Artin Approximation Compatible with a Change of Variables

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Abstract. We propose a version of the classical Artin approximation that allows us to perturb the variables of the approximated solution. Namely, it is possible to approximate a formal solution of a Nash equation by a Nash solution in a compatible way with a given Nash change of variables. This result is closely related to the so-called *nested Artin approximation* and becomes false in the analytic setting. We provide local and global versions of this approximation in real and complex geometry together with an application to the Right-Left equivalence of Nash maps.

Let $F(x,z) \in \mathbb{K}\{x,z\}$ be a convergent power series in two sets of variables $x = (x_1, \ldots, x_m)$ and $z = (z_1, \ldots, z_n)$, where $\mathbb{K}$ denotes the field of real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$. M. Artin [1] proved in 1968 that any formal power series $\Phi \in \mathbb{K}[[x]]^n$ solution of $F(x, \Phi(x)) = 0$ can be approximated in the $\mathfrak{m}$-adic topology, where $\mathfrak{m} = (x)$, by convergent power series $\widetilde{\Phi} \in \mathbb{K}\{x\}^n$ solution of $F(x, \widetilde{\Phi}(x)) = 0$. The result remains valid, replacing the analytic power series by formal power series that are algebraic over the polynomials, also called *Nash power series* and denoted by $\mathbb{K}_{\mathrm{alg}}[[x]]$.

There exists an important literature dealing with Artin approximation and its various generalisations (*cf.* [9] for a recent survey). In this paper, we are interested in a version of Artin approximation involving a modification of the source variables of the function to approximate. Roughly speaking, given a convergent power series $G \in \mathbb{K}\{x\}$ and $F \in \mathbb{K}\{x, y, z\}$, can one approximate formal solutions $\Phi \in \mathbb{K}[[x]]$ and $\Psi \in \mathbb{K}[[y]]$ of the equation $F(x, \Phi(x), \Psi \circ G(x)) = 0$ by convergent power series solutions ?

The answer to that question is negative in general, as can be seen from Gabrielov's counter-example to the nested version of Artin approximation [5]. Nevertheless, we prove in Theorem 1.1 that the answer is positive if F and G are of class Nash rather than only analytic. We propose moreover global versions of this approximation result in the real (Theorem 2.2) and complex (Theorem 2.5) algebraic settings. A particular case of this modified version of Artin approximation has already been used by author M. S., in the local case as well as in the global real case, to prove that the Right-Left analytic equivalence of real Nash maps implies their Right-Left Nash equivalence [11]. The approach proposed in this paper enables us to recover these results as direct and more conceptual consequences of Artin-type approximation.

The proof of this version of Artin approximation is based on nested approximation, which enables us to approximate the solutions by keeping the dependence of the

solutions in certain variables (*cf.* [13], for example). We show, in fact, that the version presented in the paper is equivalent to nested approximation. Note that the nested approximation, which is known to hold true in the Nash setting but not in the analytic one, is based on General Néron Desingularisation of regular ring morphisms [8,12]. For the global versions, we use in the real setting the work by Coste, Ruiz, and Shiota [2], where the regular ring homomorphism considered is the inclusion morphism of Nash functions in analytic functions on a compact Nash manifold; whereas, in the complex setting we use the work of Lempert [6] who considers the regular ring homomorphism given by the inclusion of the constant functions in the analytic ones, in the case of compact polynomial polyhedra.

The paper is organised as follows. In the first section we deal with the simplest situation of the local case, showing that formal solutions of Nash equations can be approximated by Nash solutions. The main ingredient is nested approximation. In the second section, we tackle the global version, whose proof needs some additional ingredients to achieve the approximation of analytic solutions of Nash equations by Nash solutions.

1 The Local Case

Let $\mathbb{K}$ denote a field. Denote by $\mathbb{K}[[x_1, \ldots, x_m]]$ (resp. $\mathbb{K}_{\mathrm{alg}}[[x_1, \ldots, x_m]]$) the rings of formal powers series (resp. algebraic power series) (over the polynomials $\mathbb{K}[x_1, \ldots, x_m]$), endowed with the $\mathfrak{m}$-adic topology where $\mathfrak{m}$ denotes the maximal ideal generated by $(x_1, \ldots, x_m)$. Let us recall also that, in the case $\mathbb{K} = \mathbb{R}$, the ring $\mathbb{R}_{\mathrm{alg}}[[x_1, \ldots, x_m]]$ is just the ring of Nash function germs $\mathbb{R}^m \to \mathbb{R}$ at the origin.

Theorem 1.1 *Let $l, m, n, p, q \in \mathbb{N}$ and $x = (x_1, \ldots, x_m)$, $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_p)$, and $t = (t_1, \ldots, t_l)$. Let $F \in \mathbb{K}_{\mathrm{alg}}[[x, y, z]]^q$ and $G \in \mathbb{K}_{\mathrm{alg}}[[x]]^l$ be such that $F(0) = 0$ and $G(0) = 0$.*

Assume that there exists $\Phi \in \mathbb{K}[[x]]^n$ and $\Psi \in \mathbb{K}[[t]]^p$ such that $\Phi(0) = \Psi(0) = 0$ and $F(x, \Phi(x), \Psi \circ G(x)) = 0$. Then we can approximate Φ by $\widetilde{\Phi} \in \mathbb{K}_{\mathrm{alg}}[[x]]^n$ and Ψ by $\widetilde{\Psi} \in \mathbb{K}_{\mathrm{alg}}[[t]]^p$ such that $F(x, \widetilde{\Phi}(x), \widetilde{\Psi} \circ G(x)) = 0$.

Before proving this result, let us show an immediate application to the so-called Right-Left equivalence relation of Nash function maps, in the case $\mathbb{K} = \mathbb{R}$. We say that the germs at the origin of the Nash maps $f \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^n \to \mathbb{R}^m$ are formally Right-Left (resp. Nash Right-Left) equivalent if there exist some smooth (resp. Nash) diffeomorphisms $\phi \colon \mathbb{R}^n \to \mathbb{R}^n$ and $\psi \colon \mathbb{R}^m \to \mathbb{R}^m$ such that $g \circ \phi$ agrees with $\psi \circ f$ at the level of Taylor expansions (resp. at the level of Nash germs) at the origin. One recovers the following corollary (originally proved in [11]).

Corollary 1.2 *Let $f \colon \mathbb{R}^n \to \mathbb{R}^m$ and $g \colon \mathbb{R}^n \to \mathbb{R}^m$ be two Nash germs at the origin. Then f and g are formally Right-Left equivalent if and only if they are Nash Right-Left equivalent.*

Proof Assume that f and g are formally Right-Left equivalent, so that there exist smooth diffeomorphisms ϕ and ψ such that $g \circ \phi$ agrees with $\psi \circ f$ at the level of Taylor

expansions. Denoting $F(y, z) = g(y) - z$, we obtain that the Taylor expansions of ϕ and ψ are formal solutions of $F(\phi(x), \psi \circ f(x)) = 0$. Using Theorem 1.1, it is possible to approximate ϕ and ψ by Nash map germs $\widetilde{\phi}$ and $\widetilde{\psi}$, which remain diffeomorphisms by (close enough) approximation, such that

$$F\big(\widetilde{\phi}(x), \widetilde{\psi} \circ f(x)\big) = 0.$$

Hence $g \circ \widetilde{\phi} = \widetilde{\psi} \circ f$, so that f and g are Nash Right-Left equivalent. The converse implication is immediate. ∎

Remark 1.3 The result is no longer true if we replace algebraic power series with analytic power series. A counter-example in the real case is exhibited in [11], based on a famous example of Osgood and Gabrielov [5]. More precisely, there exist two germs of analytic maps $f, g \colon (\mathbb{R}^2, 0) \to (\mathbb{R}^4, 0)$ that are Left equivalent as smooth germs, but not Left equivalent as analytic germs.

Let us come back now to the proof of Theorem 1.1, which is a consequence of the nested version of Artin approximation in the local case (see [13]).

Proof of Theorem 1.1 Write $F = (F_1, \ldots, F_q)$. We translate the equation

$$F\big(x, \Phi(x), \Psi \circ G(x)\big) = 0$$

into the conditions that

$$\begin{cases} F_k\big(x, \Phi(x), \Psi(t)\big) = 0 & \forall k \in \{1, \ldots, q\}, \\ t_j = G_j(x) & \forall j \in \{1, \ldots, l\}. \end{cases}$$

Since $G(0) = 0$, the set

$$\big\{x_1, \ldots, x_m, t_1 - G_1(x), \ldots, t_l - G_l(x)\big\}$$

can serve as a new set of variables for $\mathbb{K}[[x, t]]$. In particular, considering the F_k's as formal power series, we obtain, for any $k \in \{1, \ldots, p\}$, a unique decomposition

$$F_k\big(x, \Phi(x), \Psi(t)\big) = \sum_{(a,b) \in (\mathbb{N} \cup \{0\})^m \times (\mathbb{N} \cup \{0\})^l} c_{a,b} x^a \big(t - G(x)\big)^b$$

with $c_{a,b} \in \mathbb{K}$, $x^a = \prod_{i=1}^{m} x_i^{a_i}$ and $(t - G(x))^b = \prod_{j=1}^{l} (t_j - G_j(x))^{b_j}$. Moreover, the coefficients $c_{a,0}$ must vanish, since Φ and Ψ are solutions of

$$F\big(x, \Phi(x), \Psi \circ G(x)\big) = 0.$$

So we can rewrite these decompositions of the F_k's as finite sums

$$F_k\big(x, \Phi(x), \Psi(t)\big) = \sum_{j=1}^{l} \xi_{k,j}(x, t)\big(t_j - G_j(x)\big)$$

for some $\xi_{k,j} \in \mathbb{K}[[x, t]]$. Consider the system of equations in $\mathbb{K}_{\mathrm{alg}}[[x, t]]$ with variables

$$\big\{U_i, V_r, W_{k,j}\big\}_{(i,j,k,r) \in \{1,\ldots,n\} \times \{1,\ldots,l\} \times \{1,\ldots,q\} \times \{1,\ldots,p\}}$$

defined by:

$$\begin{cases} F_1\big(x, U_1, \ldots, U_n, V_1, \ldots, V_p\big) - \sum_{j=1}^{l} W_{1,j}\big(t_j - G_j(x)\big) = 0 \\ \qquad\qquad\qquad \vdots \\ F_q\big(x, U_1, \ldots, U_n, V_1, \ldots, V_p\big) - \sum_{j=1}^{l} W_{q,j}\big(t_j - G_j(x)\big) = 0. \end{cases}$$

This system of equations in $\mathbb{K}_{\mathrm{alg}}[[x, t]]$ admits

$$\begin{cases} (U_1, \ldots, U_n) = \Phi(x) \\ (V_1, \ldots, V_p) = \Psi(t) \\ W_{k,j} = \xi_{k,j}(x, t), \qquad (j, k) \in \{1, \ldots, l\} \times \{1, \ldots, q\} \end{cases}$$

as a solution in $\mathbb{K}[[x, t]]$. By the nested approximation theorem ([13]), we can approximate this solution by a solution

$$\begin{cases} (U_1, \ldots, U_n) = \widetilde{\Phi}(x, t) \\ (V_1, \ldots, V_p) = \widetilde{\Psi}(t) \\ W_{k,j} = \widetilde{\xi}_{k,j}(x, t), \qquad (j, k) \in \{1, \ldots, l\} \times \{1, \ldots, q\} \end{cases}$$

in $\mathbb{K}_{\mathrm{alg}}[[x, t]]$. As a consequence,

$$F_k\big(x, \widetilde{\Phi}(x, t), \widetilde{\Psi}(t)\big) = \sum_{j=1}^{l} \widetilde{\xi}_{k,j}(x, t)\big(t_j - G_j(x)\big)$$

for any $k \in \{1, \ldots, q\}$, and therefore

$$F\big(x, \widetilde{\Phi}\big(x, G(x)\big), \widetilde{\Psi}\big(G(x)\big)\big) = 0.$$

It remains to say that $\widetilde{\widetilde{\Phi}}(x) = \widetilde{\Phi}(x, G(x))$ approximates $\underset{\approx}{\Phi}(x)$ in $\mathbb{K}_{\mathrm{alg}}[[x]]^n$ since $\widetilde{\Phi}(x, t)$ approximates $\widetilde{\Phi}(x)$ in $\mathbb{K}_{\mathrm{alg}}[[x, t]]^n$, and therefore $\widetilde{\Phi}$ and $\widetilde{\Psi}$ give the desired solutions. ∎

Remark 1.4 If the nested Artin approximation is the key tool for proving Theorem 1.1; conversely, one may also note that the nested Artin approximation for two sets of variables x_1 and x_2 could be seen as a consequence of Theorem 1.1. Indeed, let $F \in \mathbb{K}_{\mathrm{alg}}[[x_1, x_2, y, z]]^q$ be such that $F(0) = 0$. Assume that there exist $\Psi \in \mathbb{K}[[x_1]]$ and $\Phi \in \mathbb{K}[[x_1, x_2]]$ such that $\Phi(0) = \Psi(0) = 0$ and

$$F\big(x_1, x_2, \Psi(x_1), \Phi(x_1, x_2)\big) = 0.$$

This last equation is equivalent to

$$F\big(x_1, x_2, \Psi_2 \circ G(x_1, x_2), \Phi(x_1, x_2)\big) = 0,$$

where $G(x_1, x_2) = (x_1, 0)$ and Ψ_2 is just Ψ viewed in $\mathbb{K}[[x_1, x_2]]$. By Theorem 1.1, one can approximate $\Psi_2 \in \mathbb{K}[[x_1, x_2]]$ (resp. $\Phi \in \mathbb{K}[[x_1, x_2]]$) by $\widetilde{\Psi}_2 \in \mathbb{K}_{\mathrm{alg}}[[x_1, x_2]]$ (resp. $\widetilde{\Phi} \in \mathbb{K}_{\mathrm{alg}}[[x_1, x_2]]$) such that

$$F\big(x_1, x_2, \widetilde{\Psi}_2 \circ G(x_1, x_2), \widetilde{\Phi}(x_1, x_2)\big) = 0,$$

or in other words

$$F\big(x_1, x_2, \widetilde{\Psi}_2(x_1, 0), \widetilde{\Phi}(x_1, x_2)\big) = 0.$$

Since $\widetilde{\Psi}_2(x_1, x_2)$ approximates $\Psi(x_1)$ in $\mathbb{K}[[x_1, x_2]]$, then $\widetilde{\Psi}_2(x_1, 0)$ also approximates $\Psi(x_1)$.

In other words, nested Artin approximation for a set of two variables and Artin approximation compatible with a change on a set of two variables are closely related notions.

One may readily generalize Theorem 1.1 to a change on several sets of variables.

Theorem 1.5 *Let $m, m_i, l_i, q \in \mathbb{N}$ and consider some sets of variables $x = (x_1, \ldots, x_m)$, $y_i = (y_{i,1}, \ldots, y_{i,m_i})$, for $i \in \{0, \ldots, n\}$. Let $F \in \mathbb{K}_{\mathrm{alg}}[[x, y_0, \ldots, y_n]]^q$ and let $G_i \in \mathbb{K}_{\mathrm{alg}}[[x]]^{l_i}$ be such that $F(0) = 0$ and $G_i(0) = 0$ for $i \in \{0, \ldots, n\}$.*

Assume there exist some n_i-tuples of formal power series Ψ_i such that, for each $i \in \{0, \ldots, n\}$, one has $\Psi_i(0) = 0$ and

$$F\Big(x, \Psi_0(x), \Psi_1\big(G_1(x)\big), \Psi_2\big(G_1(x), G_2(x)\big), \ldots, \Psi_n\big(G_1(x), \ldots, G_n(x)\big)\Big) = 0.$$

Then one can approximate all the Ψ_i's by algebraic n_i-tuples of power series $\widetilde{\Psi}_i$ such that

$$F\Big(x, \widetilde{\Psi}_0(x), \widetilde{\Psi}_1\big(G_1(x)\big), \widetilde{\Psi}_2\big(G_1(x), G_2(x)\big), \ldots, \widetilde{\Psi}_n\big(G_1(x), \ldots, G_n(x)\big)\Big) = 0.$$

As has already been noted in Remark 1.4, this result is equivalent to nested Artin approximation.

2 The Global Case

The global versions of Artin approximation that we deal with in this paper are very similar in the real and complex settings. We give more details for the real case with which the reader may be less familiar. The common idea is to translate the modification in the source variables of the solution in terms of inclusion of Nash sets. Then we use Cartan Theorem B in order to describe this inclusion by global equations, increasing the number of variables. Then we apply nested approximation to obtain the desired solution in the same way as in the local case.

2.1 The Real Case

The interest for Nash functions and Nash manifolds in real geometry comes from the original work of Nash [7], whose purpose was to equip any analytic manifold with a real algebraic structure. The use of Nash functions is now classical, in an intermediate step between polynomials and analytic functions (*cf.* [10]). Concerning Artin approximation more specifically, the first global version was used in [2] in order to solve difficult questions on Nash functions, whereas its nested version was used in [3] in order to approximate an analytic resolution of the singularities of a Nash function by a Nash resolution.

In what follows, the approximations are considered with respect to the C^∞ topology.

For the convenience of the reader, we recall [3, Proposition 3.1] (in the particular case where $M_i = X_i$ are compact Nash manifolds, for $i = 1, \ldots, m$), which will be the key to prove the main result of this section and can be seen as a global counterpart

of the nested approximation theorem. For a Nash manifold M, we denote by $\mathcal{O}$ (resp. $\mathcal{M}$), its sheaf of real analytic (resp. Nash) function germs.

Theorem 2.1 *Let $M_1, \ldots, M_m$ be compact Nash manifolds and let $l_1, \ldots, l_m$, $n_1, \ldots, n_m \in \mathbb{N}$. Let $F_i \in \mathcal{N}(M_1 \times \cdots \times M_i \times \mathbb{R}^{l_1} \times \cdots \times \mathbb{R}^{l_i})^{n_i}$ and $f_i \in \mathcal{O}(M_1 \times \cdots \times M_i)^{l_i}$, for $i = 1, \ldots, m$, be such that*

$$F_i\big(x_1, \ldots, x_i, f_1(x_1), \ldots, f_i(x_1, \ldots, x_i)\big) = 0,$$

as elements of $\mathcal{O}(M_1 \times \cdots \times M_i)^{n_i}$. Then there exist $\widetilde{f_i} \in \mathcal{N}(M_1 \times \cdots \times M_i)^{l_i}$, for $i = 1, \ldots, m$, close to f_i in the C^∞ topology, such that

$$F_i\big(x_1, \ldots, x_i, \widetilde{f_1}(x_1), \ldots, \widetilde{f_i}(x_1, \ldots, x_i)\big) = 0$$

in $\mathcal{N}(M_1 \times \cdots \times M_i)^{n_i}$.

The next result can be seen as an Artin approximation compatible with a change of variables, or in other words, as a global counterpart in the real setting of Theorem 1.1.

Theorem 2.2 *Let M, N, L, and P be compact Nash manifolds, let $q \in N$ be an integer, and let $F \colon M \times N \times P \to \mathbb{R}^q$ and $G \colon M \to L$ be Nash maps.*

Assume there exist analytic maps $\Phi \colon M \to N$ and $\Psi \colon L \to P$ such that $F(x, \Phi(x), \Psi \circ G(x)) = 0$ for any $x \in M$. Then there exist Nash maps $\widetilde{\Phi} \colon M \to N$ and $\widetilde{\Psi} \colon L \to P$, close to Φ and Ψ, such that $F(x, \widetilde{\Phi}(x), \widetilde{\Psi} \circ G(x)) = 0$ for any $x \in M$.

Before entering into the details of the proof, we state a classical auxiliary lemma.

Lemma 2.3 *Let M be a Nash manifold, let $X \subset M$ be a Nash subset, and take $x \in X$. Denote by $I \subset \mathcal{N}_{M,x}$ the ideal of Nash germs whose complexification vanishes on the complexification $X_{\mathbb{C}}$ of X in a neighbourhood of x. Then $I\mathcal{O}_{M,x}$ is equal to the ideal of analytic germs whose complexification vanishes on $X_{\mathbb{C}}$ in a neighbourhood of x.*

Proof Let us denote by J the ideal in $\mathcal{O}_{M,x}$ of analytic germs whose complexification vanishes on $X_{\mathbb{C}}$ in a neighbourhood of x. Then the ideals I and J have the same complex zero set germ at x. Note also that we have the obvious inclusion $I\mathcal{O}_{M,x} \subset J$. Let us show now the reverse inclusion.

By Ruckert's Nullstellensatz, $J\mathcal{O}_{M,x}^{\mathbb{C}} \subset \sqrt{I\mathcal{O}_{M,x}^{\mathbb{C}}}$, where $\mathcal{O}^{\mathbb{C}}$ denotes the sheaf of complex analytic functions.

Since we have a sequence of regular ring homomorphisms $\mathcal{N}_{M,x} \to \mathcal{O}_{M,x} \to \mathcal{O}_{M,x}^{\mathbb{C}}$ and I is radical in $\mathcal{N}_{M,x}$, we get that the ideal $I\mathcal{O}_{M,x}^{\mathbb{C}}$ is also radical in $\mathcal{O}_{M,x}^{\mathbb{C}}$. An alternative proof of this fact would be to use that our rings are G-rings (and even more , they are excellent rings) with same completion (a ring of formal power series).

Then one gets the inclusion $J\mathcal{O}_{M,x}^{\mathbb{C}} \subset I\mathcal{O}_{M,x}^{\mathbb{C}}$ and hence the equality $J\mathcal{O}_{M,x}^{\mathbb{C}} = I\mathcal{O}_{M,x}^{\mathbb{C}}$. Finally, the equality $J = I\mathcal{O}_{M,x}$ follows by faithful flatness of $\mathcal{O}_{M,x}^{\mathbb{C}}$ over $\mathcal{O}_{M,x}$. ∎

Proof of Theorem 2.2 We are going to translate the condition that Φ satisfies the equation

$$F\big(x, \Phi(x), \Psi \circ G(x)\big) = 0$$

for any $x \in M$ in such a way as to be able to use nested approximation. The idea is to note that this condition is equivalent to the fact that the image of the graph of G by the map

$$(\mathrm{Id}, \Phi) \times \Psi \colon M \times L \longrightarrow M \times N \times P$$
$$(x, y) \longmapsto \big(x, \Phi(x), \Psi(y) \big)$$

is included in the set $F^{-1}(0)$.

Assume that M, N, L, and P are respectively embedded in $\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^l$, and $\mathbb{R}^p$ as closed Nash manifolds, and let $h_N \colon \mathbb{R}^n \to \mathbb{R}$ and $h_P \colon \mathbb{R}^p \to \mathbb{R}$ be some global equations for N and P, respectively (global equations exist by [10]). Let $\Gamma \subset M \times L$ denote the graph of G, and let $\mathcal{J}$ be the sheaf of $\mathcal{N}_{M \times L}$-ideals of germs whose complexifications vanish on the complexification $\Gamma^{\mathbb{C}}$ of Γ. Then we know by [2] that there exist global generators $g_1, \ldots, g_s$ of $\mathcal{J}$.

We want to approximate the analytic maps Φ and Ψ keeping the property that

$$\big((\mathrm{Id}, \Phi) \times \Psi \big)(\Gamma) \subset F^{-1}(0).$$

By Lemma 2.3 applied to $\Gamma \subset M \times L$, we can use Cartan Theorem B to deduce the surjectivity of the homomorphism

$$H^0(M \times L, \mathcal{O}_{M \times L}^s) \longrightarrow H^0(M \times L, \mathcal{J}\mathcal{O}_{M \times L})$$

defined by $(\xi_1, \ldots, \xi_s) \mapsto \sum_{j=1}^s \xi_j g_j$ (note that this step is the major change with respect to what has been done in the local case). In particular, considering the global cross-section $F\big((\mathrm{Id}, \Phi) \times \Psi \big)$ of $\mathcal{J}\mathcal{O}_{M \times L}$, we obtain the existence of analytic functions $\xi_{k,1}, \ldots, \xi_{k,s}$ on $M \times L$ such that

$$\forall k \in \{1, \ldots, q\}, \quad \forall (x, y) \in M \times L, \quad F_k\big(x, \Phi(x), \Psi(y) \big) = \sum_{j=1}^s \xi_{k,j}(x, y) g_j(x, y).$$

Moreover, $h_N(\Phi(x)) = 0$ for $x \in M$ and $h_P(\Psi(y)) = 0$ for $y \in L$. Now consider the system of equations

$$\begin{cases} F_1(x, U_1, \ldots, U_n, V_1, \ldots, V_p) - \sum_{j=1}^s W_{1,j} g_j(x, y) = 0 \\ \qquad\qquad \vdots \\ F_q(x, U_1, \ldots, U_n, V_1, \ldots, V_p) - \sum_{j=1}^s W_{q,j} g_j(x, y) = 0 \\ h_N(U_1, \ldots, U_n) = 0 \\ h_P(V_1, \ldots, V_p) = 0 \end{cases}$$

with variables

$$\big(U_i, V_r, W_{k,j} \big)_{(i,j,k,r) \in \{1,\ldots,n\} \times \{1,\ldots,s\} \times \{1,\ldots,q\} \times \{1,\ldots,p\}}.$$

This system of equations involves Nash maps and functions, and it admits analytic solutions given by

$$\begin{cases} (U_1, \ldots, U_n) = \Phi(x) \\ (V_1, \ldots, V_p) = \Psi(y) \\ W_{k,j} = \xi_{k,j}(x, y) \qquad \text{for all } (j, k) \in \{1, \ldots, s\} \times \{1, \ldots, q\}. \end{cases}$$

By nested approximation in the Nash case (Theorem 2.1 applied to $(x_1, x_2) = (y, x)$), we can approximate these analytic solutions by Nash solutions

$$
\begin{cases}
(U_1, \ldots, U_n) = \widetilde{\Phi}(x, y) \\
(V_1, \ldots, V_p) = \widetilde{\Psi}(y) \\
W_{k,j} = \widetilde{\xi}_{k,j}(x, y), & \text{for all } (j, k) \in \{1, \ldots, s\} \times \{1, \ldots, q\}.
\end{cases}
$$

In particular, $\widetilde{\Phi}$ and $\widetilde{\Psi}$ take values in N and P respectively, and moreover, evaluating at $y = G(x)$, we obtain for any $x \in M$ the equality

$$
F\Big(x, \widetilde{\Phi}\big(x, G(x)\big), \widetilde{\Psi}\big(G(x)\big)\Big) = 0.
$$

We conclude by remarking that the Nash map $x \mapsto \widetilde{\Phi}(x, G(x))$ remains an approximation of Φ, and together with $\widetilde{\Psi}$, they give the desired solution. ∎

Remark 2.4 (i) Similarly to the local case, we deduce easily from Theorem 2.2 the main result in [11], namely that Nash maps on a compact Nash manifold are Nash Right-Left equivalent if they are analytic Right-Left equivalent.

(ii) Theorem 2.2 can be refined to the case of Nash map germs, defined on germs of compact semialgebraic sets in Nash manifolds (*cf.* [11] for example).

2.2 The Complex Case

In the complex setting, the result is very similar, provided one uses the correct analogue of the global nested Artin approximation. By [6], we can approximate holomorphic solutions of a Nash algebraic equation on a compact polynomial polyhedron (namely the intersection of a finite number of sets of the form $|f| \leq 1$ for f a complex polynomial; note that the ring of holomorphic functions on such a set is Noetherian by [4]) by Nash algebraic solutions. Here by Nash algebraic, we follow the terminology of [6], which defined a Nash algebraic map to be a holomorphic map between affine complex algebraic varieties whose components satisfy polynomial equations. Therefore, we obtain next result by a proof completely similar to that of Theorem 2.2.

Theorem 2.5 *Let M, N, L, and P be non singular affine complex algebraic varieties, and let $F: M \times N \times P \to \mathbb{C}^q$ and $G: M \to L$ be regular maps.*

Assume that there exist some holomorphic maps $\Phi: M \to N$ and $\Psi: L \to P$ such that $F(x, \Phi(x), \Psi \circ G(x)) = 0$ for any $x \in M$. Let $A \subset M$ and $B \subset L$ be compact polynomial polyhedra satisfying $G(A) \subset B$. Then $\Phi_{|A}$ and $\Psi_{|B}$ can be approximated by Nash algebraic maps $\widetilde{\Phi}: A \to N$ and $\widetilde{\Psi}: B \to P$ such that $F(x, \widetilde{\Phi}(x), \widetilde{\Psi} \circ G(x)) = 0$ for $x \in A$.

References

[1] M. Artin, *On the solutions of analytic equations.* Invent. Math. 5(1968), 277–291.
http://dx.doi.org/10.1007/BF01389777

[2] M. Coste, J. M. Ruiz, and M. Shiota, *Approximation in compact Nash manifolds.* Amer. J. Math.
117(1995), no. 4, 905–927. http://dx.doi.org/10.2307/2374953

[3] G. Fichou and M. Shiota, *On almost blow-analytic equivalence.* Proc. Lond. Math. Soc. (3)
103(2011), no. 4, 676–709. http://dx.doi.org/10.1112/plms/pdq031

[4] J. Frisch, *Points de platitude d'un morphisme d'espaces analytiques complexes*. Invent. Math.
     4(1967), 118–138.   http://dx.doi.org/10.1007/BF01425245
[5] A. M. Gabrièlov, *Formal relations between analytic functions*. (Russian) Funkctional Anal.
     Prilozen **5**(1971), no. 4, 64–65.
[6] L. Lempert, *Algebraic approximations in analytic geometry*. Invent. Math. **121**(1995), no. 2,
     335–353.   http://dx.doi.org/10.1007/BF01884302
[7] J. Nash, *Real algebraic manifolds*. Ann. of Math. **56**(1952), 405–421.
     http://dx.doi.org/10.2307/1969649
[8] D. Popescu, *Global Neron desingularisation*. Nagoya Math. J. **100**(1985), 97–126.
[9] G. Rond, *Artin approximation*. arxiv:1506.04717
[10] M. Shiota, *Nash manifolds*. Lecture Notes in Mathematics, 1269, Springer-Verlag, Berlin, 1987.
[11] M. Shiota, *Analytic and Nash equivalence relations of Nash maps*. Bull. Lond. Math. Soc.
     42(2010), no. 6, 1055–1064.   http://dx.doi.org/10.1112/blms/bdq064
[12] M. Spivakovsky, *A new proof of D. Popescu's theorem on smoothing of ring homomorphisms*. J.
     Amer. Math. Soc. **12**(1999), no. 2, 381–444.   http://dx.doi.org/10.1090/S0894-0347-99-00294-5
[13] B. Teissier, *Résultats récents sur l'approximation des morphismes en algèbre commutative d'après
     Artin, Popescu et Spivakovsky*. Séminaire Bourbaki, 1993/94, Astérisque **784**(1994), 259–282.

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