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The Tangent Bundle of an Almost Complex Manifold

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Abstract. Motivated by deformation theory of holomorphic maps between almost complex manifolds we endow, in a natural way, the tangent bundle of an almost complex manifold with an almost complex structure. We describe various properties of this structure.

1 Introduction

Almost complex manifolds used to occupy a place in the backwater of geometry, their structure deemed too meager to support serious analysis of any depth or use. This situation has dramatically changed in 1985 with the appearance of Gromov's paper [3]. Since that time almost complex manifolds have become a—if not *the*—most powerful tool in symplectic geometry. This note has been motivated by deformation theory of holomorphic maps between almost complex manifolds.

The question we address here is whether one can endow, in a natural way, the tangent bundle TM of an almost complex manifold (M, J) with an almost complex structure. (For definitions, see [4], and Sections 3, 4.) When (M, J) is $\mathbb{C}^n$, TM is naturally identified with $\mathbb{C}^{2n}$, so the answer is yes. Since a general *complex* manifold (M, J) is locally isomorphic to $\mathbb{C}^n$, its tangent bundle is locally isomorphic to $\mathbb{C}^{2n}$, and it follows that even in this case the answer is yes—a fact well understood since the earliest days of the theory of complex manifolds. Now a general almost complex manifold (M, J) is isomorphic to $\mathbb{C}^n$ in first order neighborhoods of its points. This implies that TM is isomorphic to $\mathbb{C}^{2n}$ in zero order neighborhoods, a property apparently not sufficient to guarantee the existence of an almost complex structure on TM.

In spite of these heuristics it turns out that TM always admits a natural almost complex structure. This, then, is our main result, Theorem 3.2. The almost complex structure we construct on TM can be characterized by a simple property, see the end of Section 3. The proof of Theorem 3.2 is not difficult; one ingredient is understanding the second tangent bundle TTM directly in terms of M, which we achieve in Section 2. Section 3 proves the main theorem, and in Section 4 we entertain the question whether TM $\to M$ is an almost holomorphic vector bundle. When this notion is properly defined, the answer turns out to be yes.

The way all this is related to deformation theory is as follows. Suppose S is a Riemann surface and $f\colon S \to M$ a holomorphic map. Holomorphic deformations

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of f can be understood in terms of sections of the induced bundle $f^*\,\mathrm{TM}$ that are annihilated by a certain Cauchy-Riemann operator $\overline{\partial}_f$, see [3], [5]. Thus the induced bundles $f^*\,\mathrm{TM}$ are endowed with some extra structure coming from the operators $\overline{\partial}_f$, in addition to their complex vector bundle structure. The results of this note show that this extra structure is nothing but what is induced from the almost holomorphic vector bundle structure $\mathrm{TM} \to M$. Such a point of view allows one to introduce the notions of positivity/negativity of TM and related bundles, and offers a differential geometric approach to proving vanishing theorems for $\overline{\partial}_f$.

In the present note all manifolds, maps, *etc.*, are taken to be smooth, *i.e.*, C^∞. Our theorems have counterparts for finite differentiability: for example, if M is a C^r-manifold, $r = 2, 3, \ldots$, with an almost complex structure J of class C^{r-1} then the natural almost complex structure on TM is of class C^{r-2}. The proof would follow the same path as in the paper, with the exception that Proposition 3.1 with optimal regularity would be a bit more difficult to prove.

2 Jets and the Second Tangent Bundle

Tangent vectors and complexified tangent vectors to a manifold M can be identified with certain jets of mappings $\mathbb{R} \to M$ resp. $\mathbb{C} \to M$. In this section we shall interpret tangent vectors to TM analogously, in terms of jets of mappings into M.

We shall start with jets. Suppose P, Q are smooth manifolds, $p \in P$, $q \in Q$, and k, l are nonnegative integers.

Definition 2.1 If $u, v \in C^\infty(P \times Q)$, we shall write

$$u \equiv v \mod(k, l) \quad \text{at } (p, q)$$

if for arbitrary integers κ, λ, $0 \le \kappa \le k$, $0 \le \lambda \le l$, and vector fields $X_1, \ldots, X_{\kappa+\lambda}$ on $P \times Q$, the function $X_1 \cdots X_{\kappa+\lambda}(u - v)$ vanishes at (p, q), provided $X_1, \ldots, X_\kappa$ resp. $X_{\kappa+1}, \ldots, X_{\kappa+\lambda}$ are tangential to the fibers $P \times \{q'\}$ resp. $\{p'\} \times Q$, $p' \in P$, $q' \in Q$. When $\kappa = \lambda = 0$, this vanishing is understood to mean $u(p, q) = v(p, q)$.

Further down we shall need the following simple result. Suppose $Q = Q_1 \times Q_2$, $q = (q_1, q_2)$, where we think of Q_1, Q_2 embedded in Q as $Q_1 \times \{q_2\}$ resp. $\{q_1\} \times Q_2$.

Proposition 2.2 *Suppose $u, v \in C^\infty(P, Q)$ satisfy*

$$u|_{P\times Q_1} \equiv v|_{P\times Q_1} \quad \textit{and} \quad u|_{P\times Q_2} \equiv v|_{P\times Q_2} \mod(k, 1)$$

at (p, q_1) resp. (p, q_2). Then $u \equiv v \mod(k, 1)$ at (p, q).

Indeed, this follows from the fact that any vector field on Q is the sum of two vector fields, one tangential to the fibers $Q_1 \times \{q_2'\}$, the other to the fibers $\{q_1'\} \times Q_2$.

Now suppose M is one more differential manifold.

Definition 2.3 For a pair $f, g \colon P \times Q \to M$ of mappings we shall write

$$f \equiv g \mod(k, l) \quad \text{at } (p, q) \tag{2.1}$$

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if for arbitrary $u \in C^\infty(M)$ we have $u \circ f \equiv u \circ g \bmod(k,l)$ at (p,q) in the sense of Definition 2.1.

By taking differentials of a mapping $f\colon P \times Q \to M$ along P resp. Q we obtain vector bundle homomorphisms $d_1 f\colon \mathrm{TP} \times Q \to \mathrm{TM}$, $d_2 f\colon P \times \mathrm{TQ} \to \mathrm{TM}$. In general, when we are given a homomorphism φ between real vector bundles, we will use the same symbol φ to denote its complexified action between the complexifications of the bundles. For example, $d_1 f$ will also act between $(\mathbb{C} \otimes \mathrm{TP}) \times Q$ and $\mathbb{C} \otimes \mathrm{TM}$. Next we shall introduce $\bmod(k,l)$ equivalence classes for such homomorphisms. Suppose $\varphi, \psi\colon (\mathbb{C} \otimes \mathrm{TP}) \times Q \to \mathbb{C} \otimes \mathrm{TM}$ are vector bundle homomorphisms.

Definition 2.4 We shall write

$$\varphi \equiv \psi \bmod(k,l) \quad \text{at } (p,q) \tag{2.2}$$

if for any vector field X on P and 1-form ω on M, we have

$$\omega\big(\varphi(X,\cdot)\big) \equiv \omega\big(\psi(X,\cdot)\big) \bmod(k,l) \quad \text{at } (p,q),$$

as mappings $P \times Q \to \mathbb{C}$.

We define the relation (2.2) in an analogous manner for bundle morphisms $\varphi, \psi\colon P \times (\mathbb{C} \otimes \mathrm{TQ}) \to \mathbb{C} \otimes \mathrm{TM}$. We shall call $\bmod(k,l)$ equivalence classes of mappings and bundle morphisms (k,l) jets. Note that if (2.1) holds for $f, g\colon P \times Q \to M$ then

$$d_1 f \equiv d_1 g \bmod(k-1,l) \quad \text{and} \quad d_2 f \equiv d_2 g \bmod(k,l-1)$$

also hold at (p,q).

Given a mapping $f\colon P \times Q \to M$ and $\sigma \in T_pP$, $\tau \in T_qQ$, we can construct a vector $f_\#(\sigma,\tau) \in \mathrm{TTM}$ by considering the mapping $h = d_2 f(\cdot,\tau)\colon P \to \mathrm{TM}$ and then putting

$$f_\#(\sigma,\tau) = dh(\sigma). \tag{2.3}$$

The same construction would associate with $\sigma \in \mathbb{C} \otimes T_pP$, $\tau \in T_qQ$ a vector $f_\#(\sigma,\tau) \in \mathbb{C} \otimes T(\mathrm{TM})$. Note that the footpoint of $f_\#(\sigma,\tau)$ is $d_2 f(p,\tau) \in \mathrm{TM}$, and, with $\pi\colon \mathrm{TM} \to M$ denoting the bundle projection,

$$d\pi\, f_\#(\sigma,\tau) = d_1 f(\sigma,q). \tag{2.4}$$

For fixed σ, τ, the vector $f_\#(\sigma,\tau)$ depends only on the $(1,0)$ jet of $d_2 f$, hence on the $(1,1)$ jet of f at (p,q).

In the sequel we shall take P, Q to be real or complex Euclidean spaces, and $p = q = 0$. It will therefore be understood even without mentioning that jets and jet equivalence relations (2.1), (2.2) will always be taken with reference to $(0,0)$.

Let first $P = \mathbb{R}$, $Q = \mathbb{R}$, and denote the coordinates on P and Q by s resp. t.

Proposition 2.5 *The mapping*

$$f \mapsto f_\#\left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0\right) \in \mathrm{TTM}$$

induces a bijection between $(1,1)$ *jets of maps* $f\colon \mathbb{R} \times \mathbb{R} \to M$ *and* TTM.

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Proof We can assume $M = \mathbb{R}^m$ with coordinates $x_1, \ldots, x_m$. Coordinates on $T\mathbb{R}^m$ are obtained by associating the $2m$-tuple $x_1, \ldots, x_m$, $y_1, \ldots, y_m$ with the vector $\Sigma y_\mu \partial/\partial x_\mu \in T_{(x_1,\ldots,x_m)}\mathbb{R}^m$. Let $\alpha + \beta s + \gamma t + \delta st + \cdots$ be the second order Taylor polynomial of $f\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$; the $(1,1)$ jets of such f's are in one to one correspondence with quadruples $\alpha = (\alpha_\mu), \ldots, \delta = (\delta_\mu) \in \mathbb{R}^m$. Since

$$f_\# \left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0 \right) = \sum \left(\beta_\mu \frac{\partial}{\partial x_\mu} + \delta_\mu \frac{\partial}{\partial y_\mu} \right) \in T_{(\alpha,\gamma)}\mathbb{R}^{2m},$$

the proposition follows.

When $M = \mathbb{R}^m$, by associating with the vectors $v = \sum \beta_\mu \partial/\partial x_\mu + \sum \delta_\mu \partial/\partial y_\mu \in T_{(\alpha,\gamma)}\mathbb{R}^{2m}$ the mapping $f_{\alpha,\beta,\gamma,\delta}(s,t) = \alpha + \beta s + \gamma t + \delta st$ we obtain a smooth family of mappings $\mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$. Hence:

Proposition 2.6 *There is a smooth family of maps $f_v\colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$, $v \in \mathrm{TT}\,\mathbb{R}^m$, such that $f_{v\#}(\partial/\partial s|_0, \partial/\partial t|_0) = v$.*

That f_v is a smooth family means that the map $(s,t,v) \mapsto f_v(s,t)$ is C^∞.

Next let $P = \mathbb{C}$, $Q = \mathbb{R}$, let $s = s_1 + is_2$ denote the complex coordinate on $\mathbb{C}$ and, as usual, $\partial/\partial s = (1/2)(\partial/\partial s_1 - i\partial/\partial s_2)$. The next proposition is proved the same way as Proposition 2.5.

Proposition 2.7 *The mapping*

$$f \mapsto f_\# \left(\frac{\partial}{\partial s}\Big|_0, \frac{\partial}{\partial t}\Big|_0 \right) \in \mathbb{C} \otimes T(\mathrm{TM})$$

induces a bijection between $(1,1)$ jets of maps $f\colon \mathbb{C} \times \mathbb{R} \to M$ and $\mathbb{C} \otimes T(\mathrm{TM})$.

In the situation of Propositions 2.5 and 2.7 we shall abbreviate $f_\#(\partial/\partial s|_0, \partial/\partial t|_0)$ as $f_\#$. It is straightforward that when $f\colon \mathbb{C} \times \mathbb{R} \to M$

$$\mathrm{Re}\, f_\# = (f|_{\mathbb{R}\times\mathbb{R}})_\#. \tag{2.5}$$

3 Almost Complex Manifolds

An almost complex manifold is given by a differential manifold M and an almost complex structure tensor J on it. Thus J is an automorphism of TM, mapping each tangent space T_aM into itself, and satisfying $J^2 = -\mathrm{id}$. Alternatively, an almost complex structure can be defined by a splitting $\mathbb{C} \otimes \mathrm{TM} = T' \oplus T''$, with T' and its complex conjugate T'' complex subbundles. If the almost complex structure is given in terms of the tensor J, the corresponding splitting is obtained by putting

$$T' = \{v - iJv : v \in \mathrm{TM}\},$$

and this correspondence is easily seen to be reversible.

A map f between two almost complex manifolds (M_1, J_1) and (M_2, J_2) is called holomorphic if its differential intertwines J_1 and J_2: $df \circ J_1 = J_2 \circ df$. An equivalent

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requirement is that df should map $T'M_1$ into $T'M_2$. For a general map $f\colon M_1 \to M_2$ we define a bundle homomorphism $\overline{\partial} f\colon \mathrm{TM}_1 \to \mathrm{TM}_2$ by

$$\overline{\partial} f = \frac{1}{2}(df + J_2 \circ df \circ J_1).$$

It is immediate that $\overline{\partial} f \circ J_1 = -J_2 \circ \overline{\partial} f$, and $\overline{\partial} f = 0$ precisely when f is holomorphic. More generally, if N is a differential manifold and $f\colon M_1 \times N \to M_2$, $g\colon N \times M_1 \to M_2$ are C^1 maps, $\overline{\partial}_1 f\colon \mathrm{TM}_1 \times N \to \mathrm{TM}_2$ and $\overline{\partial}_2 g\colon N \times \mathrm{TM}_1 \to \mathrm{TM}_2$ are defined through the partial differentials $d_1 f$, $d_2 g$.

Proposition 3.1 *Given an almost complex structure on $\mathbb{R}^m$ and $g\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^m$ there exists an $f\colon \mathbb{C} \times \mathbb{R}^k \to \mathbb{R}^m$ such that $f|_{\mathbb{R}\times\mathbb{R}^k} = g$ and $\overline{\partial}_1 f \equiv 0 \bmod (0,1)$. The $(1,1)$ jet of f is uniquely determined by the $(1,1)$ jet of g. If $g_\nu\colon \mathbb{R} \times \mathbb{R}^k \to \mathbb{R}^m$, $\nu \in \mathbb{R}^l$, is a smooth family of mappings then the corresponding $f_\nu\colon \mathbb{C} \times \mathbb{R}^k \to \mathbb{R}^m$ can also be chosen to form a smooth family, and even to satisfy $\overline{\partial}_1 f_\nu = 0$ on $\mathbb{R} \times \mathbb{R}^k$.*

Proof The almost complex structure of $\mathbb{R}^m$ is given by a mapping $J\colon \mathbb{R}^m \to \mathrm{GL}(m,\mathbb{R})$. Suppose f is an extension of g to $\mathbb{C} \times \mathbb{R}^k$ with Taylor expansion

$$f(s_1 + is_2, t) = g(s_1, t) + s_2 h(s_1, t) + \cdots . \tag{3.1}$$

The equation $\overline{\partial}_1 f \equiv 0 \bmod (0,1)$ means $\partial f/\partial s_2 \equiv J(f)\partial f/\partial s_1 \bmod (0,1)$, or

$$h \equiv J(g)\partial g/\partial s_1 \mod (0,1). \tag{3.2}$$

This can be satisfied by simply choosing $h = J(g)\partial g/\partial s_1$ and then $f(s_1 + is_2, t) = g(s_1, t) + s_2 h(s_1, t)$ gives a required extension. Clearly, if $g = g_\nu$ depends smoothly on ν then $f = f_\nu$ thus constructed also depends smoothly on ν and $\overline{\partial}_1 f_\nu = 0$ on $\mathbb{R} \times \mathbb{R}^k$. Lastly, by (3.2) the $(1,1)$ jet of g determines the $(0,1)$ jet of h, hence, by (3.1), the $(1,1)$ jet of f.

Theorem 3.2 *The tangent bundle of an almost complex manifold (M, J) can be endowed with a canonical almost complex structure. This structure has the following properties:*

(a) The projection $\pi\colon \mathrm{TM} \to M$ is holomorphic;
(b) The embedding $\epsilon\colon M \to \mathrm{TM}$ of M as the zero section is holomorphic;
(c) If (N, I) is another almost complex manifold and $\Phi\colon N \to M$ is a holomorphic map then $d\Phi\colon \mathrm{TN} \to \mathrm{TM}$ is also holomorphic;
(d) With (N, I) as above if a mapping $\Phi\colon N \times \mathbb{R} \to M$ has the property that for each $t \in \mathbb{R}$, $\Phi_t = \Phi(\cdot, t)$ is holomorphic, then so is $d\Phi_t/dt\colon N \to \mathrm{TM}$.

Proof We shall define the almost complex structure on $\mathcal{M} = \mathrm{TM}$ in terms of a splitting $\mathbb{C} \otimes T\mathcal{M} = \mathcal{T}' \oplus \mathcal{T}''$. Put

$$\mathcal{T}' = \{f_\# | f : \mathbb{C} \times \mathbb{R} \to M, \overline{\partial}_1 f \equiv 0 \bmod (0,1)\}. \tag{3.3}$$

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We claim that $\mathcal{T}'$ defines a smooth subbundle of $\mathbb{C}\otimes T\mathcal{M}$, and that $\mathcal{T}'$ and $\mathcal{T}'' = \overline{\mathcal{T}'}$ complement each other. Equivalently, for each $v \in T\mathcal{M}$ there is a unique $w = \gamma(v) \in \mathcal{T}'$ with $\operatorname{Re} w = v$, and the map γ is a smooth homomorphism $T\mathcal{M} \to \mathbb{C} \otimes T\mathcal{M}$ of real vector bundles. This needs to be verified only under the assumption we now make that $M = \mathbb{R}^m$.

Given $v \in \mathrm{TT}\,\mathbb{R}^m$, choose $g_v \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}^m$ such that $g_{v\#} = v$, *cf.* Proposition 2.5. A vector $w \in \mathcal{T}'$ has v as real part if and only if $w = f_\#$ with some $f \colon \mathbb{C} \times \mathbb{R} \to \mathbb{R}^m$ that satisfies $\overline{\partial}_1 f \equiv 0 \operatorname{mod}(0,1)$ and $f|_{\mathbb{R}\times\mathbb{R}} \equiv g_v \operatorname{mod}(1,1)$, *cf.* (3.3), (2.5), and Proposition 2.5. By Proposition 3.1 these two conditions uniquely specify the $(1,1)$ jet of f, hence, by Proposition 2.7, also $w = \gamma(v)$.

If v varies then by Proposition 2.6 g_v can be chosen to form a smooth family, and by Proposition 3.1 the mappings $f = f_v$ can also be chosen to form a smooth family. It follows that $\gamma \colon v \mapsto w = f_{v\#}$ is a smooth map. Further, γ maps the fibers $T_a T\mathbb{R}^n$ to fibers $\mathbb{C} \otimes T_a(T\mathbb{R}^n)$, and is easily checked to be homogeneous: if $c \in \mathbb{R}$ then $\gamma(cv) = c\gamma(v)$. Since any smooth homogeneous map between vector spaces is linear, γ is indeed a smooth homomorphism of vector bundles, and thus $\mathcal{T}'$ defines an almost complex structure on TM.

We now proceed to verify properties (a)–(d).

(a) Let $f \colon \mathbb{C} \times \mathbb{R} \to M$ represent a vector $f_\# \in \mathcal{T}'$. Thus $\overline{\partial}_1 f \equiv 0 \operatorname{mod}(0,1)$, so that taking (2.4) into account

$$T' \ni d_1 f(\partial/\partial s|_0, 0) = d\pi(f_\#).$$

This means $d\pi$ maps $\mathcal{T}'$ to T', and π is indeed holomorphic.

(b) For any $\xi \in \mathbb{C} \otimes \mathrm{TM}$ there is a $g \colon \mathbb{C} \to M$ with $\xi = dg(\partial/\partial s|_0)$. We shall assume $\xi \in T'$, so that $\overline{\partial} g(\partial/\partial s|_0) = 0$. Put $f(s,t) = g(s)$, $s \in \mathbb{C}$, $t \in \mathbb{R}$. Then $\overline{\partial}_1 f \equiv 0 \operatorname{mod}(0,1)$, and also

$$\epsilon g(s) = d_2 f(s, \partial/\partial t|_0).$$

In view of (2.3) and (3.3) therefore $d\epsilon(\xi) = f_\# \in \mathcal{T}'$. We conclude $d\epsilon$ maps T' to $\mathcal{T}'$, *i.e.*, ϵ is holomorphic.

(c) With $\mathcal{N} = \mathrm{TN}$, and $\mathcal{T}'\mathcal{M}$, $\mathcal{T}'\mathcal{N}$ denoting the bundles that define the almost complex structures on $\mathcal{M}$ resp. $\mathcal{N}$, suppose $f \colon \mathbb{C} \times \mathbb{R} \to N$ represents a vector $f_\# \in \mathcal{T}'\mathcal{N}$. It follows that $\overline{\partial}_1(\Phi \circ f) \equiv 0 \operatorname{mod}(0,1)$, whence $d(d\Phi)(f_\#) = (\Phi \circ f)_\# \in \mathcal{T}'\mathcal{M}$. Again, this means that $d(d\Phi) \colon \mathbb{C} \otimes T\mathcal{N} \to \mathbb{C} \otimes T\mathcal{M}$ maps $\mathcal{T}'\mathcal{N}$ to $\mathcal{T}'\mathcal{M}$, and so $d\Phi \colon \mathcal{N} \to \mathcal{M}$ is holomorphic.

(d) It will suffice to prove that $\varphi = d\Phi_t/dt|_{t=0}$ is holomorphic. To this end, pick a vector $\xi \in \mathbb{C} \otimes \mathrm{TN}$ and a map $g \colon \mathbb{C} \to N$ such that $\xi = dg(\partial/\partial s|_0)$. If $\xi \in T^{1,0}N$, which we shall henceforward assume, $\overline{\partial} g(\partial/\partial s|_0) = 0$. Define $f(s,t) = \Phi\big(g(s), t\big)$. It follows that $\overline{\partial}_1 f(\partial/\partial s|_0, t) = 0$, $t \in \mathbb{R}$; in particular

$$\overline{\partial}_1 f \equiv 0 \quad \operatorname{mod}(0,1). \tag{3.4}$$

Unwrapping the definitions we obtain

$$d\varphi(\xi) = d(\varphi \circ g)\left(\frac{\partial}{\partial s}\Big|_0\right) = d\left(d_2 f\left(\cdot, \frac{\partial}{\partial t}\Big|_0\right)\right)\left(\frac{\partial}{\partial s}\Big|_0\right) = f_\#$$

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by (2.3). Thus (3.4) implies $d\varphi(\xi) \in \mathcal{T}'\mathcal{M}$. This being true for arbitrary $\xi \in T'N$, φ is indeed holomorphic.

If $M = \mathbb{R}^m$ with coordinates $x_1, \dots, x_m$ and $x_1, \dots, y_m$ are the corresponding coordinates on $T\mathbb{R}^m = \mathbb{R}^{2m}$ as in the proof of Proposition 2.5, the matrix of the almost complex structure $\mathcal{J}$ on TM can be written down in terms of the matrix of J. Identifying linear maps with their matrices we have in block form

$$\mathcal{J} = \begin{pmatrix} J & 0 \\ \sum y_\mu \partial J/\partial x_\mu & J \end{pmatrix}. \tag{3.5}$$

We conclude this section by noting that the almost complex structure on TM introduced above is the unique such structure that has property (d) in Theorem 3.2, even if (d) is required only for (N, I) the unit disc $\Delta \subset \mathbb{C}$ with its standard complex structure. Indeed, one can show that if $\mathcal{T}'$ is given by (3.3), and $w \in \mathcal{T}'$ is in an appropriate neighborhood of the zero section, then there is in fact an $f\colon \mathbb{C} \times \mathbb{R} \to M$ with $f_\# = w$ that satisfies $\overline{\partial}_1 f = 0$ on $\Delta \times \mathbb{R}$, *cf.* [7]. Therefore if $\mathbb{C} \otimes T\mathcal{M} = \mathcal{S}' \oplus \mathcal{S}''$ defines an almost complex structure for which (d) holds, this property applied with $\Phi = f|_{\Delta \times \mathbb{R}}$ implies that any w as above belongs to $\mathcal{S}'$, and so $\mathcal{S}' = \mathcal{T}'$.

4 Is TM $\to M$ an almost holomorphic vector bundle?

That is, when M is an almost complex manifold and TM is endowed with the almost complex structure described in the previous section? Before answering the question let us clarify the concept of an almost holomorphic bundle.

Suppose $\pi_i\colon E_i \to M$ are holomorphic submersions between almost complex manifolds, $i = 1, 2$. The fiber product $E_1 \times_M E_2$, the preimage of the diagonal in $M \times M$ under the map $\pi_1 \times \pi_2\colon E_1 \times E_2 \to M \times M$, is an almost complex submanifold of $E_1 \times E_2$; in particular it is an almost complex manifold. Now an almost holomorphic vector bundle should certainly mean a surjective holomorphic submersion $\pi\colon E \to M$ between almost complex manifolds, which is also a vector bundle. The structure of a vector bundle is encoded in a mapping $\alpha\colon E \times_M E \to E$ (fiberwise addition); if it is a complex vector bundle, there is also the map $\mu\colon \mathbb{C} \times E \to E$ of fiberwise multiplication by a complex number. It would seem natural to require of an almost holomorphic vector bundle that both α and μ should be holomorphic; in this way we would get the notion of what de Bartolomeis and Tian call a "bundle almost complex structure" in [1]. However, TM would not pass this requirement: indeed, the multiplication map $\mu(c, \cdot)\colon$ TM $\to$ TM by any given $c \in \mathbb{C} \setminus \mathbb{R}$ is holomorphic if and only if the structure of M is integrable. There are various ways to see this; *e.g.* it can be read off from the formula in [5, Remark 3.3.1]. The kernel of the operator D_u there describes holomorphic sections of the restriction of TM to holomorphic curves, by virtue of Theorem 3.2(d). However, this kernel is invariant under multiplication by c if and only if the Nijenhuis tensor vanishes. Therefore we settle for the following concept.

Definition 4.1 An almost holomorphic vector bundle is a real vector bundle $\pi\colon E \to M$, with E, M almost complex manifolds, the projection π and the map $\alpha\colon E \times_M E \to$

E of fiberwise addition holomorphic.

The fibers E_a of an almost holomorphic vector bundle $\pi\colon E \to M$ are not only real vector spaces, but also almost complex submanifolds of E. In fact, the almost complex structure of E_a is translation invariant, since α is holomorphic. Now on the one hand a translation invariant almost complex structure on a real vector space V is uniquely determined by the structure on T_0V, on the other all complex structures on T_0V arise from a unique *complex* vector space structures on V. It follows that each fiber E_a has a natural structure of a complex vector space: E is in fact a complex vector bundle. The map $\mu\colon \mathbb{C} \times E \to E$ of fiberwise multiplication is thus defined; in general, it will not be holomorphic. However, its restriction to fibers $\mathbb{C} \times E_a$ will be. The definition also implies that $\mu(c,\cdot)\colon E \to E$ is holomorphic if $c \in \mathbb{R}$. Indeed, when $c > 0$ is an integer, the map $e \mapsto \mu(c,e) = e + e + \cdots$ is holomorphic because α is. When $c = p/q$ is rational with p, q positive integers, $\mu(c,\cdot) = \mu\big(p, \mu(q,\cdot)^{-1}\big)$ is again holomorphic, and by passing to limits we obtain $\mu(c,\cdot)$ holomorphic for any $c \geq 0$. Finally, when $c < 0$, $\mu(c,\cdot)$ is gotten by solving the holomorphic equation $\alpha\big(\mu(c,\cdot), \mu(-c,\cdot)\big) = \mu(0,\cdot)$, hence it is holomorphic, too.

Theorem 4.2 *If M is an almost complex manifold and* TM *is endowed with the almost complex structure described in Section 3 then $\pi\colon$ TM $\to M$ is an almost holomorphic vector bundle.*

To prove the theorem it will be convenient to think of TM $\times_M$ TM as a submanifold of $T(M \times M)$. More generally, let M_1, M_2 be almost complex manifolds, and let $\mathrm{pr}_j\colon M_1 \times M_2 \to M_j$ denote the projections, $j = 1, 2$.

Proposition 4.3

(a) *The map*
$$d\,\mathrm{pr}_1 \times d\,\mathrm{pr}_2\colon T(M_1 \times M_2) \to \mathrm{TM}_1 \times \mathrm{TM}_2$$
is a biholomorphism.

(b) *When $M_1 = M_2 = M$ and $\Delta(M) \subset M \times M$ is the diagonal, the biholomorphism above maps $T(M \times M)|_{\Delta(M)}$ to* TM $\times_M$ TM.

Proof The map in (a) is clearly a diffeomorphism; it is also holomorphic by Theorem 3.2(c) since the pr_j are. Part (b) is immediate from the definition of TM $\times_M$ TM.

Accordingly, in the sequel we shall identify TM $\times_M$ TM with $T(M \times M)|_{\Delta(M)}$.

Proof of Theorem 4.2 By Theorem 3.2(a), $\pi\colon$ TM $\to M$ is holomorphic. To prove that α is holomorphic, let us first look at a map $g = (g_1, g_2)\colon \mathbb{R} \to M \times M$, and notice that $\xi = dg(\partial/\partial t|_0) \in T(M \times M)$ is in fact in $T(M \times M)|_{\Delta(M)}$ precisely when $g_1(0) = g_2(0)$. In this case $\alpha(\xi) = dg_1(\partial/\partial t|_0) + dg_2(\partial/\partial t|_0)$ can be gotten as follows. Construct $g_{12}\colon \mathbb{R} \times \mathbb{R} \to M$ such that $g_{12}(t,0) = g_1(t)$, $g_{12}(0,t) = g_2(t)$, and put $G(t) = g_{12}(t,t)$. Then $\alpha(\xi) = dG(\partial/\partial t|_0)$.

Next suppose $v \in \mathbb{C} \otimes T\big(T(M \times M)|_{\Delta(M)}\big)$. Thus there is a map $h\colon \mathbb{C} \to T(M \times M)|_{\Delta(M)}$ such that $v = dh(\partial/\partial s|_0)$. In particular, h maps into $T(M \times M)$, hence there is a map $f = (f_1, f_2)\colon \mathbb{C} \times \mathbb{R} \to M \times M$ with
$$h(s) = d_2 f(s, \partial/\partial t|_0);$$

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clearly $\nu = f_\#$.

In view of what has been said above, $f_1(s,0) = f_2(s,0)$ for all $s \in \mathbb{C}$. Construct a map $f_{12}\colon \mathbb{C} \times (\mathbb{R} \times \mathbb{R}) \to M$ such that $f_{12}(s,t,0) = f_1(s,t)$, $f_{12}(s,0,t) = f_2(s,t)$ and put $F(s,t) = f_{12}(s,t,t)$. Then $\alpha\big(h(s)\big) = d_2F(s, \partial/\partial t|_0)$, whence

$$d\alpha(\nu) = F_\#.$$

If, in addition, $\nu \in \mathcal{T}'\big(T(M \times M)\big)$, then $\overline{\partial}_1 f \equiv 0 \bmod (0,1)$, and so $\overline{\partial}_1 f_j \equiv 0 \bmod (0,1)$, $j = 1,2$. Proposition 2.2 and Definition 2.4 imply $\overline{\partial}_1 f_{12} \equiv 0 \bmod (0,1)$, therefore $\overline{\partial}_1 F \equiv 0 \bmod (0,1)$. Thus $d\alpha(\nu) = F_\# \in \mathcal{T}'(\mathrm{TM})$, which means α is holomorphic.

On the tangent spaces T_aM of an almost complex manifold we thus have two complex structures: one induced from the inclusion $T_aM \subset \mathrm{TM}$, as explained above, the other gotten by defining multiplication by i to agree with the action of J.

Proposition 4.4 *These two complex structures on T_aM agree.*

Proof We first verify that the multiplication map $\nu = \mu(\cdot,\xi)\colon \mathbb{C} \to T_aM$ is holomorphic for an arbitrary $\xi \in T_aM$, when T_aM is endowed with the complex structure induced from TM. Suppose $g\colon \mathbb{R} \to M$ is such that $\xi = dg(\partial/\partial t|_0)$, and construct $G\colon \mathbb{C} \to M$ with $G|_{\mathbb{R}} = g$, $\overline{\partial} G = 0$ at $0 \in \mathbb{C}$, *cf.* Proposition 3.1. Thus, if $J_{\mathbb{C}}$ denotes the almost complex tensor of $\mathbb{C}$, and $\tau \in T_0\mathbb{C}$, we have $dG(J_{\mathbb{C}}\tau) = JdG(\tau)$. In particular if $\tau = \partial/\partial t|_0 \in T_0\mathbb{R} \subset T_0\mathbb{C}$ then $dG(J_{\mathbb{C}}\partial/\partial t|_0) = J\xi$. With arbitrary $s \in \mathbb{C}$, $t \in \mathbb{R}$ define $f(s,t) = G(ts)$; it follows that

$$\begin{aligned}\nu(s) = \mu(s,\xi) &= (\operatorname{Re} s)\xi + (\operatorname{Im} s)J\xi \\ &= (\operatorname{Re} s)dG(\partial/\partial t|_0) + (\operatorname{Im} s)dG(J\partial/\partial t|_0) \\ &= d_2 f(\xi, \partial/\partial t|_0),\end{aligned}$$

and so $d\nu(\partial/\partial s|_0) = f_\#$. Now with $\sigma \in T\mathbb{C}$ we have $\overline{\partial}_1 f(\sigma,t) = t\overline{\partial} G(t\sigma)$, whence $\overline{\partial}_1 f \equiv 0 \bmod (0,1)$. This means $d\nu(\partial/\partial s|_0) = f_\# \in \mathcal{T}'$, *i.e.*, ν is holomorphic at $s = 0$. Since the complex structure of T_aM is translation invariant, it follows that ν is everywhere holomorphic.

We conclude by choosing a (complex) basis $\xi_1,\dots,\xi_n$ of T_aM and setting up a map $\Phi\colon \mathbb{C}^n \to T_aM$, $\Phi(z_1,\dots,z_n) = \sum \mu(z_j,\xi_j)$. This is a biholomorphism for both structures on T_aM: for the structure induced from TM by virtue of what we have proved so far, for the structure determined by the complex vector space structure of T_aM simply because Φ is complex linear. Thus the two complex manifold structures on T_aM indeed coincide.

5 Related Results

Given two almost complex manifolds M, N, consider the manifold $J^1(N,M)$ of those 1-jets of maps $N \to M$ that are holomorphic. In [2] Gauduchon constructs a canonical almost complex structure on $J^1(N,M)$. Specializing to the case when $N = \mathbb{C}$, it is easy to see that the set of 1-jets at $0 \in \mathbb{C}$ constitutes an almost complex submanifold $J^1(\mathbb{C}, 0, M) \subset J^1(\mathbb{C}, M)$ that can be naturally identified with $T'M$, hence also with TM. Thus Gauduchon's construction also endows TM with the structure of an almost complex manifold. That structure differs from the one introduced in this paper—in particular it does not have the crucial property (d) in Theorem 3.2—unless M is a complex manifold. (This probably means that it will not be of relevance to deformation theory of holomorphic maps between almost complex manifolds.)

One way to see why this is so is to notice that for any complex number $c \neq 0$ the biholomorphism $\mathbb{C} \ni s \mapsto cs \in \mathbb{C}$ induces a biholomorphism of $J^1(\mathbb{C}, M)$ and also of $T'M$, TM, when this latter is endowed with Gauduchon's structure. On TM this biholomorphism is nothing but fiberwise multiplication $\mu(c, \cdot)$. However, when $c \notin \mathbb{R}$ $\mu(c, \cdot)$ is not holomorphic with respect to our structure, unless M is a complex manifold, see the discussion in the second papragraph of Section 4; thus the two almost complex structures are in general indeed different.

Now in [6] Muller discusses, among other things, Gauduchon's almost complex tensor on $J^1(\mathbb{C}, M)$, and proposes an expression for it in local coordinates. This expression, when restricted to $J^1(\mathbb{C}, 0, M) \cong$ TM, agrees with our formula (3.5), seemingly implying that Gauduchon's structure and the one introduced in this paper are the same after all. The error is in the first paragraph on page 229 in [6]. It is incorrectly claimed that there is a unique almost complex structure on $J^1(\mathbb{C}, M)$ with the property that for any holomorphic map $f\colon \Delta \to M$ its 1-jet $j^1(f)\colon \Delta \to J^1(\mathbb{C}, M)$ is also holomorphic (here $\Delta \subset \mathbb{C}$ is the unit disc). Thus the formula in [6, Exercise 6.1.2] does not describe Gauduchon's structure; rather it anticipates ours.

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