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A NOTE ON THE LEVITZKI RADICAL OF A NEAR-RING

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Abstract

It is known that in a near-ring N the Levitzki radical $L(N)$, that is, the sum of all locally nilpotent ideals, is the intersection of all the prime ideals P in N such that N/P has zero Levitzki radical. The purpose of this note is to prove that $L(N)$ is the intersection of a certain class of prime ideals, called l-prime ideals. Every l-prime ideal P is such that N/P has zero Levitzki radical. We also introduce an l-semi-prime ideal and show that P is an l-semi-prime ideal if and only if N/P has zero Levitzki radical. We get another characterization of the Levitzki radical of the near-ring as the intersection of all the l-semi-prime ideals.

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1. Definition and preliminaries

A near-ring is an algebraic system, $(N, +, \cdot)$ satisfying
(i) $(N, +)$ is a group,
(ii) $(N, \cdot)$ is a semigroup and
(iii) $(x + y)z = xz + yz$ for all x, y, z in N.
We abbreviate $(N, +, \cdot)$ by N.

If S and T are subsets of N, we denote the set $\{st\colon s \in S; t \in T\}$ by ST. For $n \in \mathfrak{N}$, the definition of S^n is then clear. A normal subgroup I of $(N, +)$ is called an ideal of N ($I \lhd N$) if $IN \subseteq I$ and $n(n' + i) - nn' \in I$ for all $n, n' \in N$ and all $i \in I$. An ideal P of N is called a prime ideal if for any ideals I and J of N, $IJ \subseteq P$ implies either $I \subseteq P$ of $J \subseteq P$. An ideal I of N is called a semiprime ideal if for any ideal J of N, $J^2 \subseteq I$ implies $J \subseteq I$.

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As in Bhandari and Saxena [1] we call a near-ring N locally nilpotent if every finite subset of N is nilpotent. If we denote the sum of all locally nilpotent ideals in N by $L(N)$, it follows from Bhandari and Saxena [1] that the class of locally nilpotent near-rings is a hereditary radical class. Also, if $L(N)$ denotes the Levitzki radical of N, then by Theorem 2 of Bhandari and Saxena [1] we have $L(N) = \bigcap \{P \mid P \text{ is a prime ideal with } L(N/P) = (0)\}$.

In Van der Walt [4] it was shown that, for associative rings, $L(R)$ coincides with a certain class of prime ideals, called l-prime ideals. In [2] Le Roux introduced the concept of l-semi-prime ideals and proved that the intersection of all the l-semi-prime ideals in the ring R coincides with $L(R)$.

We now extend these results to near-rings.

Definition 1 (see Van der Walt [4]). A set of elements L of a near-ring is called an *l-system* if to every element $a \in L$ is assigned a finite number of elements $a_1, a_2, \ldots, a_{n(a)}$ in the principal ideal generated by the element a, such that the following condition is satisfied: If $a, b \in L$ then for every $n > 1$ ($n \in \mathfrak{N}$) there exists a product of $N \geqslant n$ factors, consisting of a_i's and b_j's, which is in L. $\varnothing$ is defined to be an l-system.

Definition 2 An ideal P in N is an *l-prime ideal* if and only if the complement $C(P)$ of P in N is an l-system.

Definition 3 (see Le Roux [2]). A set of elements W of a near-ring N is called a *w-system* if to every element $a \in W$ is assigned a finite number of elements $a_1, a_2, \ldots, a_{n(a)}$ such that the following is satisfied: If $a \in W$, then for every $n > 1$ ($n \in \mathfrak{N}$) there exists a product of $N \geqslant n$ factors, consisting of the a_i's, which is in W. $\varnothing$ is defined to be a w-system.

Definition 4. An ideal Q of N is an *l-semi-prime ideal* if and only if the complement $C(Q)$ of Q in N is a w-system.

Clearly every l-prime ideal is an l-semi-prime ideal and every l-semi-prime ideal is a semi-prime ideal.

Lemma 1.1. *Let $L[W]$ be an l-system [w-system] in N, and A an ideal which does not meet $L[W]$. Then A is contained in an ideal P which is maximal in the class of ideals not meeting $L[W]$. P is necessarily an l-prime [l-semi-prime] ideal.*

Proof. By using Lemma 1 of Van der Walt [5] the proof follows similarly to that for rings in Van der Walt [4].

DEFINITION 5. The *l-radical* [w-radical] $l(A)$ [$w(A)$] of the ideal A of the near-ring N is the set of all elements $r \in N$ with the property that every l-system [w-system] which contains r, contains an element of A.

THEOREM 1.2. *Let A be any ideal in the near-ring N. Then $l(A)$ [$w(A)$] is the intersection of all the l-prime [l-semi-prime] ideals which contain A.*

PROOF. We shall prove the theorem for the l-radical. The proof for the w-radical is quite analogous.

Let $r \in l(A)$ and suppose $r \in \mathcal{C}(P)$ where P is an l-prime ideal containing A. Now $\mathcal{C}(P) \cap A = \varnothing$, contradicting the definition of $l(A)$. Thus $l(A)$ is contained in the intersection of all the l-prime ideals containing A in N.

Now let $r \notin l(A)$. Hence, by the definition of $l(A)$, there exists an l-system L containing r such that $L \cap A = \varnothing$. From Lemma 1.1 there exists an l-prime ideal P such that $A \subseteq P$ and $L \cap P = \varnothing$, that is, $r \notin P$. Thus r cannot be in the intersection of all the l-prime ideals in N containing A, and the proof is completed.

DEFINITION 6. The l-radical [w-radical] of the near-ring N is $l((0))$ [$w((0))$].

THEOREM 1.3. *Let N be any near-ring. $l((0))$ [$w((0))$] coincides with the Levitzki radical $L(N)$ of the near-ring N.*

PROOF. By making the necessary adjustments and using similar techniques, the proof follows as in [4], Theorem 2.

THEOREM 1.4. *Let N be any near-ring. If Q is an ideal in N, then Q is l-semi-prime if and only if N/Q contains no non-zero locally nilpotent ideals.*

PROOF. Suppose Q is l-semi-prime. Hence $\mathcal{C}(Q)$ is a w-system. Let A/Q be any non-zero ideal of N/Q. Since A/Q is non-zero there exists an $a \in A$, $a \notin Q$. Because Q is an l-semi-prime ideal and $a \notin Q$ there exist elements $a_1, a_2, \ldots, a_{n(a)} \in (a)$ such that for every $n > 1$ there is a product of $N \geqslant n$ factors consisting of the a_i's which is not in Q. There thus exists a finite set $\{a_1 + Q, a_2 + Q, \ldots, a_{n(a)} + Q\} \subseteq A/Q$ such that $\{a_1 + Q, \ldots, a_{n(a)} + Q\}^m \neq Q$ for every m. Hence A/Q is not locally nilpotent.

Now suppose N/Q contains no non-zero locally nilpotent ideals. Let $x \in \mathcal{C}(Q)$ be arbitrary. Since $(0) \neq (x)/(x) \cap Q \lhd N/Q$, it follows from our assumption that $(x)/(x) \cap Q$ is not locally nilpotent. Hence there exist $x_1, x_2, \ldots, x_n \in (x)$, $x_i \notin Q$, such that $\{x_1 + Q, x_2 + Q, \ldots, x_n + Q\}$ is not nilpotent. Therefore, for every $x \in \mathcal{C}(Q)$, we can find elements $x_1, x_2, \ldots, x_n \in (x)$ such that for every $n > 1$ ($n \in \mathfrak{N}$) there exists a product of $N \geq n$ factors consisting of the x_i's which is in $\mathcal{C}(Q)$. Hence $\mathcal{C}(Q)$ is a w-system.

Lemma 1.5. *If $(S_k)_{k \in K}$ is a family of l-semi-prime ideals in N then $S = \bigcap_{k \in K} S_k$ is also l-semi-prime.*

Proof. For each $k \in K$, $\mathcal{C}(S_k)$ is a w-system. Let $a \in \mathcal{C}(S) = \mathcal{C}(\bigcap_{k \in K} S_k) = \bigcup_{k \in K} \mathcal{C}(S_k)$ be arbitrary. There exists an element $t \in K$ such that $a \in \mathcal{C}(S_t)$. Since $\mathcal{C}(S_t)$ is a w-system it follows easily that $\mathcal{C}(S)$ is also a w-system. Hence S is an l-semi-prime ideal.

Corollary. *Any intersection of l-prime ideals is l-semi-prime.*

Theorem 1.6. *Let N be any near-ring. Q is an l-semi-prime ideal in N if and only if $l(Q) = Q$.*

Proof. If $l(Q) = Q$ it follows from Theorem 1.2 and the corollary to Lemma 1.5 that Q is an l-semi-prime ideal.

Suppose now Q is an l-semi-prime ideal. From the definition of $l(Q)$ we have $Q \subseteq l(Q)$. Furthermore, it follows from Theorems 1.3 and 1.4 that $l(Q) \subseteq Q$.

We now make the following general conclusions. We have the following characterization of the Levitzki radical $L(N)$ of the near-ring N.

Corollary 1.7. *If N is any near-ring, then $L(N)$ coincides with the intersection of all the l-semi-prime ideals in N, that is, $L(N)$ is an l-semiprime ideal which is contained in every l-semi-prime ideal in N.*

Corollary 1.8 (see Bhandari and Saxena [1]). *$L(N)$ is the smallest ideal I of N such that N/I has no non-zero locally nilpotent ideals.*

Corollary 1.9. *$L(N) = (0)$ if and only if N has no non-zero locally nilpotent ideals.*

Proof. The proof follows from the definition of $L(N)$.

Theorem 1.10. *If N is a near-ring and I any ideal of N, the Levitzki radical of the ring I is $I \cap L(N)$.*

PROOF. Let P be any l-semi-prime ideal in N. $\mathcal{C}(P)$ is a w-system and it is easy to show that $\mathcal{C}(P) \cap I$ is a w-system in I. Hence $P \cap I$ is an l-semi-prime ideal in I. From Theorem 1.3 it now follows that, if we denote the l-radical of the ring I by K, then $K \subseteq l(N) \cap I$. Conversely, if $a \in I \cap l(N)$, then every l-system in N which contains a, also contains 0. In particular, every l-system in I which contains a, also contains 0. Hence $a \in K$, and $I \cap l(N) \subseteq K$. We have, therefore, shown that $K = I \cap l(N) = I \cap L(N)$ and the proof is completed.

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