AN EFFECTIVE UPPER BOUND FOR ANTI-CANONICAL VOLUMES OF SINGULAR FANO THREEFOLDS

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Abstract   For a real number $0 < \epsilon < 1/3$, we show that the anti-canonical volume of an ϵ-klt Fano threefold is at most $3{,}200/\epsilon^4$, and the order $O(1/\epsilon^4)$ is sharp.

Contents

1. Introduction

Throughout this paper, we work over the field of complex numbers $\mathbb{C}$.

A normal projective variety X is a *Fano* variety if $-K_X$ is ample. According to the minimal model program, Fano varieties form a fundamental class in the birational classification of algebraic varieties. One recent breakthrough in birational geometry is the proof of the Borisov–Alexeev–Borisov conjecture by Birkar [3, 4], which states that for a fixed positive integer d and a positive real number ϵ, the set of d-dimensional Fano varieties with ϵ-klt singularities forms a bounded family. During the proof, one important step is to establish the upper bound for the anti-canonical volume $(-K_X)^d$ for an ϵ-klt Fano variety X of dimension d ([3, Theorem 1.6]).

Motivated by the classification theory of threefolds, we mainly focus on the anti-canonical volume $(-K_X)^3$ for an ϵ-klt Fano threefold X. In this direction, Lai [13] gave an upper bound for those X which are $\mathbb{Q}$-factorial and of Picard rank 1, which is over $O((\frac{4}{\epsilon})^{384/\epsilon^5})$; later, the first author [10] showed the existence of a nonexplicit upper bound; recently, Birkar [5] gave the first explicit upper bound, which is about $O(\frac{2^{1536/\epsilon^3}}{\epsilon^9})$.

The main goal of this paper is to provide a reasonably small explicit upper bound with a sharp order, for the anti-canonical volume of an ϵ-klt Fano threefold. Here, we state the result for a larger class of varieties containing ϵ-klt Fano threefolds. Recall that a normal projective variety X is said to be *of ϵ-Fano type* if there exists an effective $\mathbb{R}$-divisor B, such that (X,B) is ϵ-klt and $-(K_X + B)$ is ample.

Theorem 1.1. *Fix a real number $0 < \epsilon < \frac{1}{3}$. Let X be a threefold of ϵ-Fano type. Then*

$$\mathrm{Vol}(X, -K_X) < \frac{3200}{\epsilon^4}.$$

The following example shows that the order $O(\frac{1}{\epsilon^4})$ in Theorem 1.1 is sharp.

Example 1.2. Ambro [2, Example 6.3] showed that for each positive integer q, there exists a projective toric threefold pair (X,B), such that

- (X,B) is $\frac{1}{q}$-lc,
- $-(K_X + B)$ is ample (in fact, $-K_X$ is ample as $\rho(X) = 1$), and
- $(-K_X)^3 > (-(K_X + B))^3 = \frac{u_{4,q}}{q^4} = O(q^4)$, where $u_{4,q} = (q^2 + q + 1)^2(q^2 + q)^2 + (q^2 + q + 1)(q^2 + q)$.

Remark 1.3. The ideas of this paper originate from [9, 10], which we briefly explain in the following. Given a threefold X of ϵ-Fano type, in order to give an upper bound of the anti-canonical volume of X, we may reduce to the case that X admits a Mori fiber structure $X \to T$. Then we can split the discussion according to $\dim T \in \{0,1,2\}$. The case $\dim T = 0$ was solved in [17] (or [13]) and other two cases were solved in [10]. The main obstructions of getting a reasonably small explicit upper bound in [10] are the following two issues:

(1) when $\dim T = 1$, we reduce the upper bound problem to finding a lower bound of certain log canonical thresholds on surfaces, called $\mu(2,\epsilon)$, but the lower bound in [10] is extremely small so that the resulting upper bound is extremely large;

(2) when $\dim T = 2$, we reduce the upper bound problem to the boundedness of such surfaces T (more precisely, the existence of very ample divisors on T with bounded self-intersection numbers), but the geometry of those T is quite complicated, which makes the upper bound nonexplicit; in fact, even if we can classify those surfaces T, then the resulting upper bound will be explicitly computable but still extremely large.

This paper is devoted to solving these two issues, and we put two main ingredients into the recipe.

The first one is a new reduction originated from [11] (Proposition 4.1), which shows that we may further assume that X admits a better fibration structure $X \to S$ so that if $\dim S = 2$, then there exists a free divisor on S with small self-intersection number. This solves the second issue.

The second one is a more detailed estimate on the lower bound $\mu(2, \epsilon)$ (Theorem 3.1), which solves the first issue. We significantly improve the lower bound from about $O(2^{-64/\epsilon^3})$ in [10] to $O(\epsilon^3)$ (Theorem 3.1).

Remark 1.4. We shall also compare our result with [5]. The method used in [5, Theorem 1.2] is a slight modification of [3, Theorem 1.6], which is different from our method (but it is a more general strategy that works in any dimension). One thing we share in common is that, [5, Theorem 1.2] also reduces the problem to finding some kind of lower bound of log canonical thresholds on surfaces ([5, Lemma 2.2]). Here, note that the constant $\mu(2, \epsilon)$ in [5, Lemma 2.2] is $\frac{1}{\mu(2,\epsilon)}$ in our terminology. If we replace [5, Lemma 2.2] by Theorem 3.1 in the proof of [5, Theorem 1.2], then we get an explicit upper bound about $O(\frac{1}{\epsilon^{12}})$.

2. Preliminaries

We adopt standard notation and definitions in [12] and will freely use them. We use $\sim_{\mathbb{Q}}, \sim_{\mathbb{R}}, \equiv$ to denote $\mathbb{Q}$-linear equivalence, $\mathbb{R}$-linear equivalence, and numerical equivalence respectively.

2.1. Singularities of pairs

Definition 2.1. A *pair* (X, B) consists of a normal variety X and an effective $\mathbb{R}$-divisor B on X, such that $K_X + B$ is $\mathbb{R}$-Cartier.

Definition 2.2. Let (X, B) be a pair. Let $f : Y \to X$ be a log resolution of (X, B), write

$$K_Y = f^*(K_X + B) + \sum a_i E_i,$$

where E_i are distinct prime divisors on Y satisfying $f_*(\sum a_i E_i) = -B$. The number $a_i + 1$ is called the *log discrepancy* of E_i with respect to (X, B), and is denoted by $a(E_i, X, B)$. The pair (X, B) is called

(1) *Kawamata log terminal* (*klt*, for short) if $a_i + 1 > 0$ for all i;

(2) *log canonical* (*lc*, for short) if $a_i + 1 \geq 0$ for all i;

(3) ϵ-*klt* if $a_i + 1 > \epsilon$ for all i and for some $0 < \epsilon < 1$;

(4) ϵ-*lc* if $a_i + 1 \geq \epsilon$ for all i and for some $0 < \epsilon < 1$;

(5) *terminal* if $a_i > 0$ for all f-exceptional divisors E_i and for all f.

Usually, we write X instead of $(X, 0)$ in the case when $B = 0$.

2.2. Varieties of Fano type

Definition 2.3. A variety X is said to be *of ϵ-Fano type* if X is projective and there exists an effective $\mathbb{R}$-divisor B, such that $-(K_X + B)$ is ample and (X, B) is ϵ-klt for some $0 < \epsilon < 1$.

2.3. Volumes

Definition 2.4. Let X be a normal projective variety of dimension n, and let D be a Cartier divisor on X. The *volume* of D is defined by

$$\mathrm{Vol}(X, D) = \limsup_{m \to \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^n/n!}.$$

Moreover, by the homogeneous property of volumes, the definition can be extended to $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors. Note that if D is a nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor, then $\mathrm{Vol}(X, D) = D^n$. We refer to [14, Section 2.2.C] for more details and properties on volumes of divisors.

3. A lower bound of log canonical thresholds on surfaces

The main goal of this section is to prove the following theorem on certain log canonical thresholds on surfaces. This is the main ingredient of this paper and the most technical part. Note that in this paper, by a *curve*, we always mean an irreducible one.

Theorem 3.1. *Fix $0 < \epsilon < \frac{1}{3}$. Let S be a smooth projective surface. Suppose that there exists a real number $0 < t < 1$ and effective $\mathbb{R}$-divisors B, D on S, such that*

- *(S, B) is ϵ-lc;*
- *$(S, (1-t)B + tD)$ is not klt;*
- *$B \sim_{\mathbb{R}} D \sim_{\mathbb{R}} -K_S$ and $-K_S$ is big.*

Then $t > \frac{3\epsilon^3}{400}$.

As an immediate corollary, we confirm the generalized Ambro's conjecture ([10, Conjecture 2.7]) in dimension 2 with a greatly improved lower bound.

Corollary 3.2 (cf. [10, Theorem 2.8]). *The generalized Ambro's conjecture ([10, Conjecture 2.7]) holds in dimension 2 with $\mu(2, \epsilon) > \frac{3\epsilon^3}{400}$.*

The reduction from Corollary 3.2 to Theorem 3.1 is standard (see [10, Section 5, Page 1583]).

Remark 3.3. While the constant term might be improved slightly, the order ϵ^3 in Theorem 3.1 is sharp. In fact, Ambro [2, Theorem 1.1, Example 6.3] showed that for each positive integer q, there exists a projective toric surface pair (X, Δ), such that

- (X, Δ) is $\frac{1}{q}$-lc,
- $-(K_X + \Delta)$ is ample, and
- there exists an effective $\mathbb{Q}$-divisor $H \sim_{\mathbb{Q}} -(K_X + \Delta)$, such that $(X, \Delta + tH)$ is not klt for $t = \frac{1}{(q+1)(q^2+q+1)} = O(\frac{1}{q^3})$.

We can modify this example to satisfy assumptions in Theorem 3.1. Take $A \sim_{\mathbb{Q}} -(K_X + \Delta)$ to be a sufficiently general ample effective $\mathbb{Q}$-divisor, such that $(X, \Delta + A)$ is still $\frac{1}{q}$-lc. Take $\pi : S \to X$ to be the minimal resolution of X, then $-K_S$ is big as $-K_X$ is big. We may write

$$K_S + B = \pi^*(K_X + \Delta + A) \sim_{\mathbb{Q}} 0$$

for some effective $\mathbb{Q}$-divisor $B \geq \pi^* A$. Then (S, B) is $\frac{1}{q}$-lc. In this case, $D := B - \pi^* A + \pi^* H$ is an effective $\mathbb{Q}$-divisor, such that $D \sim_{\mathbb{Q}} B \sim_{\mathbb{Q}} -K_S$ and the pair $(S, (1-t)B + tD)$ is not klt as

$$K_S + (1-t)B + tD = \pi^*(K_X + \Delta + tH + (1-t)A).$$

3.1. Weighted dual graphs

In this subsection, we recall basic knowledge of weighted dual graphs of resolutions of surface singularities from [1] or [12, 4.1].

Let Y be a normal surface, and let $\pi : Y' \to Y$ be a resolution with π-exceptional curves $\{E_i\}_i$. The *weighted dual graph* Γ of π is defined as the following: each vertex v_i of Γ corresponds to a π-exceptional curve E_i, and it has a positive weight $-E_i^2$; two vertices v_i and v_j are connected by an edge of weight $m = (E_i \cdot E_j)$ if $(E_i \cdot E_j) \neq 0$. If Y is klt and has a unique singular point, then Γ is a tree with simple edges and all π-exceptional curves are smooth rational curves by [1, Lemma 3.2.7] or [12, Theorem 4.7]. In this case, denote by $\overline{v_i v_j}$ the path from v_i to v_j, that is, the unique shortest chain in Γ joining v_i and v_j.

For any subgraph $\Gamma' \subset \Gamma$, define $\Delta(\Gamma')$ to be the absolute value of the determinant of the matrix $[(E_i \cdot E_j)]$, made up by vertices in Γ'. Here, $\Delta(\emptyset) = 1$ by default.

We will often use the following lemma to compute log discrepancies and multiplicities of exceptional divisors.

Lemma 3.4 (cf. [1, (3.1.10)]). *Let $\pi : Y' \to Y$ be a resolution of a klt surface singularity $P \in Y$. Suppose that the set of π-exceptional curves is $\{E_1, E_2, \ldots, E_n\}$. Denote by Γ the weighted dual graph of π.*

(1) *Then for each $1 \leq k \leq n$,*

$$a(E_k, Y, 0) = \frac{\sum_{j=1}^n (2 - (\sum_{i \neq j} E_i \cdot E_j)) \cdot \Delta(\Gamma \setminus \overline{v_k v_j})}{\Delta(\Gamma)}.$$

(2) *If C is an irreducible curve on Y, then for each $1 \le k \le n$,*

$$\mathrm{mult}_{E_k} \pi^* C = \frac{\sum_{j=1}^n (\pi_*^{-1} C \cdot E_j) \cdot \Delta(\Gamma \setminus \overline{v_k v_j})}{\Delta(\Gamma)}.$$

Proof. (1) is just [1, (3.1.10)] and (2) can be deduced in the same way by applying [1, Lemma 3.1.9] and the Cramer's rule. □

The following lemma will be used to deal with the weighted dual graph of the minimal resolution of a cyclic quotient singularity.

Lemma 3.5. *Let Γ be a chain with vertices $v_1, \ldots, v_n$ ordering in the natural sense that v_i is connected to v_{i+1} by an edge for $1 \le i \le n-1$. Suppose that for each $1 \le i \le n$, the weight of v_i is m_i with $m_i \ge 2$. Then the following assertions hold:*

(1) $\Delta(\Gamma) = m_1 \cdot \Delta(\Gamma \setminus \{v_1\}) - \Delta(\Gamma \setminus \overline{v_1 v_2})$;
(2) $\Delta(\Gamma \setminus \overline{v_1 v_k}) = m_{k+1} \cdot \Delta(\Gamma \setminus \overline{v_1 v_{k+1}}) - \Delta(\Gamma \setminus \overline{v_1 v_{k+2}})$ *for* $1 \le k \le n-2$;
(3) $\Delta(\Gamma) > \Delta(\Gamma \setminus \{v_1\}) > \Delta(\Gamma \setminus \overline{v_1 v_2}) > \cdots > \Delta(\Gamma \setminus \overline{v_1 v_n}) = 1$;
(4) $\Delta(\Gamma) \ge n+1$ *and* $\Delta(\Gamma \setminus \overline{v_1 v_k}) \ge n-k+1$ *for* $1 \le k \le n$; *moreover, the equalities hold if all $m_i = 2$;*
(5) *if* $\Delta(\Gamma) = \Delta(\Gamma \setminus \{v_1\}) + 1$, *then* $\Delta(\Gamma) = n+1$;
(6) *if* $m_{i_0} \ge 3$ *for some* $1 \le i_0 \le n$, *then* $\Delta(\Gamma) > (i_0 + 1)\Delta(\Gamma \setminus \overline{v_1 v_{i_0}})$.

Proof. Assertions (1) and (2) can be calculated easily from computing determinants. By Assertions (1)(2) and the fact that $m_i \ge 2$, we have

$$\Delta(\Gamma) \ge 2\Delta(\Gamma \setminus \{v_1\}) - \Delta(\Gamma \setminus \overline{v_1 v_2}) \tag{3.1}$$

and

$$\Delta(\Gamma \setminus \overline{v_1 v_k}) \ge 2\Delta(\Gamma \setminus \overline{v_1 v_{k+1}}) - \Delta(\Gamma \setminus \overline{v_1 v_{k+2}}) \tag{3.2}$$

for $1 \le k \le n-2$. So Assertion (3) follows inductively from the fact that

$$\Delta(\Gamma \setminus \overline{v_1 v_{n-1}}) - \Delta(\Gamma \setminus \overline{v_1 v_n}) = m_n - 1 > 0.$$

Assertion (4) follows from Assertion (3) and a direct computation.

For Assertion (5), the assumption combining with Eqs. (3.1) and (3.2) implies that

$$\Delta(\Gamma \setminus \overline{v_1 v_{k+1}}) - \Delta(\Gamma \setminus \overline{v_1 v_{k+2}}) = 1$$

for $1 \le k \le n-2$. Moreover, Eqs. (3.1) and (3.2) become equalities, and hence, all $m_i = 2$. So $\Delta(\Gamma) = n+1$ by Assertion (4).

By applying Eqs. (3.1) and (3.2) inductively, one can see that

$$\Delta(\Gamma) \ge j\Delta(\Gamma \setminus \overline{v_1 v_{j-1}}) - (j-1)\Delta(\Gamma \setminus \overline{v_1 v_j})$$

for $1 \le j \le n$. Here, we set $\Gamma \setminus \overline{v_1 v_0} = \Gamma$. Then, by Assertions (1)(2)(3), for $1 \le j \le n$, we have

$$\Delta(\Gamma) \geq j(m_j \Delta(\Gamma \setminus \overline{v_1 v_j}) - \Delta(\Gamma \setminus \overline{v_1 v_{j+1}})) - (j-1)\Delta(\Gamma \setminus \overline{v_1 v_j})$$
$$= (jm_j - j + 1)\Delta(\Gamma \setminus \overline{v_1 v_j}) - j\Delta(\Gamma \setminus \overline{v_1 v_{j+1}})$$
$$> (jm_j - 2j + 1)\Delta(\Gamma \setminus \overline{v_1 v_j}).$$

Here, we set $\Delta(\Gamma \setminus \overline{v_1 v_{n+1}}) = 0$. In particular, if $m_{i_0} \geq 3$, then $\Delta(\Gamma) > (i_0 + 1)$ $\Delta(\Gamma \setminus \overline{v_1 v_{i_0}})$. $\qquad\square$

3.2. Geometric structure of δ-lc surface pairs

Lemma 3.6. *Fix a real number $0 < \delta < \frac{1}{6}$ and a positive integer N. Let Y be a normal surface, and let C be an irreducible curve on Y, such that $(Y,(1-\delta)C)$ is δ-lc. Let $\pi : Y' \to Y$ be the minimal resolution of Y. Suppose that $\rho(Y'/Y) \leq N$. Then $\mathrm{mult}_E \pi^* C > \frac{\delta}{N+1}$ for any prime divisor E on Y', such that E is π-exceptional and $\pi(E) \in C$.*

Proof. By shrinking Y if necessary, we may assume that $P \in Y$ is the only singular point on Y and $P \in C$. Denote by Γ the weighted dual graph of π. For any π-exceptional prime divisor E, clearly $\mathrm{mult}_E \pi^* C$ is a positive rational number and its denominator divides $\Delta(\Gamma)$ by Lemma 3.4. So it suffices to show that $\Delta(\Gamma) < \frac{N+1}{\delta}$.

By [16, Corollary 6.0.9], (Y,C) is lc. Then the weighted dual graph Γ of the minimal resolution $\pi : Y' \to Y$ are classified into three cases, as in [12, Theorem 4.15]. We split the discussion into these three cases. Denote by $m = \rho(Y'/Y)$ the number of π-exceptional curves on Y'.

As $(Y,(1-\delta)C)$ is δ-lc, $a(E,Y,(1-\delta)C) \geq \delta$ for any prime divisor E over Y. We will apply this fact to some specially chosen E.

Case (1): For the case [12, Theorem 4.15(1)], Γ is a chain with vertices $v_1, \cdots, v_m$ corresponding to π-exceptional curves $E_1, \cdots, E_m$, such that $\pi_*^{-1} C$ intersects E_1 and E_m.

If $m \geq 2$, by Lemma 3.4, we have

$$a(E_1, Y, 0) = \mathrm{mult}_{E_1} \pi^* C = \frac{\Delta(\Gamma \setminus \{v_1\}) + \Delta(\emptyset)}{\Delta(\Gamma)}.$$

Then

$$a(E_1, Y, (1-\delta)C) = a(E_1, Y, 0) - (1-\delta) \mathrm{mult}_{E_1} \pi^* C$$
$$= \delta \cdot \frac{\Delta(\Gamma \setminus \{v_1\}) + 1}{\Delta(\Gamma)}.$$

Therefore, $a(E_1, Y, (1-\delta)C) \geq \delta$ implies that $\Delta(\Gamma \setminus \{v_1\}) + 1 \geq \Delta(\Gamma)$. So, by Lemma 3.5(5), $\Delta(\Gamma) = m + 1 \leq N + 1$.

If $m = 1$, then a similar computation by Lemma 3.4 shows that

$$a(E_1, Y, (1-\delta)C) = \delta \cdot \frac{2\Delta(\Gamma \setminus \{v_1\})}{\Delta(\Gamma)} = \frac{2\delta}{\Delta(\Gamma)}.$$

Therefore, $a(E_1, Y, (1-\delta)C) \geq \delta$ implies that $\Delta(\Gamma) \leq 2$.

Case (2): For the case [12, Theorem 4.15(2)], we have $m \geq 3$.

If $m \geq 4$, then Γ is a tree with only one fork. Let E_{f} be the π-exceptional curve corresponding to the fork vertex v_{f}. Then $\Gamma \setminus \{v_{\mathrm{f}}\} = \Gamma' \cup \{v_1\} \cup \{v_2\}$, where v_1, v_2 correspond to (-2)-curves intersecting E_{f} and Γ' is the chain corresponding to curves connecting E_{f} and $\pi_*^{-1}C$.

Then, by Lemma 3.4,

$$
\begin{aligned}
&a(E_{\mathrm{f}}, Y, 0)\\
&= \frac{\Delta(v_1) \cdot \Delta(v_2) - \Delta(\Gamma') \cdot \Delta(v_1) \cdot \Delta(v_2) + \Delta(\Gamma') \cdot \Delta(v_2) + \Delta(\Gamma') \cdot \Delta(v_1)}{\Delta(\Gamma)}\\
&= \frac{4}{\Delta(\Gamma)}
\end{aligned}
$$

and

$$
\mathrm{mult}_{E_{\mathrm{f}}} \pi^* C = \frac{\Delta(v_1) \cdot \Delta(v_2)}{\Delta(\Gamma)} = \frac{4}{\Delta(\Gamma)}.
$$

Then $a(E_{\mathrm{f}}, Y, (1-\delta)C) = \frac{4\delta}{\Delta(\Gamma)}$. Therefore, $a(E_{\mathrm{f}}, Y, (1-\delta)C) \geq \delta$ implies that $\Delta(\Gamma) \leq 4$.

If $m = 3$, then Γ is a chain consisting of three vertices v_1, v_2, v_3 corresponding to E_1, E_2, E_3, such that E_1 and E_3 are (-2)-curves and $\pi_*^{-1}C$ intersects E_2. Then a similar computation by Lemma 3.4 shows that $a(E_2, Y, (1-\delta)C) = \frac{4\delta}{\Delta(\Gamma)}$. Therefore, $a(E_2, Y, (1-\delta)C) \geq \delta$ implies that $\Delta(\Gamma) \leq 4$.

Case (3): For the case [12, Theorem 4.15(3)], Γ is a chain with vertices $v_1, \cdots, v_m$ corresponding to π-exceptional curves $E_1, \cdots, E_m$, such that $\pi_*^{-1}C$ intersects E_1. If all E_i are (-2)-curves for $1 \leq i \leq m$, then by Lemma 3.5(4), $\Delta(\Gamma) = m + 1 \leq N + 1$. If $-E_{i_0}^2 \geq 3$ for some $1 \leq i_0 \leq m$, take i_0 to be the minimal one, then by Lemmas 3.4 and 3.5(4),

$$
\begin{aligned}
a(E_{i_0}, Y, (1-\delta)C) &= \frac{\Delta(\Gamma \setminus \overline{v_{i_0} v_m})}{\Delta(\Gamma)} + \delta \cdot \frac{\Delta(\Gamma \setminus \overline{v_1 v_{i_0}})}{\Delta(\Gamma)}\\
&= \frac{i_0}{\Delta(\Gamma)} + \delta \cdot \frac{\Delta(\Gamma \setminus \overline{v_1 v_{i_0}})}{\Delta(\Gamma)}\\
&< \frac{i_0}{\Delta(\Gamma)} + \frac{\delta}{i_0 + 1}.
\end{aligned}
$$

Here, the last inequality is by Lemma 3.5(6). Therefore, $a(E_{i_0}, Y, (1-\delta)C) \geq \delta$ implies that $\Delta(\Gamma) < \frac{i_0 + 1}{\delta} \leq \frac{N+1}{\delta}$. $\square$

Lemma 3.7 (cf. [10, Claim 2]). *Fix $0 < \delta < 1$. Let X be a smooth projective surface, and let (X, B) be a δ-lc pair, such that $K_X + B \equiv 0$. Then $C^2 \geq -\frac{2}{\delta}$ for any irreducible curve C on X.*

Proof. We may assume that $C^2 < 0$. Then, by the genus formula,

$$
\begin{aligned}
-2 \leq 2p_a(C) - 2 &= (K_X + C) \cdot C\\
&= \delta C^2 + (K_X + (1-\delta)C) \cdot C\\
&\leq \delta C^2 + (K_X + B) \cdot C = \delta C^2.
\end{aligned}
$$

$\square$

Lemma 3.8. *Let X be a normal projective surface, such that K_X is $\mathbb{Q}$-Cartier and not pseudo-effective. Then either $X \simeq \mathbb{P}^2$ or X is covered by a family of rational curves C, such that $(-K_X \cdot C) \leq 2$.*

Proof. Take $\pi : X' \to X$ to be the minimal resolution of X, then $K_{X'} + G = \pi^* K_X$, where G is an effective $\mathbb{Q}$-divisor. Then $K_{X'}$ is not pseudo-effective as K_X is not pseudo-effective. Suppose that $X \not\simeq \mathbb{P}^2$, then clearly $X' \not\simeq \mathbb{P}^2$. By the standard minimal model program, there exists a morphism $X' \to T$ whose general fibers are $\mathbb{P}^1$. Therefore, X' is covered by a family of rational curves C', such that $(-K_{X'} \cdot C') = 2$. So $(-K_X \cdot \pi(C')) \leq (-K_{X'} \cdot C') = 2$. □

3.3. Proof of Theorem 3.1

In this subsection, we give the proof of Theorem 3.1.

By [8, Lemma 3.1], there is a birational morphism $g : S \to S'$, where S' is $\mathbb{P}^2$ or the n-th Hirzebruch surface $\mathbb{F}_n$ with $n \leq \frac{2}{\epsilon}$. Since $B \sim_{\mathbb{R}} D \sim_{\mathbb{R}} -K_S$, we may write

$$K_S + B = g^*(K_{S'} + g_* B);$$
$$K_S + (1-t)B + tD = g^*(K_{S'} + (1-t)g_* B + tg_* D).$$

Hence, $(S', g_* B)$ is ϵ-lc and $(S', (1-t)g_* B + tg_* D)$ is not klt. By replacing the triple (S, B, D) with $(S', g_* B, g_* D)$, we may assume that S is $\mathbb{P}^2$ or $\mathbb{F}_n$ for some $n \leq \frac{2}{\epsilon}$.

Fix a positive real number $0 < \delta < \epsilon$, such that $\delta < \frac{1}{6}$. Take

$$t_0 = \max\{s \in \mathbb{R} \mid (S, (1-s)B + sD) \text{ is } \delta\text{-lc}\}.$$

Then clearly $0 < t_0 < t$. In the following, we will show that

$$t_0 > \frac{\delta^2(\epsilon - \delta)}{16 + 4\delta + \delta^2(\epsilon - 1)}. \tag{3.3}$$

In particular, we can take $\delta = \frac{\epsilon}{2}$, then Eq. (3.3) implies that $t > t_0 > \frac{3\epsilon^3}{400}$.

By the definition of t_0, there exists a prime divisor E over S, such that

$$a(E, S, (1-t_0)B + t_0 D) = \delta.$$

If E is a prime divisor on S, then as (S, B) is ϵ-lc, we have

$$\mathrm{mult}_E B \leq 1 - \epsilon;$$
$$\mathrm{mult}_E((1-t_0)B + t_0 D) = 1 - \delta.$$

So

$$t_0 \geq \frac{\epsilon - \delta}{\mathrm{mult}_E D - 1 + \epsilon} \geq \frac{\epsilon(\epsilon - \delta)}{2 + 3\epsilon + \epsilon^2},$$

which implies Eq. (3.3). Here, we used the fact that

$$\mathrm{mult}_E D \leq \begin{cases} 3 & \text{if } S = \mathbb{P}^2; \\ n + 4 \leq \frac{2}{\epsilon} + 4 & \text{if } S = \mathbb{F}_n, \end{cases}$$

by applying [10, Lemma 3.3] to a general point on E.

So, from now on, we may assume that E is exceptional over S. By [6, Corollary 1.4.3], there exists a projective birational morphism $f : Y \to S$, such that E is the unique π-exceptional divisor on Y. We have

$$K_Y + (1-t_0)B_Y + t_0 D_Y + (1-\delta)E = f^*(K_S + (1-t_0)B + t_0 D). \qquad (3.4)$$

Here, B_Y and D_Y are strict transforms of B and D on Y. Write

$$K_Y + B_Y + bE = f^*(K_S + B) \equiv 0,$$
$$K_Y + D_Y + dE = f^*(K_S + D) \equiv 0.$$

Then $(1-t_0)b + t_0 d = 1 - \delta$. Since $b \leq 1 - \epsilon$ as (S,B) is ϵ-lc, we have

$$t_0 \geq \frac{\epsilon - \delta}{d + \epsilon - 1}. \qquad (3.5)$$

So, in order to bound t_0 from below, we need to bound d from above.

By Eq. (3.4), we know that $(Y, (1-t_0)B_Y + t_0 D_Y + (1-\delta)E)$ is δ-lc, and

$$K_Y + (1-t_0)B_Y + t_0 D_Y + (1-\delta)E \equiv 0.$$

We can run a $(K_Y + (1-t_0)B_Y + t_0 D_Y)$-MMP (which is also a $(-E)$-MMP) on Y to get a Mori fiber space $Y' \to Z$, such that E' is ample over Z, where E' is the strict transform of E on Y'. Here, $(Y', (1-\delta)E')$ is again δ-lc by the negativity lemma. We have $-K_{Y'} \equiv D_{Y'} + dE'$, where $D_{Y'}$ is the strict transform of D_Y on Y'.

If $\dim Z = 1$, then a general fiber of $Y' \to Z$ is a smooth rational curve. By restricting $-K_{Y'} \equiv D_{Y'} + dE'$ on a general fiber, we get $d \leq 2$.

If $\dim Z = 0$, then Y' is of Picard rank 1. In this case, $-K_{Y'} \equiv eE'$ for some $e \geq d$. If $Y' \simeq \mathbb{P}^2$, then clearly $d \leq e \leq 3$. So, we may assume that $Y' \not\simeq \mathbb{P}^2$. By Lemma 3.8, there is a general rational curve C, such that $(-K_{Y'} \cdot C) \leq 2$. Take $\pi' : Y'_{\min} \to Y'$ to be the minimal resolution of Y', and take $Y_{\min}$ to be the minimal resolution of Y. Then the morphism $Y_{\min} \to Y'$ factors through $Y'_{\min}$.

Claim 3.9. *We have* $\rho(Y_{\min}/Y) \leq \frac{8}{\delta} - 1$.

We grant Claim 3.9 for this moment and continue the proof of Theorem 3.1. The proof of Claim 3.9 will be provided later. By Claim 3.9,

$$\rho(Y'_{\min}/Y') \leq \rho(Y_{\min}/Y') = \rho(Y_{\min}/Y) + \rho(Y) - 1$$
$$\leq \rho(Y_{\min}/Y) + 2 \leq \frac{8}{\delta} + 1.$$

Here, we used the fact that $\rho(Y) = \rho(S) + 1 \leq 3$. Recall that $(Y', (1-\delta)E')$ is δ-lc and $\delta < \frac{1}{6}$, then, by Lemma 3.6, $\pi'^* E'$ is an effective $\mathbb{Q}$-divisor with all coefficients larger than $\frac{\delta^2}{8+2\delta}$. As C is general, by the projection formula, $(E' \cdot C) > \frac{\delta^2}{8+2\delta}$, which implies that

$$\frac{e\delta^2}{8+2\delta} < (eE' \cdot C) = (-K_{Y'} \cdot C) \leq 2.$$

Hence, $d \leq e \leq \frac{16+4\delta}{\delta^2}$.

In summary, we always have $d \leq \frac{16+4\delta}{\delta^2}$. Therefore, by Eq. (3.5),

$$t_0 \geq \frac{\epsilon - \delta}{d + \epsilon - 1} \geq \frac{\delta^2(\epsilon - \delta)}{16 + 4\delta + \delta^2(\epsilon - 1)}.$$

Proof of Claim 3.9. Denote by $\pi : Y_{\min} \to Y$ the minimal resolution of Y. Denote by E_0 the strict transform of E on $Y_{\min}$. Denote by $f_0 = f \circ \pi : Y_{\min} \to S$ the induced morphism.

Denote $P = f(E) \in S$. Then $Y \setminus E \simeq S \setminus \{P\}$ is smooth. Therefore, $Y_{\min}$ and Y are isomorphic over $Y \setminus E$ and

$$\operatorname{Exc}(f_0) = f_0^{-1}(P) = \operatorname{Exc}(\pi) \cup E_0.$$

Note that $f_0 : Y_{\min} \to S$ can be decomposed into successive blow-ups along smooth points and $\operatorname{Exc}(\pi)$ does not contain any (-1)-curves, so E_0 is the unique (-1)-curve in $\operatorname{Exc}(f_0)$. In other words, if we denote the last blow-up by $Y_{\min} \to S_1$, then E_0 is the exceptional divisor over S_1.

Denote by Γ the weighted dual graph of f_0, and denote by v_0 the vertex corresponding to E_0. Then Γ is a tree and $\Gamma \setminus \{v_0\}$ is the weighted dual graph of π. Since the weighted dual graph of $S_1 \to S$ is also a tree, $\Gamma \setminus \{v_0\}$ has at most two connected components (which implies that Y has at most two singular points).

Claim 3.10. Γ *is a chain.*

Proof of Claim 3.10. Take Γ' to be a connected component of $\Gamma \setminus \{v_0\}$, then it corresponds to exceptional curves over a singular point on Y. Recall that $(Y, (1 - \delta)E)$ is δ-lc by Eq. (3.4). As $\delta < \frac{1}{6}$, (Y, E) is lc by [16, Corollary 6.0.9]. Then the weighted dual graph Γ' and its relation with v_0 are classified into three cases, as in [12, Theorem 4.15]. We shall rule out [12, Theorem 4.15(1)(2)]. In the case of [12, Theorem 4.15(1)], Γ contains a loop, which is absurd; in the case of [12, Theorem 4.15(2)], Γ contains a fork with two (-2)-curves on two tails of the fork, so, by contracting (-1)-curves in the graph successively, we will reach some model with two (-1)-curves over $P \in S$, which is also absurd. Hence, we conclude that Γ' is a chain connecting to v_0 by one edge at one end. Therefore, Γ is a chain. $\qquad\square$

If $\Gamma \setminus \{v_0\}$ is empty, then clearly Y is smooth and $\rho(Y_{\min}/Y) = 0$. So, in the following, we split the discussion into two cases, depending on the number of connected components of $\Gamma \setminus \{v_0\}$.

Case (a). $\Gamma \setminus \{v_0\}$ has one connected component.

In this case, denote the π-exceptional curves by $E_1, \ldots, E_N$, such that E_0 intersects E_1, where $N = \rho(Y_{\min}/Y)$. Then all E_i are (-2)-curves for $1 \leq i \leq N$.

Suppose that $Y_{\min} \to S$ is decomposed into successive blow-ups at smooth points as

$$Y_{\min} = Y_0 \to Y_1 \to \cdots \to Y_N \to Y_{N+1} = S,$$

then E_i is the strict transform of the exceptional divisor of $Y_i \to Y_{i+1}$. For each i, denote by G_i the strict transform of $(1 - t_0)B + t_0 D$ on Y_i and denote by $P_i \in Y_i$ the blow-up center on Y_i.

Write

$$K_{Y_{\min}} + G_0 + (1-\delta)E_0 + \sum_{i=1}^{N} b_i E_i = f_0^*(K_S + (1-t_0)B + t_0 D). \qquad (3.6)$$

Then, the coefficient of E_0 is computed from G_1 and E_1 by the formula

$$1 - \delta = \operatorname{mult}_{P_1} G_1 + b_1 - 1.$$

Since $b_1 \leq 1-\delta$ as $(S, (1-t_0)B + t_0 D)$ is δ-lc, we have $\operatorname{mult}_{P_1} G_1 \geq 1$, hence, $\operatorname{mult}_{P_i} G_i \geq \operatorname{mult}_{P_1} G_1 \geq 1$ for $1 \leq i \leq N+1$. Hence, the intersection number G_i^2 decreases at least by 1 after each blow-up for $1 \leq i \leq N+1$. Therefore,

$$N + 1 \leq G_{N+1}^2 - G_0^2 = K_S^2 - G_0^2. \qquad (3.7)$$

Since $Y_{\min}$ is the minimal resolution of Y, we have $b_i \geq 0$ for $1 \leq i \leq N$ in Eq. (3.6). In particular, $(Y_{\min}, G_0 + (1-\delta)E_0 + \sum_{i=1}^{N} b_i E_i)$ is a δ-lc pair, such that

$$K_{Y_{\min}} + G_0 + (1-\delta)E_0 + \sum_{i=1}^{N} b_i E_i \equiv 0.$$

Write $G_0 = \sum_k c_k C_k$, where C_k are distinct prime divisors, then $c_k \leq 1-\delta$. By Lemma 3.7, $C_k^2 \geq -\frac{2}{\delta}$.

If $S = \mathbb{F}_n$, then $\sum_k c_k \leq 4$ by [10, Lemma 3.3]. Hence

$$G_0^2 = \left(\sum_k c_k C_k\right)^2 \geq \left(\sum_k c_k^2\right) \cdot \left(-\frac{2}{\delta}\right) \geq \left(\sum_k c_k\right) \cdot (1-\delta) \cdot \left(-\frac{2}{\delta}\right)$$

$$\geq 4(1-\delta) \cdot \left(-\frac{2}{\delta}\right) = 8 - \frac{8}{\delta}.$$

Combining with Eq. (3.7), we have

$$N + 1 \leq K_S^2 - 8 + \frac{8}{\delta} = \frac{8}{\delta}.$$

If $S = \mathbb{P}^2$, then we have $\sum_k c_k \leq 3$, and by the same argument, we get

$$N + 1 \leq K_S^2 - G_0^2 \leq 9 - 6 + \frac{6}{\delta} \leq \frac{8}{\delta}.$$

Case (b). $\Gamma \setminus \{v_0\}$ has two connected components.

In this case, suppose that the two connected components are two chains Γ_1, Γ_2 consisting of vertices $v_1, \ldots, v_p$ and $u_1, \ldots, u_q$, corresponding to exceptional divisors $E_1, \ldots, E_p$ and $F_1, \ldots, F_q$, respectively, where E_1 and F_1 intersect E_0. Here, $p + q = \rho(Y_{\min}/Y)$.

Set $m_i = -E_i^2$ and $l_j = -F_j^2$ for $1 \leq i \leq p$ and $1 \leq j \leq q$. Recall that Γ is the weighted dual graph of a resolution over the smooth point $P \in S$, so after blowing down E_0, there is exactly one (-1)-curve among the strict transforms of E_1 and F_1. So without loss of generality, we may assume that $m_1 \geq 3$ and $l_1 = 2$. Again, by the fact that a contraction

of a (-1)-curve in the graph induces another unique (-1)-curve, we know that $l_j = 2$ for $1 \leq j \leq m_1 - 2 \leq q$, and $l_{m_1-1} \geq 3$ if $m_1 - 1 \leq q$.

Since $Y_{\min}$ is the minimal resolution of Y, we may write

$$K_{Y_{\min}} + G' = f_0^*(K_S + (1-t_0)B + t_0 D) \equiv 0,$$

where $(Y_{\min}, G')$ is a δ-lc pair. By Lemma 3.7, we conclude that $m_1 = -E_1^2 \leq \frac{2}{\delta}$.

By Lemma 3.4, we have

$$a(E_i, Y, (1-\delta)E) = \frac{\Delta(\Gamma_1 \setminus \overline{v_i v_p})}{\Delta(\Gamma_1)} + \delta \cdot \frac{\Delta(\Gamma_1 \setminus \overline{v_1 v_i})}{\Delta(\Gamma_1)}; \qquad (3.8)$$

$$a(F_j, Y, (1-\delta)E) = \frac{\Delta(\Gamma_2 \setminus \overline{u_j u_q})}{\Delta(\Gamma_2)} + \delta \cdot \frac{\Delta(\Gamma_2 \setminus \overline{u_1 u_j})}{\Delta(\Gamma_2)} \qquad (3.9)$$

for $1 \leq i \leq p$ and $1 \leq j \leq q$.

To finish the proof, we need to show the following claim.

Claim 3.11. *We have $p \leq \frac{1}{\delta}$ and $q \leq \frac{3}{\delta} - 2$.*

Proof of Claim 3.11. Recall that $(Y, (1-\delta)E)$ is δ-lc.

First, we show that $p \leq \frac{1}{\delta}$. We may assume that $p \geq 2$. By Eq. (3.8),

$$a(E_1, Y, (1-\delta)E) = \frac{1}{\Delta(\Gamma_1)} + \delta \cdot \frac{\Delta(\Gamma_1 \setminus \{v_1\})}{\Delta(\Gamma_1)}.$$

Therefore, $a(E_1, Y, (1-\delta)E) \geq \delta$ implies that

$$\frac{1}{\delta} \geq \Delta(\Gamma_1) - \Delta(\Gamma_1 \setminus \{v_1\}) > \Delta(\Gamma_1 \setminus \{v_1\}) \geq p.$$

Here, the second inequality is from Lemma 3.5(6) with $i_0 = 1$ and the third is from Lemma 3.5(4).

Next, we show that $q \leq \frac{3}{\delta} - 2$. If $q = m_1 - 2$, then clearly $q \leq \frac{2}{\delta} - 2$. If $q > m_1 - 2$, then take $j_0 = m_1 - 1$, we have $l_{j_0} \geq 3$. By Eq. (3.9),

$$
\begin{aligned}
a(F_{j_0}, Y, (1-\delta)E) &= \frac{\Delta(\Gamma_2 \setminus \overline{u_{j_0} u_q})}{\Delta(\Gamma_2)} + \delta \cdot \frac{\Delta(\Gamma_2 \setminus \overline{u_1 u_{j_0}})}{\Delta(\Gamma_2)} \\
&= \frac{j_0}{\Delta(\Gamma_2)} + \delta \cdot \frac{\Delta(\Gamma_2 \setminus \overline{u_1 u_{j_0}})}{\Delta(\Gamma_2)}.
\end{aligned}
$$

Therefore, $a(F_{j_0}, Y, (1-\delta)E) \geq \delta$ implies that

$$\frac{j_0}{\delta} \geq \Delta(\Gamma_2) - \Delta(\Gamma_2 \setminus \overline{u_1 u_{j_0}}) > j_0 \Delta(\Gamma_2 \setminus \overline{u_1 u_{j_0}}) \geq j_0(q - j_0 + 1).$$

Here, the second inequality is from Lemma 3.5(6) with $i_0 = j_0$ and the third is from Lemma 3.5(4). Recall that $m_1 = j_0 + 1$, so $q \leq \frac{1}{\delta} + m_1 - 2 \leq \frac{3}{\delta} - 2$. $\qquad \square$

In summary, $\rho(Y_{\min}/Y) = p + q \leq \frac{4}{\delta} - 2$. $\qquad \square$

4. Upper bound of anti-canonical volumes

In this section, we prove the main theorem.

4.1. A reduction step

The following proposition is a refinement of [10, Theorem 4.1] by the idea of [11, Proposition 4.1].

Proposition 4.1. *Fix $0 < \epsilon < 1$. Let X be a threefold of ϵ-Fano type. Then X is birational to a normal projective threefold W satisfying the following:*

(1) *W is $\mathbb{Q}$-factorial terminal;*

(2) *$\mathrm{Vol}(X, -K_X) \leq \mathrm{Vol}(W, -K_W)$;*

(3) *W is of ϵ-Fano type;*

(4) *there exists a projective morphism $f : W \to Z$ with connected fibers, such that one of the following conditions holds:*

    (a) *Z is a point and W is a $\mathbb{Q}$-factorial terminal Fano threefold with $\rho(W) = 1$;*

    (b) *$Z = \mathbb{P}^1$;*

    (c) *Z is a del Pezzo surface with at worst Du Val singularities and $\rho(Z) = 1$, and general fibers of f are $\mathbb{P}^1$.*

Proof. By [10, Theorem 4.1], X is birational to W with a Mori fiber structure (see [10, Definition 2.1]), in particular, W satisfies Properties (1)(2)(3). So here, we only need to explain how to get Property (4) by the proof of [11, Proposition 4.1(5)].

Denote by $W \to T$ the Mori fiber structure on W. Note that $\dim T \in \{0,1,2\}$. By [18, Theorem 1], W is rationally connected, which implies that T is also rationally connected.

If $\dim T = 0$, then take $Z = T$ and W is a $\mathbb{Q}$-factorial terminal Fano threefold with $\rho(W) = 1$. In this case, we get (a).

If $\dim T = 1$, then $T \simeq \mathbb{P}^1$. In this case, we get (b).

If $\dim T = 2$, then T is a rational surface as it is rationally connected, and T has at worst Du Val singularities by [15, Theorem 1.2.7]. We can run a K_T-MMP on T which ends up with a surface T', such that either

- T' is a del Pezzo surface with at worst Du Val singularities and $\rho(T') = 1$, or
- there is a morphism $T' \to \mathbb{P}^1$ with connected fibers.

In the former case, take $Z = T'$ and take $f : W \to Z$ to be the induced morphism $W \to T \to T'$, then general fibers of f are smooth rational curves as $-K_W$ is ample over T, then we get (c). In the latter case, take $Z = \mathbb{P}^1$ and take $f : W \to Z$ to be the induced morphism $W \to T \to T' \to \mathbb{P}^1$, then we get (b). □

4.2. Proof of Theorem 1.1

According to Proposition 4.1, we can split the discussion into three cases. We essentially follow the proof in [10].

Proposition 4.2 (cf. [10, Corollary 6.3]). *Suppose that $0 < \epsilon < \frac{1}{3}$. Keep the setting as in Proposition 4.1. Assume that case (b) holds. Then*

$$\mathrm{Vol}(W, -K_W) < \frac{3200}{\epsilon^4}.$$

Proof. By [10, Theorem 6.1],

$$\mathrm{Vol}(W, -K_W) \leq \frac{6M(2,\epsilon)}{\mu(2,\epsilon)},$$

where $M(2,\epsilon) \leq \frac{2}{\epsilon} + 4 + \frac{2}{3} < \frac{4}{\epsilon}$ by [8] or [10, Corollary 4.5] and $\mu(2,\epsilon) > \frac{3\epsilon^3}{400}$ by Corollary 3.2. $\qquad\square$

Proposition 4.3 (cf. [10, Theorem 6.6]). *Keep the setting as in Proposition 4.1. Assume that case (c) holds. Then*

$$\mathrm{Vol}(W, -K_W) \leq \frac{1152}{\epsilon^2}.$$

Proof. By the classification of del Pezzo surfaces with at worst Du Val singularities and Picard rank 1 (see [7, Theorem 8.3.2]), there exists a base point free linear system $\mathcal{H}$ on Z which defines a generically finite map, such that $\mathcal{H}^2 \leq 6$ (see also [11, Proposition 4.3]). Take a general element $H \in \mathcal{H}$ and denote $G = f^{-1}(H)$. Consider the self-intersection number $d = H^2$.

By [10, Lemma 6.5], we have $\mathrm{Vol}(G, -K_W|_G) \leq \frac{8(d+2)}{\epsilon}$. Then by [10, Theorem 6.6], $\mathrm{Vol}(W, -K_W) \leq \frac{144(d+2)}{\epsilon^2}$. This proves the proposition as $d \leq 6$.

Here, we remark that in [10, Lemma 6.5, Theorem 6.6], the assumptions are

- $W \to Z$ is a Mori fiber space and $\dim Z = 2$;
- $\mathcal{H}$ is very ample.

But those assumptions can be slightly weakened as in our setting without any other changes to the proofs:

- general fibers of $W \to Z$ are $\mathbb{P}^1$;
- $\mathcal{H}$ is base point free and defines a generically finite map. $\qquad\square$

Proof of Theorem 1.1. By Proposition 4.1, it suffices to bound $\mathrm{Vol}(W, -K_W)$. If case (a) holds, then $\mathrm{Vol}(W, -K_W) = (-K_W)^3 \leq 64$ by [17]. If case (b) or (c) holds, then the conclusion follows from Propositions 4.2 and 4.3. $\qquad\square$

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