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THE DEGREE OF THE GRADIENT OF A COERCIVE FUNCTIONAL

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Abstract

An elementary proof is given of a theorem of Castro and Lazer that the degree of the gradient of a coercive functional on a large ball of $\mathscr{R}^n$ is one.

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In a recent review article on variational and topological methods in nonlinear problems, Nirenberg (1981), p. 276 cited the following result due to Castro and Lazer (1979) and remarked that although the result is intuitively clear he did not know of an elementary proof of it.

THEOREM 1 (Castro and Lazer). *Let $f \in C^2(\mathscr{R}^n, \mathscr{R})$ and suppose that f has only a finite number of critical points and that $f(x) \to \infty$ as $\|x\| \to \infty$. Then for R large*

$$\deg(\nabla f, B_R, 0) = 1$$

where $B_R = \{x \in \mathscr{R}^n \,|\, \|x\| < R\}$.

The author has recently given a simple proof of this result (McKenzie (1982), p. 156). However, Professor Nirenberg has drawn to the author's attention a recent paper by Amann (1982) which contains an elementary proof of the following generalisation of Theorem 1:

THEOREM 2 (Amann). *Let H be a real Hilbert space and suppose that the gradient ∇f: $H \to H$ of a given function $f \in C^1(H, \mathscr{R})$ is a compact vector field. Suppose*

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that $\nabla f(x) \neq 0$ *for* $\|x\| \geqslant r_0$ *and some* $r_0 > 0$. *Then there is a number* $r_1 \geqslant r_0$ *such that*

$$\deg(\nabla f, B_r, 0) = 1$$

for all $r \geqslant r_1$.

Although the result below due to the author is less general than Amann's, the proof is more direct and transparent, particularly as it shows that this result is basically topological.

THEOREM 3. *Suppose that* $f \in C^2(\mathscr{R}^n, \mathscr{R})$, $f(x) \to \infty$ *as* $\|x\| \to \infty$, *and that there exists S such that* $\nabla f(x) \neq 0$ *for* $\|x\| \geqslant S$. *Then for R large*

$$\deg(\nabla f, B_R, 0) = 1.$$

PROOF. If $f^d = \{x \in \mathscr{R}^n | f(x) \leqslant d\}$ then since $f(x) \to \infty$ as $\|x\| \to \infty$, f^d is bounded for each d. Thus f^d is a closed and bounded subset of $\mathscr{R}^n$ and so is compact. Let $a = \sup_{\|x\| \leqslant S} f(x)$. It is now shown that if $b \geqslant a$ then f^b is contractible to a point.

If $y \in f^b$ then since $\mathscr{R}^n$ is contractible to a point there exists a deformation retraction F: $I \times \mathscr{R}^n \to \mathscr{R}^n$ which maps $\mathscr{R}^n$ onto y. If $G = F | I \times f^b$ then since $I \times f^b$ is compact, $G(I \times f^b)$ is compact and f is bounded on this set so there exists $c < \infty$ such that

$$G(I \times f^b) \subset f^c.$$

Since $\nabla f \neq 0$ for $x \notin f^b$ it follows from a standard construction of Morse Theory (Milnor (1963), p. 12) that f^b is a deformation retraction of f^c. Thus, there exists a retraction f: $f^c \to f^b$. The map H: $I \times f^b \to f^b$ defined by $H = r \circ G$ is then a deformation retraction of f^b onto y.

Let the family of real valued functions $\{g_a\}_{a \in \mathscr{R}^n}$ be defined by

$$g_a(x) = f(x) + \langle a, x \rangle$$

where $\langle\ ,\ \rangle$ is the standard inner product on $\mathscr{R}^n$. Let $\varepsilon = \inf_{x \in \partial f^b} \|\nabla f\|$. Then by Morse's Theorem (Guillemin and Pollack (1974), p. 43) $c \in \mathscr{R}^n$ can be chosen so that $\|c\| < \varepsilon$ and g_c has non-degenerate critical points. If K: $\mathscr{R}^n \times I \to \mathscr{R}^n$ is defined by $K(x, t) = (1 - t)\nabla f + t\nabla g_c = \nabla f + tc$ then $K(x, t)$ is nonzero for all $t \in I$, $x \in \partial f^b$. Thus, since K is a homotopy joining ∇f and ∇g_c

$$\deg(\nabla f, f^b, 0) = \deg(\nabla g_c, f^b, 0).$$

Clearly ∇g_c points outwards on the boundary of f^b, that is,

$$\langle \nabla g_c(x), \nabla f(x) \rangle > 0 \quad \text{for } x \in \partial f^b.$$

Let $\{x^1,\ldots,x^N\}$ be the set of isolated zeroes of ∇g_c in f^b and $i(\nabla g_c, x^j)$ denote the index of the zero at x^j. Then, by the Poincaré-Hopf index theorem (Milnor (1965), p. 35).

$$\deg(\nabla g_c, f^b, 0) = \sum_{j=1}^{N} i(\nabla g_c, x_j) = \chi(f^b)$$

where $\chi(f^b)$ is the Euler characteristic of f^b. Since f^b is contractible to a point $\chi(f^b) = 1$. If $R \geqslant \sup_{x\in f^b}\|x\|$ then by the excision property of the degree

$$\deg(\nabla f, B_R, 0) = \deg(\nabla f, f^b, 0) = \deg(\nabla g_c, f^b, 0) = 1.$$

This work was done while the author was a student in the Department of Theoretical Physics, Faculty of Science, Australian National University.

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