THE SPECTRUM OF ORTHOGONAL SUMS OF SUBNORMAL PAIRS

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Introduction. This note provides yet another example of the difficulties that arise when one wants to extend the spectral theory of subnormal operators to subnormal tuples. Several basic properties of a subnormal operator Y remain true for tuples; e.g. the existence and uniqueness of its minimal normal extension N, the spectral inclusion $\sigma(N) \subset \sigma(Y)$-proved for n-tuples in **[4]** and generalized to infinite tuples in **[5]**. However, neither the invariant subspace theorem nor the spectral mapping theorem in the "strong form" as in **[3]** is known so far for subnormal tuples.

The present note shows that even such a well known equality as

$$\sigma\left(\bigoplus_{n=1}^{\infty} Y_n\right) = \left(\bigcup_{n=1}^{\infty} \sigma(Y_n)\right)^{-}, \tag{*}$$

valid for bounded sequences $\{Y_n\}$ of subnormals fails to have a multiparameter analogue. Namely, we shall construct a sequence of subnormal pairs (S_n, T_n) for which the equality (1) below fails.

$$\sigma(\oplus S_n, \oplus T_n) = (\bigcup \sigma(S_n, T_n))^{-}. \tag{1}$$

Here σ stands for the joint spectrum in the sense of J. L. Taylor **[8, 2]**, the bar denotes closure in the natural topology of $\mathbb{C}^2$. By subnormal we mean a pair being a restriction of two commuting, bounded normal operators to one of their common invariant subspaces.

Remarks. The containment "$\supset$" in (1) does always take place. However, each of the following conditions suffices for the equality there for pairs, or even tuples of arbitrary commuting operators.

(a) The sequence (S_n, T_n) is constant beginning from some k: for all $n \geqslant k$.

(b) $\|S_n\| \to 0$ and $\|T_n\| \to 0$ as $n \to \infty$.

(c) $S_n = T_n$ and the S_n are subnormal for n large enough.

(To prove (b) use the semicontinuity of σ, cf. **[6]**. (c) follows from (*) if we identify $\sigma(S, S)$ with $\sigma(S)$.)

It would be interesting to know more nontrivial sufficient conditions for the equality in (1), since this equality may be applied to solving certain linear equations, a technique developed in the proof of our main result.

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Preliminaries. Let S and T be a pair of bounded, commuting operators on a Hilbert space H. Then, by definition, $(0.0) \notin \sigma(S, T)$ iff the mappings: $f \to Sf \oplus Tf$ and $(g \oplus h) \to Sg - Th$ form a short exact sequence $0 \to H \to H \oplus H \to H \to 0$. If this is the case, then the "Laplacian" $X := SS^* + TT^*$ is invertible and for $A := S^*X^{-1}$, $B := T^*X^{-1}$ we have

$$SA + TB = I, \tag{2}$$

where I denotes the identity operator on H. See [**2**]. Conversely, if (2) has a solution with A, B in a commutative algebra containing S and T then $(0, 0) \notin \sigma(S, T)$. Generally, $\sigma(S, T)$ is defined as $\{(z, w) \in \mathbb{C}^2; (0, 0) \in \sigma(S - zI, T - wI)\}$.

If S, T are multiplication operators on a Hardy space, the equality (2) looks like a solution of the corona equation. This formal similarity lies behind a deep relationship between (1) and the corona problem; cf. [**5**]. In that setting instead of Taylor's joint spectrum I have considered the so called extended spectrum of subnormal representations of $H^\infty(\Omega)$, the algebra of all bounded analytic functions on a (Runge) domain Ω in $\mathbb{C}^2$. Then (1) for that type of spectrum is equivalent to the corona theorem for $H^\infty(\Omega)$ and so is not always true. This follows from N. Sibony's counterexample, given in [**7**]. Here we use the following modification of this example: there is a Runge domain G contained in the unit bidisc $\mathbb{D}^2$ such that $\mathbb{D}^2 \not\subset G$ but $H^\infty(G) = H^\infty(\mathbb{D}^2)$ i.e., any $f \in H^\infty(G)$ extends analytically onto $\mathbb{D}^2$ with the same norm. The construction in [**7**] is simple.

Choose a discrete subset $\alpha = \{\alpha_m; m < \infty\}$ of the unit disc $\mathbb{D}$ such that $|u(z)| \leq \sup_m |u(\alpha_m)|$ for all $u \in H^\infty(\mathbb{D})$, $z \in \mathbb{D}$. Next construct a non-negative bounded subharmonic function V on $\mathbb{D}$ such that $\alpha = V^{-1}\{0\}$. Let $G := \{(z, w) \in \mathbb{D}^2; |w| < \exp(-V(z))\}$; then

$$H^\infty(G) = H^\infty(\mathbb{D}^2). \tag{3}$$

However, we need here such an extension for some Hilbert space of analytic functions on G in place of $H^\infty(G)$. Unfortunately, I cannot prove that this extension takes place for the Hardy or Bergman space over G, but only for the Lumer–Hardy space $LH^2(G)$ which does not seem to be a Hilbert space. To overcome this difficulty I introduce some technical L^2-norms.

The construction. Let G be as in (3) a domain related to the set $\alpha = \bigcup_{n=2}^\infty C_n$, where each C_n is a collection of n^4 points equidistributed at the circle $|z| = 1 - (1/n)$. Choose an exhaustion of G by a sequence of smoothly bordered domains of holomorphy Q_n (cf. [**5**]) such that $\bar{Q}_n \subset Q_{n+1} \subset \ldots \subset G = \bigcup Q_n$ and for which $(z, w) \in Q_n$ if either $z \in C_n$, $|w| \leq 1 - (1/n)$ or if $|z| \leq 1 - (1/2n)$ and $|w| \leq \frac{1}{4}$.

Notation:

- v volume i.e. 4-dimensional Lebesgue measure on $\mathbb{C}^2$
- μ_n the equidistributed probability measure on C_n and
- v' the planar Lebesgue measure on $\mathbb{C}$.

Let us define the measure v_n on Q_n by the following formula:

$$\int f\,dv_n = \int_{Q_n} f\,dv + \int_{C_n} d\mu_n(z) \int_{|w|<1-(1/n)} f(z, w)\,dv'(w). \tag{4}$$

As (S_n, T_n), we shall take the multiplication by z and by w operators on the space H_n, defined as the closure in $L^2(v_n)$ of all complex polynomials $p(z, w)$. In other words, $(S_nf)(z, w) = zf(z, w)$ and $(T_nf)(z, w) = wf(z, w)$. Note that H_n is nothing else but the renormed Bergman space with a norm equivalent to the $L^2(v)$-norm. The point of this modification is that the norms of elements $h \in H_n$ will be more influenced as $n \to \infty$ by the values $h(z, w)$ for $z \in C_n$, helping to "enlarge" radii of convergence in the variable w. More precisely, for $0 \leq r \leq 1$, having a function a analytic on $r\mathbb{D}$, put

$$\|a\|_r := \left(\int_{|w|<r} |a(w)|^2\,dv'(w) \right)^{1/2}.$$

Then for $s := 1 - (1/n)$, $r := 1 - (1/2n)$ our basic estimate will be

$$\|a\|_s \leq \pi \left(\int_{C_n} |a|^2\,d\mu_n \right)^{1/2} + Cn^{-2}\,\|a\|_r, \tag{5}$$

where C is independent of n, a. The proof of (5) will be given later on. Let us also note that if $a(w) = \sum a_k w^k$, then using polar coordinates one may easily estimate the Fourier coefficients a_k as follows

$$|a_k| \leq M_k\,\|a\|_t, \tag{6}$$

where $t = \frac{1}{4}$ and $M_k^2 = 4^{2k+2}(k+1)/\pi$.

The key property of the sequence $\{H_n\}$ is contained in the following result.

Lemma. *If $f_n \in H_n$ form a sequence converging pointwise on G with the $L^2(v_n)$-norms of f_n bounded by some $M < \infty$, then $f := \lim f_n$ extends analytically onto $\mathbb{D}^2$.*

Assuming the lemma and (5) for a moment, we shall prove our main result.

Proposition. *The (S_n, T_n) are subnormal pairs of contractions for which the equality* (1) *fails.*

Proof. The subnormality is obvious—the same formulae define normal extensions on $L^2(v_n)$ of these pairs. $\|S_n\| \leq 1$, since $|z| < 1$ on Q_n and similarly $\|T_n\| \leq 1$. It is easy to see that $\sigma(S_n, T_n) \subset \bar{G}$. Indeed, for $(z', w') \notin \bar{G}$ there exist functions f, g analytic on G such that $(z - z')f + (w - w')g \equiv 1$. Therefore the equation (2) has solutions in the algebra of multiplication operators by functions from $H^\infty(Q_n)$ and the criterion (2′) is applicable (cf. [**5**]).

Let us fix a point $(z', w') \in \mathbb{D}^2 \setminus \bar{G}$. If (1) were true, we would have $(0, 0) \notin \sigma(S, T)$, where $S := \oplus (S_n - z'I)$, $T := \oplus (T_n - w'I)$, and there would exist operators X, A, B of the form $A = \oplus A_n$, $B = \oplus B_n$ with $\|A_n\|$, $\|B_n\|$ bounded, which would solve (2).

The constant polynomial 1 has its $L^2(v_n)$-norms bounded as $n \to \infty$ and so have the functions $f_n := A_n 1$ and $g_n := B_n 1$. We may apply a normal family argument and assume that these functions converge on G to certain functions f, g respectively. These functions satisfy $(z-z')f+(w-w')g=1$ (by (2)) for any $(z, w)\in G$ and, after analytic continuation (by the Lemma)—also for any $(z, w)\in\mathbb{D}^2$, which is absurd: taking $z=z'$, $w=w'$ we get $0=1$. ■

REMARK. We do not know if the A_n, B_n commute with S_n and T_n. If this were true, it would be a major simplification as these operators would then be multiplications by f_n and g_n, and this would provide the uniform estimates (on Q_n) for these functions. To conclude we need to prove (5) and the Lemma.

Proof. For $z\in\mathbb{C}$ with $|z|=s$ (i.e. $1-(1/n)$ let L_z be the point of C_n next to z in the clockwise direction of this circle. Then

$$|z-L_z|\leqslant 2\pi sn^{-4}:=Kn^{-4}.$$

If $r=1-(1/2n)$ and $|w|\leqslant r$, the estimate on the derivative $a'(w)$ treated as a Taylor coefficient gives $|a'(w)|\leqslant K'n^2\|a\|_r$. To see this, use (6) for the L^2-norm of a over $\{z, |z-w|<r-|w|\}$ when $r-|w|\geqslant\frac{1}{2}n$. Hence

$$|a(z)-a(L_z)|\leqslant KK'n^{2-4}\|a\|_r:=K''n^{-2}\|a\|_r. \tag{7}$$

Let μ be the normalized Lebesgue measure on the unit circle. Then it is easy to see that

$$\int_{C_n}|a|^2\,d\mu_n=\int_{|z|=1}|a(L_{sz})|^2\,d\mu.$$

Denoting the square root of the last integral by J and using (7) we get

$$J\leqslant\left(\int_{C_n}|a|^2\,d\mu_n\right)^{1/2}+K''n^{-2}\|a\|_r.$$

Now the comparison between the Bergman and Hardy norms yields $\|a\|_s\leqslant\pi^{1/2}sJ\leqslant\pi J$, which proves the estimate (5).

To prove the Lemma let us develop f and f_n in power series in the variable w, say $f(z, w)=\sum a_k(z)w^k$ for $(z, w)\in G$ and $f_n(z, w)=\sum a_{k,n}(z)w^k$ if $\{z\}\times(w\mathbb{D})\subset Q_n$, (e.g. if $|z|\leqslant r$, $|w|\leqslant\frac{1}{4}$ or if $z\in C_n$ and $|w|\leqslant s$, where r, s are as above).

Obviously, the functions $a_{k,n}$ are analytic on $|z|<1-(1/2n)$ and converging to a_k uniformly on compact subsets of D since $f_n\to f$ along with all derivatives.

From (6) we obtain

$$|a_{k,n}(z)|^2\leqslant M_k^2\int_{|w|<1/4}|f_n(z, w)|^2\,dv'(w).$$

Integration over $r\mathbb{D}$ and the definition of v_n gives the estimate $\|a_{k,n}\|_r\leqslant M_kM$, which is independent of n. Indeed, $r\mathbb{D}\times\frac{1}{4}\mathbb{D}\subset Q_n$ and $dv'(z)dv'(w)=dv(z, w)$. Similarly, the second term contributing to v_n and (6) with s in place of t gives $\int_{C_n}|a_{k,n}|^2\,d\mu_n\leqslant s^{-2k}(M'M)^2$. Now the application of (5) implies that $\|a_{k,n}\|_s\leqslant s^{-k}M''+Cn^{-2}M_kM$. Note that $\|a\|_t$ is increasing and continuous with respect to t, so we may fix $s' < 1$ and, since $s = 1 - 1/n \to 1$, letting $n \to \infty$ we shall obtain $\|a_k\|_{s'} \leq M''$. Because s' may be arbitrarily close to 1, also $\|a_k\|_1 \leq M''$. This, independent of k estimate, guarantees the convergence of $\sum a_k(z)w^k$ for all $(z, w) \in \mathbb{D}^2$, proving the Lemma.

Note added in September 1986. The example obtained in this work can be used, as noted by J. Janas, to show the following: the description of the spectrum of inductive limits given in [9] for self-adjoint operators cannot be extended to inductive limits of subnormal pairs.

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