\tau_1$ or $t \leq t_2$, the proof is similar to that of Lemma 3.3. If $t \leq t_2$, it follows that

$$\begin{aligned}
d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)[1 - a_1\overline{\phi}(t - c\tau_1) - b_1\underline{\psi}(t - c\tau_2)] \\
&\leq d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t) \\
&\leq d_1(J_1 * (e^{\lambda_1(c)\cdot}))(t) - d_1 e^{\lambda_1(c)t} - c(e^{\lambda_1(c)t})' + r_1 e^{\lambda_1(c)t} \\
&= [d_1(J_1 * (e^{\lambda_1(c)\cdot})) - d_1 - c\lambda_1(c) + r_1]e^{\lambda_1(c)t} = 0.
\end{aligned}$$

Similarly, we can obtain

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)(1 - b_2\underline{\phi}(t - c\tau_3) - a_2\overline{\psi}(t - c\tau_4)) \leq 0.$$

For $t \geq t_2 + c\tau_1$, then $t - c\tau_2 \geq t_3$ and $\underline{\psi}(t - c\tau_2) = k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_2)}$. Since $0 < \overline{\phi}(t) \leq k_1 + \varepsilon_1$ for any $t \in \mathbb{R}$, then

$$\begin{aligned}
d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)(1 - a_1\overline{\phi}(t - c\tau_1) - b_1\underline{\psi}(t - c\tau_2)) \\
&\leq d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) - c(k_1 + \varepsilon_1 e^{-\lambda t})' \\
&\quad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda(t - c\tau_1)}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_2)})] \\
&= d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t} \\
&\quad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1(k_1 + \varepsilon_1 e^{-\lambda(t - c\tau_1)}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t - c\tau_2)})] \\
&=: I_1(\lambda).
\end{aligned}$$

Obviously,

$$\begin{aligned}
I_1(0) &= d_1 J_1 * (k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) \\
&\quad + r_1(k_1 + \varepsilon_1)[1 - a_1k_1 - b_1k_2 - a_1\varepsilon_1 + b_1\varepsilon_4] \\
&= d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1) + r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1) \\
&= r_1(k_1 + \varepsilon_1)(b_1\varepsilon_4 - a_1\varepsilon_1).
\end{aligned}$$

Since $a_1\varepsilon_1 - b_1\varepsilon_4 > \varepsilon_0$, we have $I_1(0) < -r_1(k_1 + \varepsilon_1)\varepsilon_0 < 0$, and there exists a $\overline{\lambda}_1^* > 0$ such that $I_1(\lambda) < 0$ for $\lambda \in (0, \overline{\lambda}_1^*)$. Thus,

$$d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)(1 - a_1\overline{\phi}(t - c\tau_1) - b_1\underline{\psi}(t - c\tau_2)) \leq 0.$$

Similarly, there exists a $\overline{\lambda}_2^* > 0$ such that, for $\lambda \in (0, \overline{\lambda}_2^*)$,

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)(1 - b_2\underline{\phi}(t - c\tau_3) - a_2\overline{\psi}(t - c\tau_4)) \leq 0.$$

For $t_2 + c\tau_1 > t > t_2$, then $\overline{\phi}(t) = k_1 + \varepsilon_1 e^{-\lambda t}$, $\overline{\phi}'(t) = -\varepsilon_1\lambda e^{-\lambda t}$ and $\overline{\phi}(t - c\tau_1) = e^{\lambda_1(t - c\tau_1)}$.

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Since $0 < \overline{\phi}(t) \le k_1 + \varepsilon_1$ for any $t \in \mathbb{R}$, then

$$\begin{aligned}
&d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)(1 - a_1\overline{\phi}(t - c\tau_1) - b_1\underline{\psi}(t - c\tau_2)) \\
&\quad \le d_1(J_1 * (k_1 + \varepsilon_1)) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t} \\
&\qquad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1 e^{\lambda_1(t-c\tau_1)} - b_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&\quad \le d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t} \\
&\qquad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1 e^{\lambda_1(t_2-c\tau_1)} - b_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&\quad = d_1(k_1 + \varepsilon_1) - d_1(k_1 + \varepsilon_1 e^{-\lambda t}) + c\varepsilon_1\lambda e^{-\lambda t} \\
&\qquad + r_1(k_1 + \varepsilon_1 e^{-\lambda t})[1 - a_1 e^{-\lambda_1 c\tau_1}(k_1 + \varepsilon_1 e^{-\lambda t_2}) - b_1(k_2 - \varepsilon_4 e^{-\lambda(t-c\tau_2)})] \\
&\quad =: I_2(\lambda).
\end{aligned}$$

For enough small τ_1, there exists an ε^* $(0 < \varepsilon^* < \varepsilon_0/a_1(k_1 + \varepsilon_1))$ such that $e^{-\lambda_1 c\tau_1} > 1 - \varepsilon^*$. Thus,

$$\begin{aligned}
I_2(0) &= r_1(k_1 + \varepsilon_1)[1 - a_1 e^{-\lambda c\tau_1}(k_1 + \varepsilon_1) - b_1 k_2 + b_1\varepsilon_4] \\
&= r_1(k_1 + \varepsilon_1)[a_1 k_1 - a_1 e^{-\lambda c\tau_1}(k_1 + \varepsilon_1) + b_1\varepsilon_4] \\
&\le r_1(k_1 + \varepsilon_1)[a_1 k_1 - a_1(1 - \varepsilon^*)(k_1 + \varepsilon_1) + b_1\varepsilon_4] \\
&= r_1(k_1 + \varepsilon_1)[b_1\varepsilon_4 - a_1\varepsilon_1 + a_1(k_1 + \varepsilon_1)\varepsilon^*] \\
&< r_1(k_1 + \varepsilon_1)[-\varepsilon_0 + a_1(k_1 + \varepsilon_1)\varepsilon^*] < 0.
\end{aligned}$$

Thus, there exists a $\overline{\lambda}_3^* > 0$ such that for $\lambda \in (0, \overline{\lambda}_3^*)$,

$$d_1(J_1 * \overline{\phi})(t) - d_1\overline{\phi}(t) - c\overline{\phi}'(t) + r_1\overline{\phi}(t)(1 - a_1\overline{\phi}(t - c\tau_1) - b_1\underline{\psi}(t - c\tau_2)) \le 0.$$

Similarly, there exists a $\overline{\lambda}_4^* > 0$ such that, for $\lambda \in (0, \overline{\lambda}_4^*)$,

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)(1 - b_2\underline{\phi}(t - c\tau_3) - a_2\overline{\psi}(t - c\tau_4)) \le 0.$$

Next, we prove that $(\underline{\phi}(t), \underline{\psi}(t)) \in C(\mathbb{R}, \mathbb{R}^2)$ is a lower solution of (3.8).

For $t \le t_1$ and $t \ge t_1 + c\tau_1$, the proof of a lower solution $(\underline{\phi}(t), \underline{\psi}(t))$ of (3.8) is similar to that of Lemma 3.4.

Now, for $t_1 < t < t_1 + c\tau_1$, it is easy to see that $\underline{\phi}(t) = k_1 - \varepsilon_2 e^{-\lambda t}$, $\underline{\phi}(t - c\tau_1) = e^{\lambda_1(t-c\tau_1)} - q e^{\eta\lambda_1(t-c\tau_1)}$ and $\overline{\psi}(t - c\tau_2) = e^{\lambda_3(t-c\tau_2)}$. Then

$$\begin{aligned}
&d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t - c\tau_1) - b_1\overline{\psi}(t - c\tau_2)) \\
&\quad = d_1(J_1 * \underline{\phi})(t) - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + \lambda\varepsilon_2 e^{-\lambda t} \\
&\qquad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\left\{1 - a_1[e^{\lambda_1(t-c\tau_1)} - q e^{\eta\lambda_1(t-c\tau_1)}] - b_1 e^{\lambda_3(t-c\tau_2)}\right\} \\
&\quad \ge d_1\int_{t_1}^{\infty} J_1(t - s)(k_1 - \varepsilon_2 e^{-\lambda s})\, ds - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + \lambda\varepsilon_2 e^{-\lambda t} \\
&\qquad + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\{1 - a_1[e^{\lambda_1 t_1} - q e^{\eta\lambda_1 t_1}] - b_1 e^{\lambda_3 t_4}\}
\end{aligned}$$

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$$= d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2 e^{-\lambda s})\,ds - d_1(k_1 - \varepsilon_2 e^{-\lambda t}) + \lambda \varepsilon_2 e^{-\lambda t} + r_1(k_1 - \varepsilon_2 e^{-\lambda t})\{1 - a_1(k_1 - \varepsilon_2 e^{-\lambda t_1}) - b_1(k_2 + \varepsilon_3 e^{-\lambda t_4})\} =: I_3(\lambda).$$

Thus

$$\begin{aligned} I_3(0) &= d_1 \int_{t_1}^{\infty} J_1(t-s)(k_1 - \varepsilon_2)\,ds - d_1(k_1 - \varepsilon_2) \\ &\quad + r_1(k_1 - \varepsilon_2)[1 - a_1(k_1 - \varepsilon_2) - b_1(k_2 + \varepsilon_3)] \\ &\geq d_1(k_1 - \varepsilon_2)(1 - \varepsilon') - d_1(k_1 - \varepsilon_2) + r_1(k_1 - \varepsilon_2)(a_1\varepsilon_2 - b_1\varepsilon_3) \\ &\geq -d_1(k_1 - \varepsilon_2)\varepsilon' + r_1(k_1 - \varepsilon_2)\varepsilon_0 > 0. \end{aligned}$$

Hence, there exists a $\overline{\lambda}_5^* > 0$ such that, for $\lambda \in (0, \overline{\lambda}_5^*)$,

$$d_1(J_1 * \underline{\phi})(t) - d_1\underline{\phi}(t) - c\underline{\phi}'(t) + r_1\underline{\phi}(t)(1 - a_1\underline{\phi}(t) - b_1\,\overline{\psi}(t - c\tau_2)) \geq 0.$$

In a similar way, there exists a $\overline{\lambda}_6^* > 0$ such that for $\lambda \in (0, \overline{\lambda}_6^*)$,

$$d_2(J_2 * \overline{\psi})(t) - d_2\overline{\psi}(t) - c\overline{\psi}'(t) + r_2\overline{\psi}(t)(1 - b_2\underline{\phi}(t - c\tau_3) - a_2\,\overline{\psi}(t - c\tau_4)) \geq 0.$$

This completes the proof. □

By Theorem 2.2 and Lemma 3.7, we have the following result.

THEOREM 3.8. *Assume that* (A3), (A4) *and* (3.2) *hold and, in addition,* J_1, J_2 *are even functions. If* τ_1 *and* τ_4 *are small enough, then, for every* $c > c^*$, *Model* (3.8) *has a travelling wave solution* $(\phi(x+ct), \psi(x+ct))$, *which connects* $(0, 0)$ *and* (k_1, k_2). *Furthermore,* $\lim_{\xi\to-\infty} \phi(\xi)e^{-\lambda_1\xi} = \lim_{\xi\to-\infty} \psi(\xi)e^{-\lambda_3\xi} = 1$.

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