

BOUNDS FOR A CONE-TYPE MULTIPLIER OPERATOR OF NEGATIVE INDEX IN \mathbb{R}^3

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Abstract

In this paper we obtain some sharp $L^p - L^q$ estimates and the restricted weak-type endpoint estimates for the multiplier operator of negative order associated with conic surfaces in \mathbb{R}^3 which have finite type degeneracy.

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1. Introduction and statement of results

Let $\gamma : [-1, 1] \rightarrow \mathbb{R}$ be a smooth function. In this paper, we consider the cone multiplier problem associated with the conic surface

$$\Gamma_\gamma = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : (\xi, \tau) = \lambda(t, \gamma(t), 1), t \in [-1, 1], \lambda > 0\},$$

which is generated by the curve $C = \{(t, \gamma(t)) \in \mathbb{R}^2 : t \in [-1, 1]\}$. To do this, let us define a cone-type multiplier operator S^α of order α by

$$(S^\alpha f)^\wedge(\xi, \tau) = \frac{\phi(\tau)}{\Gamma(\alpha + 1)} \chi(\xi) \left(\xi_2 - \tau \gamma\left(\frac{\xi_1}{\tau}\right) \right)_+^\alpha \hat{f}(\xi, \tau), \quad (\xi, \tau) = (\xi_1, \xi_2, \tau) \in \mathbb{R}^2 \times \mathbb{R}, \quad (1.1)$$

where $\phi \in C_0^\infty(1, 2)$ and χ is a smooth function compactly supported in a small neighborhood of $(0, \gamma(0))$. Here $\Gamma(z)$ is the gamma function, and $r_+ = r$ if $r > 0$ and $r_+ = 0$ if $r \leq 0$. By analytic continuation, this definition makes sense even when $\text{Re}(\alpha) \leq -1$.

When Γ_γ is a subset of the light cone, S^α becomes essentially the standard cone multiplier operator. We may represent this by the smooth surface generated by the parabola $C(t) = (t, t^2)$, $t \in [-1, 1]$, which is a simple model of curves with nonvanishing

curvature. In this case, when $\alpha > 0$, the problem of L^p boundedness has been studied by several authors [4, 20, 24, 25, 27] and the most recent result in this direction is due to Garrigós and Seeger [8] (see also [12, 13] for higher dimensions). When $\alpha < 0$, Lee [16] obtained some sharp range of $L^p - L^q$ boundedness and showed that the cone multiplier operator of negative order in \mathbb{R}^3 can be bounded from $L^p(\mathbb{R}^3)$ to $L^q(\mathbb{R}^3)$ only in the range where the Bochner–Riesz operator in \mathbb{R}^2 of the same order is bounded. However, the problems of L^p and $L^p - L^q$ boundedness of the cone multiplier operator are still open for both positive and negative orders.

On the other hand, one may consider the problem of $L^p - L^q$ boundedness associated with the conic surface Γ_γ which is generated by a curve C having degeneracy at some points where the curvature of C vanishes. In fact, it turns out that the $L^p - L^q$ boundedness of $S^{-\alpha}$ of negative order $-\alpha$ depends on the degeneracy of the curve C . The purpose of this note is to show certain sharp $L^p - L^q$ estimates for $S^{-\alpha}$ when the conic surface Γ_γ is generated by a curve C whose curvature vanishes at a single point.

We need the following definition to specify the type of the curve C at a point.

DEFINITION 1.1. Let $m \geq 2$ be an integer and $a \in \mathbb{R}$. Let γ be a smooth function defined in a neighborhood of a . We say that γ is of finite type m at a if $\gamma^{(k)}(a) = 0$ for $2 \leq k < m$ and $\gamma^{(m)}(a) \neq 0$. We also say that the curve C is of finite type m at $(a, \gamma(a))$ if γ is of finite type m at a .

We may assume that γ is of finite type m at zero and $\gamma(0) = \gamma'(0) = 0$. Indeed, translation on the Fourier transform side and discarding some harmless smooth factor of the multiplier do not affect the boundedness of $S^{-\alpha}$ except for a constant multiple.

Now we introduce some notation. Let $m \geq 2$ and, for $0 < \alpha < (m + 1)/m$, let us set

$$\Delta_\alpha^m = \left\{ \left(\frac{1}{p}, \frac{1}{q} \right) \in [0, 1] \times [0, 1] : \frac{1}{p} - \frac{1}{q} \geq \frac{m\alpha}{m+1}, \frac{1}{p} > \frac{1}{4} + \frac{\alpha}{2}, \frac{1}{q} < \frac{3}{4} - \frac{\alpha}{2} \right\}.$$

Also we define points $A_\alpha, B_\alpha^m, C_\alpha^m, A'_\alpha, B'^m_\alpha$ and C'^m_α contained in $[0, 1] \times [0, 1]$ by

$$A_\alpha = \left(\frac{1}{4} + \frac{\alpha}{2}, 0 \right), \quad B_\alpha^m = \left(\frac{1}{4} + \frac{\alpha}{2}, \frac{1}{4} - \frac{(m-1)\alpha}{2m+2} \right), \quad C_\alpha^m = \left(\frac{m\alpha}{m+1}, 0 \right),$$

and

$$A'_\alpha = \left(1, \frac{3}{4} - \frac{\alpha}{2} \right), \quad B'^m_\alpha = \left(\frac{3}{4} + \frac{(m-1)\alpha}{2m+2}, \frac{3}{4} - \frac{\alpha}{2} \right), \quad C'^m_\alpha = \left(1, 1 - \frac{m\alpha}{m+1} \right).$$

(See Figure 1.) Note that if $0 < \alpha < (m + 1)/[2(m - 1)]$, Δ_α^m is the closed pentagon with vertices $A_\alpha, B_\alpha^m, B'^m_\alpha, A'_\alpha$ and $(1, 0)$, from which closed line segments $[A_\alpha, B_\alpha^m], [A'_\alpha, B'^m_\alpha]$ are removed. If $\alpha > (m + 1)/[2(m - 1)]$, Δ_α^m is the closed triangle with vertices C_α^m, C'^m_α and $(1, 0)$. If $\alpha = (m + 1)/[2(m - 1)]$, Δ_α^m is the closed triangle with vertices C_α^m, C'^m_α and $(1, 0)$, from which two points C_α^m, C'^m_α are removed.

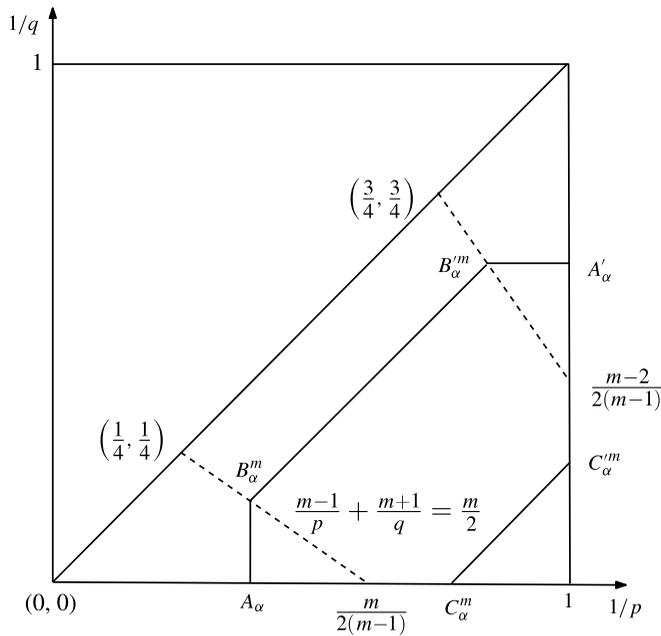


FIGURE 1. $L^p - L^q$ boundedness of $S^{-\alpha}$ with γ of finite type m .

In order to predict the mapping properties of $S^{-\alpha}$, let us consider a Bochner–Riesz type operator $T^{-\alpha}$ of negative order $-\alpha$ defined by

$$(T^{-\alpha} f)^\wedge(\xi) = \frac{(\xi_2 - \gamma(\xi_1))_+^{-\alpha}}{\Gamma(-\alpha + 1)} \chi(\xi) \hat{f}(\xi), \quad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

When C is a part of the circle (more generally, a curve with $\gamma'' \neq 0$), $T^{-\alpha}$ is essentially the Bochner–Riesz operator. In this case, the sharp range of $L^p - L^q$ boundedness was proved by Bak [1] (see also [2, 5]). There are also some restricted weak-type endpoint estimates in [11]. Recently, Lee and Seo [19] showed the sharp $L^p - L^q$ boundedness of $T^{-\alpha}$ of negative order $-\alpha$ when the curve C is of finite type m . More precisely, they showed that for $0 < \alpha < (m + 1)/m$, $\|T^{-\alpha} f\|_{L^q(\mathbb{R}^2)} \leq C \|f\|_{L^p(\mathbb{R}^2)}$ if and only if $(1/p, 1/q) \in \Delta_\alpha^m$. They also obtained some restricted weak-type results.

As was shown in [16], the cone multiplier operator of negative order is closely related to the Bochner–Riesz operator of the same order. In other words, the results of the cone multiplier operator of negative order in \mathbb{R}^3 are parallel to those of the Bochner–Riesz operator of the same order in \mathbb{R}^2 . Thus we can also expect that the type set of $S^{-\alpha}$ is the same as that of $T^{-\alpha}$ when the curve C is of finite type m (see Section 3). Furthermore, as with the Bochner–Riesz type operator $T^{-\alpha}$ of negative order $-\alpha$ when the curve C is of finite type m , it can be conjectured that for $0 < \alpha < (m + 1)/m$, $\|S^{-\alpha} f\|_{L^q(\mathbb{R}^3)} \leq C \|f\|_{L^p(\mathbb{R}^3)}$ if and only if $(1/p, 1/q) \in \Delta_\alpha^m$.

The following is a partial answer to this question. We denote by $L^{p,r}$ the Lorentz space equipped with norm $\|\cdot\|_{p,r}$.

THEOREM 1.2. *Let γ be of finite type $m \geq 2$ at zero and let $(m + 1)/[2(4m - 1)] < \alpha < (m + 1)/m$. Then the following hold.*

- (i) *When $(m + 1)/[2(4m - 1)] < \alpha < (m + 1)/[2(m - 1)]$:*
 - (a) $\|S^{-\alpha} f\|_q \leq C\|f\|_p$ if $(1/p, 1/q) \in \Delta_\alpha^m$;
 - (b) $\|S^{-\alpha} f\|_{q,\infty} \leq C\|f\|_p$ if $(1/p, 1/q) \in (B_\alpha^m, A'_\alpha)$;
 - (c) $\|S^{-\alpha} f\|_{q,\infty} \leq C\|f\|_{p,1}$ if $(1/p, 1/q) = B_\alpha^m$ or B_α^m .
- (ii) *When $(m + 1)/[2(m - 1)] \leq \alpha < (m + 1)/m$:*
 - (d) $\|S^{-\alpha} f\|_q \leq C\|f\|_p$ if $(1/p, 1/q) \in \Delta_\alpha^m \setminus \{C_\alpha^m, C_\alpha^m\}$;
 - (e) $\|S^{-\alpha} f\|_{q,\infty} \leq C\|f\|_p$ if $(1/p, 1/q) = C_\alpha^m$ and $\alpha \neq (m + 1)/[2(m - 1)]$.

To obtain the results of Theorem 1.2, we decompose $S^{-\alpha}$ dyadically away from its singularities on Γ_γ . Thus, we consider a multiplier operator T_δ whose Fourier multiplier is essentially supported in a δ -neighborhood of the cone Γ_γ . More precisely, for $0 < \delta \ll 1$, define

$$(T_\delta f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\chi(\xi)\hat{f}(\xi, \tau), \tag{1.2}$$

where $\phi \in C_0^\infty(1, 2)$, $\psi \in C_0^\infty(-1, 1)$ (or $\mathcal{S}(\mathbb{R})$) and χ is a smooth function compactly supported in a small neighborhood of $(0, \gamma(0))$. Here \mathcal{S} is the Schwartz class.

PROPOSITION 1.3. *Let T_δ be defined by (1.2) with $\psi \in C_0^\infty(-1, 1)$ and let γ be of finite type m at zero. Then for $p \geq 2$, $(m - 1)/p + (m + 1)/q \leq m/2$, $q > 5p/3$ and $(p, q) \neq (2, 2(m + 1))$, there is a constant C such that*

$$\|T_\delta f\|_q \leq C\delta^{2/p-1/2}\|f\|_p. \tag{1.3}$$

Moreover, if ψ is a smooth function satisfying the condition

$$\hat{\psi} \text{ is supported in } \{t \in \mathbb{R} : |t| \sim 1\}, \tag{1.4}$$

then for $1 \leq p < 2(m - 1)/m$, there is a constant C such that

$$\|T_\delta f\|_\infty \leq C\delta^{(m+1)/(mp)}\|f\|_p. \tag{1.5}$$

The constant C may depend on the norms $\|\gamma\|_{C^N((-1, 1))}$, $\|\psi\|_{C^N((-1, 1))}$ and $\|\phi\|_{C^N((-1, 1))}$ for some large N .

REMARK 1.4. When Γ_γ is a subset of the light cone, if $1/p + 3/q = 1$ and $14/3 < q \leq 6$ then (1.3) is due to Lee [16]. In particular, the estimate (1.3) with $m = 2$ is covered by Proposition 2.1 which additionally contains the point $(p, q) = (2, 6)$ in the (p, q) range. For this reason, it suffices to show the case $m \geq 3$ of the estimates (1.3) after proving Proposition 2.1. Moreover, (1.3) shows that the $L^p - L^q$ bounds for T_δ are not influenced by the degeneracy of the curve C when $p \geq 2$, $1/p + 1/q \leq 1/2$, $q > 5p/3$ and $(p, q) \neq (2, \infty)$.

We would like to remark that the condition $q > 5p/3$ in Proposition 1.3 has been dictated by the restriction $r > 5/3$ in the bilinear cone restriction estimate (stated below as Theorem 4.1) and the use of the L^∞ estimate $\|T_\delta f\|_\infty \leq C\delta^{-1/2}\|f\|_\infty$. One can relax this condition a little and prove some (almost optimal) estimates outside that region ($q > 5p/3$) by using the so-called ‘plate decomposition estimates’ due to Wolff [27] and Garrigós and Seeger [8] instead of the L^∞ estimate. We get the following ϵ -loss version of (1.3).

PROPOSITION 1.5. *Let T_δ be defined by (1.2) with $\psi \in C_0^\infty(-1, 1)$ and γ be of finite type m at zero. Then for all $\epsilon > 0$, there is a constant C_ϵ such that*

$$\|T_\delta f\|_q \leq C_\epsilon \delta^{2/p-1/2-\epsilon} \|f\|_p \quad (1.6)$$

in the (additional) range given by $(m-1)/p + (m+1)/q \leq m/2$ and

$$\frac{3}{5p} \leq \frac{1}{q} < \frac{3/10 - 1/p_w}{1/2 - 1/p_w} \left(\frac{1}{p} - \frac{1}{p_w} \right) + \frac{1}{p_w}. \quad (1.7)$$

REMARK 1.6 (Comments on Proposition 1.5). When $m = 2$, the estimate (1.6) can be obtained from the results due to Wolff [27] (also Garrigós and Seeger [8]) and a bilinear cone restriction estimate (stated below as Theorem 4.1).

To be more precise, Wolff established an important inequality [27, Theorem 1] on the plate decompositions related to the (circular) cone multipliers. Let us temporarily denote by W_δ the operator T_δ corresponding to the light cone (that is, $\gamma(t) = t^2$). Wolff’s inequality leads to the following almost sharp L^p bounds: for all sufficiently large p , say $p > p_w$, and all $\epsilon > 0$, there exists a constant C_ϵ such that

$$\|W_\delta f\|_p \leq C_\epsilon \delta^{2/p-1/2-\epsilon} \|f\|_p. \quad (1.8)$$

Wolff obtained these estimates for $p > p_w = 74$ (see [27, Corollary 2]). More recently, Garrigós and Seeger [8, Remark 1.4 and Corollary 1.5 in Section 1] improved this range to $p > p_w$ with $p_w = 63\frac{1}{3}$ (for a generalized version of U_δ corresponding to the case $|\gamma''(t)| \geq c > 0$). Further progress ($p > p_w$ with $p_w = 20$) has been made by Garrigós *et al.* [9].

The following $L^p - L^q$ estimate was deduced in [16] (in the case $\gamma(t) = t^2$) from (1.8) and a bilinear cone restriction estimate (stated below as Theorem 4.1):

$$\|W_\delta f\|_q \leq C_\epsilon \delta^{2/p-1/2-\epsilon} \|f\|_p$$

for some constant C_ϵ , $\epsilon > 0$, where p, q satisfy (1.7). Then, the estimate (1.6) with $m = 2$ may be easily obtained by applying the arguments in [16, Section 5].

When $m \geq 3$, one can use a scaling argument and (1.6) with $m = 2$. Notice that this does not include the critical line $(m-1)/p + (m+1)/q = m/2$, because of the presence of the $\delta^{-\epsilon}$ factor in (1.6) with $m = 2$. We omit the details of this argument, since it is similar to the argument used to prove (1.3) from Proposition 2.1 (see Section 2 below).

Furthermore, the estimates (1.6) give the following extended range of α in (a) of Theorem 1.2:

$$\alpha_{p_{w,m}} < \alpha \leq \frac{m + 1}{2(4m - 1)}$$

where

$$\alpha_{p_{w,m}} = \frac{2}{p_{w,m}} - \frac{1}{2}$$

and $(1/p_{w,m}, 1/q_{w,m})$ is the point of intersection of the lines $(m - 1)/p + (m + 1)/q = m/2$ and

$$\frac{1}{q} = \frac{3/10 - 1/p_w}{1/2 - 1/p_w} \left(\frac{1}{p} - \frac{1}{p_w} \right) + \frac{1}{p_w},$$

which is the line joining the points $(1/p_w, 1/p_w)$ and $(1/2, 3/10)$. This is because Theorem 1.2 is a consequence of Proposition 1.3 and a summation method (see Section 3 below). For more details, we refer the reader to [16, Section 5].

In order to obtain the sharp $L^p - L^q$ boundedness for $S^{-\alpha}$, the estimates (1.5) play a crucial role. However, it is impossible to prove (1.5) without imposing an additional condition (1.4) on ψ . To see this, suppose that (1.5) holds for a function ψ which does not satisfy (1.4). We may choose an interval I_0 away from zero satisfying $\gamma'' \neq 0$ on I_0 because γ is of finite type m at zero. Choose a smooth function $f \in \mathcal{S}(\mathbb{R}^3)$ satisfying

$$(T_\delta f)^\wedge(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right) \chi_{I_0}(\xi_1/\tau)$$

where $\chi_I \in C_0^\infty$ with $\chi_I = 1$ on interval I . By the simple change of variables $\xi_2 = \xi_2 + \tau\gamma(\xi_1/\tau)$ and integration in ξ_2 , we see that

$$T_\delta f(x, t) = \delta \hat{\psi}(\delta x_2) \int e^{2\pi i t \tau} I(x, \tau) \phi(\tau) d\xi_1 d\tau,$$

where $I(x, \tau) = \int e^{2\pi i(x_1 \xi_1 + x_2 \tau \gamma(\xi_1/\tau))} \chi_{I_0}(\xi_1/\tau) d\xi_1$. Since $\gamma'' \neq 0$ on I_0 and $\tau \sim 1$, by using the stationary phase method, we have $|I(x, \tau)| \geq C|x|^{-1/2}$ for sufficiently large $|x_2|$ and $|x_1| < C|x_2|$. Note that for t with $|t| \leq c$, we have $e^{2\pi i t \tau} = 1 + O(c)$. Hence, for sufficiently large R and sufficiently small c ,

$$\|T_\delta f\|_q^q \geq C\delta^q \int_{A_{R,c}} |\hat{\psi}(\delta x_2)|^q |x|^{-q/2} dx dt \geq C\delta^q \int_R^\infty |\hat{\psi}(\delta r)|^q r^{-q/2+1} dr$$

where $A_{R,c} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : |x_2| > R, |x_1| < C|x_2|, |t| \leq c\}$. This means that if $q > 4$ then $\|T_\delta f\|_q \geq C\delta$. By duality and (1.5), if $1 \leq p < 4/3$ then $C\delta \leq \|T_\delta f\|_q \leq C\delta^{(m+1)/(mp)}$ for any $q \geq 1$. This says that (1.5) is no longer valid for $1 \leq p < (m + 1)/m$. Therefore, we conclude that the condition (1.4) on ψ plays an important role in deriving the boundedness of T_δ .

REMARK 1.7. We obtain the estimates (1.5) from interpolation between the following estimates;

$$\|T_\delta f\|_\infty \leq C\delta^{(m+1)/[2(m-1)]}\|f\|_{2(m-1)/m,1}, \quad (1.9)$$

$$\|T_\delta f\|_\infty \leq C\delta^{(m+1)/m}\|f\|_1. \quad (1.10)$$

Then an interpolation between (1.3) and (1.9) gives

$$\|T_\delta f\|_q \leq C\delta^{2/p-1/2}\|f\|_p \quad (1.11)$$

for $p > 2(m-1)/m$, $q > 5p/3$, and $m/2 \geq (m-1)/p + (m+1)/q$. Note that $2/p - 1/2 = (1+1/m)(1/p - 1/q)$ if $(1/p, 1/q)$ is on the line $(m-1)/p + (m+1)/q = m/2$. By interpolation between (1.3) for p, q satisfying $(m-1)/p + (m+1)/q = m/2$ and (1.5), we see that

$$\|T_\delta f\|_q \leq C\delta^{(1+1/m)(1/p-1/q)}\|f\|_p \quad (1.12)$$

for $(1/p, 1/q) \in \Delta \setminus \{A, B\}$, where Δ is the closed triangle with vertices $A = (5m/[4(4m-1)], 3m/[4(4m-1)])$, $B = (m/[2(m-1)], 0)$ and $C = (1, 0)$, from which the line segment (A, C) is removed. In fact, we use the estimates (1.5), (1.11), and (1.12) to prove Theorem 1.2.

The organization of this paper is as follows. In Section 2 we prove Proposition 1.3. First, we obtain (1.3) from Proposition 2.1 by using a scaling argument which depends on stability of estimates (see Remark 2.2). We also verify (1.5) by showing the estimates (1.9) and (1.10). Actually, the good condition (1.4) on ψ makes it possible to prove (1.9) and (1.10) by using the kernel estimates (see Lemma 2.6). In Section 3 we give the proof of Theorem 1.2 by combining Proposition 1.3 and a dyadic decomposition of $S^{-\alpha}$ (see Lemma 3.1). Then we prove the necessary conditions for $S^{-\alpha}$. In Section 4 we give the proof of Proposition 2.1 which is similar to the arguments that were used by Lee [16] (see also [15]). We will also use the bilinear restriction estimates for some conic surfaces, which are a generalized version of the bilinear cone restriction estimates due to Wolff [28] and Tao [23].

Throughout this paper, C is a positive constant which may vary from line to line. Let $A \lesssim B$ or $A = \mathcal{O}(B)$ denote the estimates $A \leq CB$ and let $A \sim B$ denote $C^{-1}A \leq B \leq CA$ for some C . In addition to the symbol $\hat{\cdot}$, we use $\mathcal{F}(\cdot)$, $\mathcal{F}^{-1}(\cdot)$ to denote the Fourier transform and the inverse Fourier transform, respectively. Finally, $\text{supp } f$, $\text{supp } \hat{f}$ (or the support of \hat{f}) mean the support of f and the Fourier support of f , respectively.

2. Estimate for T_δ

In this section we prove Proposition 1.3. Before we begin, let us choose a smooth function χ_0 supported in $I = [-1, 1]$. For $0 < \delta \ll 1$, define U_δ by

$$(U_\delta f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\chi_0(\xi_1/\tau)\hat{f}(\xi, \tau), \quad (2.1)$$

where $\phi \in C_0^\infty(1, 2)$ and $\psi \in C_0^\infty(-1, 1)$. (Note that the only difference between U_δ and T_δ in (1.2) is in the cutoff function.) We need the following proposition which will be shown in Section 4.

PROPOSITION 2.1. *Let U_δ be defined by (2.1) and let γ be a smooth function defined on $I = [-1, 1]$ with $|\gamma''| \geq c > 0$ on I . Then for $p \geq 2$, $q > 5p/3$ and $1/p + 3/q \leq 1$,*

$$\|U_\delta f\|_{L^q(\mathbb{R}^3)} \leq C\delta^{2/p-1/2} \|f\|_{L^p(\mathbb{R}^3)} \tag{2.2}$$

where the constant C is stable under ‘small smooth perturbations’ of γ (in a sense made precise in Remark 2.2).

REMARK 2.2. The stability of estimates under small smooth perturbation of γ plays an important role in the proof of our results. Let $\tilde{\gamma}$ be a smooth function defined on $I = [-1, 1]$. Suppose that there exist a large positive integer N , a constant $B = B_N$ (depending on N) and a small $\epsilon_0 > 0$ such that for all $0 < \epsilon \leq \epsilon_0$ and $0 \leq n \leq N$,

$$\left| \frac{d^n}{dt^n} (\tilde{\gamma}(t) - \gamma(t)) \right| \leq B\epsilon, \quad t \in I. \tag{2.3}$$

Then the stability of estimates means that the constant C in the estimate in (2.2) is uniform in the functions $\tilde{\gamma}$ satisfying (2.3).

However, in our problem, we need to treat $\gamma(\xi_1/\tau)$ which is a function of two variables. Thus, in this case we need to replace (2.3) by

$$\left| \frac{\partial^{n+l}}{\partial x^n \partial y^l} (\tilde{\gamma}(x/y) - \gamma(x/y)) \right| \leq B\epsilon, \quad x/y \in I, y \in [1, 2],$$

for all $l + n \leq N$. But this follows from (2.3) by the chain rule and product rule in the given range of x, y . □

The fact that a function γ is of finite type m at a makes it possible to use a scaling argument which relies on this type of stability of estimates. More precisely, let us set

$$\gamma(a, t) = \gamma(t + a) - \gamma(a) - \gamma'(a)t.$$

For $0 < \delta \ll 1$ and γ of finite type m at a , we also set

$$\gamma_\delta(a, t) = \delta^{-m} \gamma(a, \delta t)$$

and

$$G_\delta(a, x, y) = \gamma_\delta\left(a, \frac{x}{y}\right).$$

The following lemma means that G_δ is a smooth function uniformly in a, δ . Thus, it gives the stability of estimates for γ_δ .

LEMMA 2.3. *Let γ be of finite type m at a . Then for sufficiently small $\delta > 0$, if $|t| \sim 1$,*

$$|\gamma''_{\delta}(a, t)| \geq c > 0 \tag{2.4}$$

and

$$G_{\delta}(a, \cdot) \in C^N(V)$$

uniformly in a, δ , where N is a large constant and $V = \{(x, y) : x/y \in [-1, 1], 1 \leq y \leq 2\}$.

PROOF. By a Taylor expansion, we see that

$$\gamma_{\delta}(a, t) = \frac{\gamma^m(a)}{m!} t^m + \zeta_{\delta}(t, a) \tag{2.5}$$

with $|\zeta_{\delta}(t, a)| \leq C\delta|t|^{m+1}$. This gives the estimate (2.4). If x/y is contained in $[-1, 1]$ and $1 \leq y \leq 2$, by replacing t by x/y in (2.5), we see that

$$\left| \frac{\partial^{n+l}}{\partial x^n \partial y^l} \left(G_{\delta}(a, x, y) - \frac{\gamma^m(a)}{m!} \left(\frac{x}{y} \right)^m \right) \right| \leq C_N \delta$$

for all $l, n \leq N$. This means that G_{δ} is a smooth function contained in $C^N(V)$ uniformly in a, δ . □

2.1. Proof of Proposition 1.3. As mentioned above, interpolation between (1.9) and (1.10) gives (1.5). Therefore, we only need to show (1.3), (1.9) and (1.10).

First, in order to prove (1.3), we decompose T_{δ} dyadically away from its degeneracy. We treat decomposed parts by using Proposition 2.1 and a rescaling argument. Then it remains to treat the part of T_{δ} containing the degeneracy. To control it, we need the following kernel estimate.

For $0 < \delta \ll 1$, let us define K_a by

$$(K_a)^{\wedge}(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right) \rho\left(\frac{\xi_1/\tau - a}{\delta^{1/m}}\right) \tag{2.6}$$

where $\phi \in C_0^{\infty}(1, 2)$, $\psi \in \mathcal{S}(\mathbb{R})$ and $\rho \in C_0^{\infty}[-1, 1]$.

LEMMA 2.4. *Suppose γ is finite type m at a . Then there is a constant $C = C(M)$, independent of a, δ , such that*

$$|K_a(x, t)| \leq C\delta^{(m+1)/m} (1 + |\delta^{1/m}(x_1 + \gamma'(a)x_2)| + |\delta x_2| + |t + ax_1 + \gamma(a)x_2|)^{-M} \tag{2.7}$$

for all $0 < M \leq N$ with large N .

PROOF. By the change of variables $(\xi_1, \xi_2, \tau) \rightarrow (\xi_1 + \tau a, \xi_2 + \gamma'(a)\xi_1 + \tau\gamma(a), \tau)$ and rescaling $(\xi_1, \xi_2, \tau) \rightarrow (\delta^{1/m}\xi_1, \delta\xi_2, \tau)$, we have that

$$K_a(x, t) = \delta^{(m+1)/m} \tilde{K}_a(\delta^{1/m}(x_1 + \gamma'(a)x_2), \delta x_2, t + ax_1 + \gamma(a)x_2)$$

where $\mathcal{F}(\tilde{K}_a) = \phi(\tau) \psi(\xi_2 - \tau\gamma_{\delta^{1/m}}(a, \xi_1/\tau)) \rho(\xi_1/\tau)$. Since γ is of finite type m at a , by Lemma 2.3, $\gamma_{\delta^{1/m}}$ is a smooth function contained in $C^N(V)$ uniformly in a, δ ,

where $V = \{(\xi_1, \tau) : \xi_1/\tau \in [-1, 1], 1 \leq \tau \leq 2\}$. Then by integration by parts, we see that there is a uniform constant C , independent of a, δ , such that $|\tilde{K}_a(x, t)| \leq C(1 + |(x, t)|)^{-M}$ for all $M \leq N$. This gives (2.7). \square

We now state the following interpolation lemma which is a multilinear extension of a result implicit in [3] (see also [6]). We refer the reader to [16] for a proof of the lemma. It will be used several times throughout this paper.

LEMMA 2.5 (Interpolation lemma). *Let $\epsilon_1, \epsilon_2 > 0$. Suppose $\{T_j\}$ is a sequence of linear operators satisfying that for $1 \leq p_1^i, p_2^i < \infty, i = 1, \dots, l$ (here the superscript i is not an exponent, but an index), and $1 \leq q_1, q_2 < \infty$,*

$$\|T_j(f^1, \dots, f^l)\|_{q_1} \leq M_1 2^{\epsilon_1 j} \prod_{i=1}^l \|f^i\|_{p_1^i}$$

and

$$\|T_j(f^1, \dots, f^l)\|_{q_2} \leq M_2 2^{-\epsilon_2 j} \prod_{i=1}^l \|f^i\|_{p_2^i}.$$

Then for $T = \sum_{j=-\infty}^{\infty} T_j$,

$$\|T(f^1, \dots, f^l)\|_{q, \infty} \leq CM_1^\theta M_2^{1-\theta} \prod_{i=1}^l \|f^i\|_{p_i, 1} \tag{2.8}$$

where $\theta = \epsilon_2/(\epsilon_1 + \epsilon_2), 1/q = \theta/q_1 + (1 - \theta)/q_2, 1/p^i = \theta/p_1^i + (1 - \theta)/p_2^i$, for $i = 1, \dots, l$. Furthermore, if $q_1 = q_2 = q$, then

$$\|T(f^1, \dots, f^l)\|_q \leq CM_1^\theta M_2^{1-\theta} \prod_{i=1}^l \|f^i\|_{p_i, 1}. \tag{2.9}$$

We now prove (1.3). To decompose T_δ dyadically, let $\beta \in C_0^\infty([-2, -1/2] \cup [1/2, 2])$ satisfying $\sum_{j \in \mathbb{Z}} \beta(2^j \cdot) = 1$ and let $\beta_0 \in C_0^\infty([-2, 2])$ satisfying $\sum_{2^{-j} < \lambda} \beta(2^j \cdot) = \beta_0(\cdot/\lambda)$. Then

$$1 = \beta_0(\cdot/\lambda) + \sum_{2^{-j} \geq \lambda} \beta(2^j \cdot).$$

This gives

$$T_\delta = T_\delta^0 + \sum_{2^{-j} \geq \delta^{1/m}} T_\delta^j,$$

where

$$(T_\delta^0 f)^\wedge(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau \gamma(\xi_1/\tau)}{\delta}\right) \beta_0\left(\frac{\xi_1/\tau}{\delta^{1/m}}\right) \chi(\xi) \hat{f}(\xi, \tau)$$

and, for $2^{-j} \geq \delta^{1/m}$,

$$(T_\delta^j f)^\wedge(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau \gamma(\xi_1/\tau)}{\delta}\right) \beta\left(2^j \frac{\xi_1}{\tau}\right) \chi(\xi) \hat{f}(\xi, \tau).$$

Since γ is of finite type m at zero, \hat{T}_δ^0 is supported in a cube of size $\delta \times \delta^{1/m} \times 1$. Observe that $T_\delta^0 f = K_0 * f$, where K_a is defined by (2.6). Then using Lemma 2.4 and Young’s convolution inequality yields that for $1 \leq p \leq q$,

$$\|T_\delta^0 f\|_q \leq C \delta^{(1+1/m)(1/p-1/q)} \|f\|_p.$$

Here we note that $(1 + 1/m)(1/p - 1/q) \geq 2/p - 1/2$ because $(m - 1)/p + (m + 1)/q \leq m/2$. This gives the desired bound for T_δ^0 .

Next we consider $\sum_{2^{-j} \geq \delta^{1/m}} T_\delta^j$. Let us define

$$(\tilde{T}_\lambda^j f)^\wedge(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau \gamma_{2^{-j}}(0, \xi_1/\tau)}{\lambda}\right) \beta(\xi_1/\tau) \hat{f}(\xi, \tau).$$

By setting $f_j(x, t) = f(2^j x_1, 2^{mj} x_2, t)$ and rescaling $T_\delta^j f$ by $(\xi_1, \xi_2) \rightarrow (2^{-j} \xi_1, 2^{-mj} \xi_2)$ in frequency space, we have that

$$T_\delta^j f(x, t) = \tilde{T}_{\delta 2^{mj}}^j f_j(2^{-j} x_1, 2^{-mj} x_2, t).$$

From Lemma 2.3, we see that $|\gamma_{2^{-j}}''(0, \xi_1/\tau)| \geq c > 0$ uniformly in j because $|\xi_1/\tau| \sim 1$ on the support of β . Therefore, applying Proposition 2.1 to $\tilde{T}_\lambda^j f$, we see that there is a uniform constant C , independent of j , such that

$$\|T_\delta^j f\|_q \leq C 2^{(m+1)j/q} (\delta 2^{mj})^{2/p-1/2} \|f_j\|_p.$$

Rescaling again gives

$$\|T_\delta^j f\|_q \leq C 2^{j((m-1)/p + (m+1)/q - m/2)} \delta^{2/p-1/2} \|f\|_p \tag{2.10}$$

for $p \geq 2, q > 5p/3$ and $1/p + 3/q \leq 1$.

If $(m-1)/p + (m+1)/q < m/2$, then from direct summation, we obtain the required estimate for $\sum_{2^{-j} \geq \delta^{1/m}} T_\delta^j$. If $(m-1)/p + (m+1)/q = m/2$, choose $(1/p, 1/q_1), (1/p, 1/q_2)$ satisfying $1/p + 3/q_i \leq 1, q_i > 5p/3 (i = 1, 2), p \geq 2$ and $(m-1)/p + (m+1)/q_1 > m/2 > (m-1)/p + (m+1)/q_2$. Then by (2.10), we have for $i = 1, 2$,

$$\|T_\delta^j f\|_{q_i} \leq C 2^{j((m-1)/p + (m+1)/q_i - m/2)} \delta^{2/p-1/2} \|f\|_p.$$

Applying (2.8) in Lemma 2.5 with $l = 1, \epsilon_1 = (m-1)/p + (m+1)/q_1 - m/2$ and $\epsilon_2 = -((m-1)/p + (m+1)/q_2 - m/2)$, we obtain the restricted weak-type estimate

$$\left\| \sum_{2^{-j} \geq \delta^{1/m}} T_\delta^j f \right\|_{q, \infty} \leq C \delta^{2/p-1/2} \|f\|_{p,1}$$

for $(m-1)/p + (m+1)/q = m/2, q > 5p/3$ and $p \geq 2$. Then interpolation between the restricted weak-type estimates leads to the strong-type estimates except for $(p, q) = (2, 2(m+1))$. This completes the proof of (1.3).

We now turn to (1.9) and (1.10). In order to obtain these results, we need certain estimates for a kernel which has the condition (1.4) imposed on ψ . Let us choose a smooth function supported in a small neighborhood of the origin in \mathbb{R} so that $\chi(\xi) = \chi_1(\xi_1) \chi(\xi)$. Define \hat{K}_δ by

$$\hat{K}_\delta(\xi, \tau) = \phi(\tau) \psi\left(\frac{\xi_2 - \tau \gamma(\xi_1/\tau)}{\delta}\right) \chi_1(\xi_1) \tag{2.11}$$

where $\psi \in \mathcal{S}(\mathbb{R})$ and $\phi \in C_0^\infty(1, 2)$. Since χ is a smooth function compactly supported in a small neighborhood of $(0, \gamma(0))$, we may write that $T_\delta f = K_\delta * f$. By Young's convolution inequality, it is enough to show that $\|K_\delta\|_\infty \leq C \delta^{(m+1)/m}$ and $\|K_\delta\|_{2(m-1)/(m-2), \infty} \leq C \delta^{(m+1)/[2(m-1)]}$. These are obtained from the following Lemma 2.6.

LEMMA 2.6. *Let K_δ be defined by (2.11). Suppose that ψ satisfies (1.4) and γ is of finite type m at zero. Then for $0 < \delta \ll 1$,*

$$\|K_\delta\|_\infty \leq C\delta^{(m+1)/m}, \tag{2.12}$$

$$\|K_\delta\|_{2(m-1)/(m-2),\infty} \leq C\delta^{(m+1)/[2(m-1)]}. \tag{2.13}$$

PROOF. We first consider the case $m = 2$. By the change of variables $\xi_2 \rightarrow \xi_2 + \tau\gamma(\xi_1/\tau)$ and integration in ξ_2 ,

$$K_\delta(x, t) = \delta\hat{\psi}(\delta x_2) \int e^{2\pi i(x_1\xi_1 + t\tau + x_2\tau\gamma(\xi_1/\tau))} \chi_1(\xi_1)\phi(\tau) d\xi_1 d\tau.$$

Set $\Phi(\xi_1, \tau) = \tau\gamma(\xi_1/\tau)$. Then Φ is homogeneous of degree one and the Hessian matrix of Φ has rank one because γ is of finite type 2 at zero. From the well-known oscillatory integral estimates, we see that

$$\left| \int e^{2\pi i(x_1\xi_1 + t\tau + x_2\tau\gamma(\xi_1/\tau))} \chi_1(\xi_1)\phi(\tau) d\xi_1 d\tau \right| \leq C|(x, t)|^{-1/2}$$

if $|x_2| \geq C(x_1^2 + t^2)^{1/2}$. Therefore,

$$\|K_\delta\|_\infty \leq C\delta^{3/2} \tag{2.14}$$

because $\hat{\psi}$ is supported in $\{x_2 \in \mathbb{R} : |x_2| \sim \delta^{-1}\}$.

Turning to the cases $m \geq 3$, the proof of (2.12) is similar to the argument that was used to prove (1.3). As before, we use β and β_0 to decompose K_δ dyadically away from its degeneracy. Then

$$K_\delta = K_\delta^0 + \sum_{2^{-j} \geq \delta^{1/m}} K_\delta^j$$

where

$$\begin{aligned} \hat{K}_\delta^0(\xi, \tau) &= \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\beta_0\left(\frac{\xi_1/\tau}{\delta^{1/m}}\right)\chi_1(\xi_1), \\ \hat{K}_\delta^j(\xi, \tau) &= \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\beta\left(2^j\frac{\xi_1}{\tau}\right)\chi_1(\xi_1) \quad \text{for } 2^{-j} \geq \delta^{1/m}. \end{aligned}$$

Since γ is of finite type m at zero, Lemma 2.4 shows that K_δ^0 satisfies (2.12).

Now consider $\sum_{2^{-j} \geq \delta^{1/m}} K_\delta^j$. By rescaling $(\xi_1, \xi_2) \rightarrow (2^{-j}\xi_1, 2^{-mj}\xi_2)$,

$$K_\delta^j(x, t) = 2^{-j(m+1)}\tilde{K}_{\delta 2^{mj}}(2^{-j}x_1, 2^{-mj}x_2, t) \tag{2.15}$$

where

$$\tilde{K}_\lambda = \mathcal{F}^{-1}\left(\phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma_{2^{-j}}(0, \xi_1/\tau)}{\lambda}\right)\beta(\xi_1/\tau)\right).$$

By (2.14) we see that there is a uniform constant C , independent of j , such that $\|\tilde{K}_\lambda\|_\infty \leq C\lambda^{3/2}$, because, from Lemma 2.3, $|\gamma''_{2^{-j}}(0, \xi_1/\tau)| \geq c > 0$ on the support of β uniformly in j . Therefore,

$$\|K_\delta^j\|_\infty \leq C\delta^{3/2}2^{(m-2)j/2}. \tag{2.16}$$

Since $m \geq 3$, we may sum over $2^{-j} \geq \delta^{1/m}$, and thus

$$\sum_{2^{-j} \geq \delta^{1/m}} \|K_\delta^j\|_\infty \leq C\delta^{(m+1)/m}.$$

This gives the required bound (2.12).

Similarly, we can prove (2.13) by using an analogous argument. To see this, define K_δ^0, K_δ^j and \tilde{K}_λ as before. From Lemma 2.4, one can easily see that K_δ^0 satisfies the estimate (2.13).

By Plancherel’s theorem, we have $\|\tilde{K}_\lambda\|_2 \leq C\lambda^{1/2}$. Then we obtain from (2.15) that

$$\|K_\delta^j\|_2 \leq C\delta^{1/2}2^{-j/2}. \tag{2.17}$$

Using Hölder’s inequality together with (2.16) and (2.17),

$$\|K_\delta^j\|_q \leq C\delta^{3/2-2/q}2^{-j((m-1)/q-(m-2)/2)}.$$

Therefore

$$\left\| \sum_{2^{-j} \geq \delta^{1/m}} K_\delta^j \right\|_{2(m-1)/(m-2), \infty} \leq C\delta^{(m+1)/[2(m-1)]}$$

by Lemma 2.5 and hence (2.13) holds. □

Furthermore, we claim that Proposition 1.3 is true if we replace $\psi \in C_0^\infty$ by $\psi \in \mathcal{S}(\mathbb{R})$. To see this, it is sufficient to show that (1.3) holds when $\psi \in \mathcal{S}(\mathbb{R})$ takes the place of $\psi \in C_0^\infty$, because we only used the condition (1.4) to prove (1.9) and (1.10). For $\psi \in \mathcal{S}(\mathbb{R})$, let us define

$$(S_0f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\beta_0\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\chi(\xi)\hat{f}(\xi, \tau)$$

and, for $2^{-j} \geq \delta$,

$$(S_jf)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\beta\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{2^{-j}}\right)\chi(\xi)\hat{f}(\xi, \tau),$$

where β, β_0 are the same functions as in the proof of (1.3). Then

$$T_\delta = S_0 + \sum_{2^{-j} \geq \delta} S_j.$$

By (1.3), we have $\|S_0f\|_q \leq C\delta^{2/p-1/2}\|f\|_p$.

It remains to consider $\sum_{2^{-j} \geq \delta} S_j$. Set $\psi_{2^{-j}}(t) = \beta(t/2^{-j})\psi(t/\delta)$. Then for $2^{-j} \geq \delta$, the smooth function $\psi_{2^{-j}}$ is supported in $[-2^{-j}, 2^{-j}]$ and it is easy to see that $|(d^l/dt^l)\psi_{2^{-j}}(t)| \leq C2^{jl}\delta^M2^{jM}$ for any M . Observe that (1.3) is also valid if the function $\psi(\xi_2 - \tau\gamma(\xi_1/\tau)/\delta)$ in the definition (1.2) with $\psi \in C_0^\infty$ is replaced by $\psi_\delta(\xi_2 - \tau\gamma(\xi_1/\tau))$, where ψ_δ is a smooth function supported in $[-\delta, \delta]$ which satisfies $|(d^l/dt^l)\psi_\delta| \leq C_l\delta^{-l}$ for $l \geq 0$. Then we have that, for $2^{-j} \geq \delta$,

$$\|S_jf\|_q \leq C\delta^M2^{jM}2^{-j(2/p-1/2)}\|f\|_p$$

for the same p, q in Proposition 1.3. Summing these estimates shows our claim.

3. $L^p - L^q$ boundedness of $S^{-\alpha}$

In this section we prove Theorem 1.2 by using Proposition 1.3. We also show the necessary conditions for $S^{-\alpha}$.

3.1. Proof of Theorem 1.2. First, we need to decompose $S^{-\alpha}$ dyadically into T_δ whose kernel has a good localization property (1.4) to use Proposition 1.3. This is obtained by the following lemma (see, for example, [7, Lemma 2.1]). Let us define the distribution D^z by

$$\langle D^z, f \rangle = \int \frac{(\xi_2 - \tau\gamma(\xi_1/\tau))_+^z}{\Gamma(z+1)} \phi(\tau) f(\xi, \tau) d\xi d\tau, \quad \text{Re}(z) > -1.$$

When $\text{Re}(z) \leq -1$, D^z is defined by analytic continuation.

LEMMA 3.1. *For $\text{Re}(z) > 0$, there is a smooth function ψ_{-z} satisfying $\text{supp } \hat{\psi}_{-z} \subset \{t \in \mathbb{R} : |t| \sim 1\}$ such that for all $f \in \mathcal{S}$,*

$$\langle D^{-z}, f \rangle = \sum_j 2^{zj} \int \psi_{-z}\left(2^j\left(\xi_2 - \tau\gamma\left(\frac{\xi_1}{\tau}\right)\right)\right) \phi(\tau) f(\xi, \tau) d\xi d\tau.$$

By this lemma, we may write

$$S^{-\alpha} f = \sum_j K_j * f$$

where $\hat{K}_j = 2^{\alpha j} \psi_{-\alpha}(2^j(\xi_2 - \tau\gamma(\xi_1/\tau))) \phi(\tau) \chi(\xi)$ and $\text{supp } \hat{\psi}_{-\alpha} \subset \{t : |t| \sim 1\}$. Since ϕ and χ are compactly supported, by the rapid decay of $\psi_{-\alpha}$, for $2^j \leq 1$ and $1 \leq p \leq \infty$, we have $\|K_j\|_p \leq C2^{\alpha j}$. This gives, for all $p \leq q$,

$$\left\| \sum_{2^j \leq 1} K_j * f \right\|_q \leq C \|f\|_p.$$

Hence we only need to treat the part $\sum_{2^j \geq 1} K_j * f$ to prove Theorem 1.2. Since ψ satisfies the condition (1.4), from Lemma 2.6 and Young’s convolution inequality, we obtain $\|K_j * f\|_\infty \leq C2^{(\alpha-(m+1)/m)j} \|f\|_1$. Thus, for $0 < \alpha < (m+1)/m$,

$$\left\| \sum_{2^j \geq 1} K_j * f \right\|_\infty \leq C \|f\|_1.$$

Therefore, if $(m+1)/[2(4m-1)] < \alpha < (m+1)/[2(m-1)]$, it is sufficient to prove (b) and (c), because then we can obtain (a) by interpolation and duality. If $(m+1)/[2(m-1)] < \alpha < (m+1)/m$, we also obtain (d) from (e) by interpolation and duality.

We first consider case (i): $(m+1)/[2(4m-1)] < \alpha < (m+1)/[2(m-1)]$. To show (c), by duality we only need to show the restricted weak type for $\sum_{2^j \geq 1} K_j * f$ at B_α^m . From (1.11),

$$\|K_j * f\|_q \leq C2^{\alpha j} 2^{(1/2-2/p)j} \|f\|_p, \tag{3.1}$$

for $p > 2(m - 1)/m$, $(m - 1)/p + (m + 1)/q \leq m/2$ and $q > 5p/3$. Choose $(1/p_1, 1/q_1)$, $(1/p_2, 1/q_2)$ satisfying $(m - 1)/p_i + (m + 1)/q_i = m/2$ for $i = 1, 2$ and $5m/[4(4m - 1)] < 1/p_1 < 1/4 + \alpha/2 < 1/p_2 < m/[2(m - 1)]$. Then we obtain from (3.1) that

$$\|K_j * f\|_{q_i} \leq C2^{\alpha j}2^{(1/2-2/p_i)j}\|f\|_{p_i} \quad i = 1, 2.$$

An application of Lemma 2.5 yields that, for $(1/p, 1/q) = B_\alpha^m$,

$$\left\| \sum_{2^j \geq 1} K_j * f \right\|_{q, \infty} \leq C\|f\|_{p, 1}$$

because $\alpha + 1/2 - 2/p_2 < 0 < \alpha + 1/2 - 2/p_1$. This gives the desired estimates.

Next we prove (b). By duality it is sufficient to show, for $(1/r, 1/s) \in (B_\alpha^m, A_\alpha]$, that

$$\left\| \sum_{2^j \geq 1} K_j * f \right\|_s \leq C\|f\|_{r, 1}.$$

Since $(m + 1)/[2(4m - 1)] < \alpha < (m + 1)/[2(m - 1)]$, we can choose $(1/r_1, 1/s)$, $(1/r_2, 1/s)$ satisfying $(m - 1)/r_i + (m + 1)/s \leq m/2$ for $i = 1, 2$ and $5m/[4(4m - 1)] < 1/r_1 < 1/4 + \alpha/2 < 1/r_2 < m/[2(m - 1)]$. Then, by (3.1),

$$\|K_j * f\|_s \leq C2^{\alpha j}2^{(1/2-2/r_i)j}\|f\|_{r_i} \quad i = 1, 2.$$

Noting that $\alpha + 1/2 - 2/r_2 < 0 < \alpha + 1/2 - 2/r_1$ as before and using (2.9) in Lemma 2.5, we get $\|\sum_{2^j \geq 1} K_j * f\|_s \leq C\|f\|_{r, 1}$. This shows (b).

We now consider case (ii): $(m + 1)/[2(m - 1)] < \alpha < (m + 1)/m$. To show (e), by duality it is sufficient to show, for $(m + 1)/[2(m - 1)] < \alpha < (m + 1)/m$, that

$$\left\| \sum_{2^j \geq 1} K_j * f \right\|_\infty \leq C\|f\|_{(m+1)/m\alpha, 1}.$$

By (1.5), for $1 \leq p < 2(m - 1)/m$,

$$\|K_j * f\|_\infty \leq C2^{\alpha j}2^{-(m+1)j/(pm)}\|f\|_p. \tag{3.2}$$

As before, since $(m + 1)/[2(m - 1)] < \alpha < (m + 1)/m$, one can choose p_1, p_2 satisfying $m/[2(m - 1)] < 1/p_1 < m\alpha/(m + 1) < 1/p_2 < 1$. By (3.2), for $i = 1, 2$,

$$\|K_j * f\|_\infty \leq C2^{\alpha j}2^{-(m+1)j/(p_i m)}\|f\|_{p_i}.$$

Note that $m\alpha/(m + 1) - 1/p_2 < 0 < m\alpha/(m + 1) - 1/p_1$. By applying (2.9) in Lemma 2.5, we obtain the desired estimates for $\alpha \neq (m + 1)/[2(m - 1)]$.

Next we prove (d). Since duality and interpolation between (e) and the $L^1 - L^\infty$ estimate give (d) except for the case $\alpha = (m + 1)/[2(m - 1)]$, it is sufficient to prove that, for $1/p - 1/q = m/[2(m - 1)]$ and $q \neq \infty$,

$$\left\| \sum_{2^j \geq 1} K_j * f \right\|_{q, \infty} \leq C\|f\|_{p, 1}.$$

However, this can be obtained by the same argument as before by using (1.12), duality and interpolation.

3.2. Necessary conditions.

THEOREM 3.2. *Let $0 < \alpha < (m + 1)/m$ and let $S^{-\alpha}$ be defined by (1.1). If $\chi(0, 0) \neq 0$ and $\alpha \neq 1$, $\|S^{-\alpha} f\|_q \leq C\|f\|_p$ may hold only if $(1/p, 1/q) \in \Delta_\alpha^m$.*

PROOF. For $0 < \alpha < (m + 1)/m$ and $\alpha \neq 1$, we want to show that $S^{-\alpha}$ may be bounded from L^p to L^q only if

$$\frac{1}{p} > \frac{1}{4} + \frac{\alpha}{2}, \quad \frac{1}{q} < \frac{3}{4} - \frac{\alpha}{2}, \tag{3.3}$$

$$\frac{1}{p} - \frac{1}{q} \geq \frac{\alpha m}{m + 1}. \tag{3.4}$$

First we prove (3.3). By duality it is sufficient to prove this for $1/q < 3/4 - \alpha/2$. Define a smooth function b by

$$\hat{b}(\xi, \tau) = \eta\left(\frac{\xi_1}{\tau}\right)\varphi\left(\xi_2 - \tau\gamma\left(\frac{\xi_1}{\tau}\right)\right)\phi(\tau)/\chi(\xi)$$

where η is a smooth function supported away from zero such that $\gamma'' \neq 0$ on the support of η and $\varphi \in C_0^\infty$ with $\varphi = 1$ on a small interval around zero. Since $\chi(0, 0) \neq 0$, if we choose η, φ supported in a sufficiently small neighborhood of zero, $\chi \neq 0$ on the support of η and φ . Hence we have a smooth function $b \in L^p$ for all $1 \leq p \leq \infty$.

Observe that

$$S^{-\alpha} b(x, t) = L(x, t)B_\alpha(x_2),$$

where

$$L(x, t) = \int e^{2\pi i(x_1 \xi_1 + t\tau + x_2 \tau \gamma(\xi_1/\tau))} \eta\left(\frac{\xi_1}{\tau}\right) \phi(\tau) d\xi_1 d\tau$$

and

$$B_\alpha(x_2) = \int \varphi(\xi_2) \frac{(\xi_2)_+^{-\alpha}}{\Gamma(-\alpha + 1)} e^{2\pi i x_2 \xi_2} d\xi_2.$$

First, we want to show that for $|x_2| > R, |x_1| < C|x_2|$ and $|t| \leq c$,

$$|L(x, t)| \geq C|(x, t)|^{-1/2}, \tag{3.5}$$

where R is a sufficiently large constant and $c > 0$ is a sufficiently small constant. Using the change of variable $\xi_1 \rightarrow \xi_1 \tau$,

$$L(x, t) = \int e^{2\pi i t \tau} \int e^{2\pi i(x_1 \tau \xi_1 + x_2 \tau \gamma(\xi_1))} \eta(\xi_1) d\xi_1 \tau \phi(\tau) d\tau.$$

Since $\gamma'' \neq 0$ on the support of η , by a well-known asymptotic expansion, we see that if $|x_1| < C|x_2|$,

$$\int e^{2\pi i(x_1 \tau \xi_1 + x_2 \tau \gamma(\xi_1))} \eta(\xi_1) d\xi_1 = C(x_2 \tau)^{-1/2} \sum_{j=0}^N a_j(x_2 \tau)^{-j/2} + A_N(x_2 \tau),$$

where A_N satisfies $|(d/ds)^k A_N(s)| = O(s^{-k-(N+1)/2})$. Furthermore, for t with $|t| \leq c$, we have $e^{2\pi i t \tau} = 1 + O(c)$ because $\tau \sim 1$. From this and an integration for $\tau \sim 1$, it follows that

$$\begin{aligned} L(x, t) &= C(1 + O(c)) \int \left((x_2 \tau)^{-1/2} \sum_{j=0}^N a_j (x_2 \tau)^{-j/2} + A_N(x_2 \tau) \right) \tau \phi(\tau) d\tau \\ &= C(1 + O(c)) \left(x_2^{-1/2} + \sum_{j=1}^N a_j x_2^{-(1+j)/2} + A_N(x_2) \right). \end{aligned}$$

Note that if $|x_2|$ is sufficiently large,

$$|x_2|^{-1/2} \geq B \left| \sum_{j=1}^N a_j x_2^{-(1+j)/2} + A_N(x_2) \right|$$

for some $B \geq 2$. Then the desired estimate (3.5) is obtained.

Now we claim that for sufficiently large $|x_2|$,

$$|B_\alpha(x_2)| \geq C|x_2|^{-1+\alpha}. \tag{3.6}$$

We may assume that $x_2 > 0$. From the change of variable $\xi_2 \rightarrow \xi_2/x_2$,

$$B_\alpha(x_2) = x_2^{-1+\alpha} \int \varphi\left(\frac{\xi_2}{x_2}\right) \frac{(\xi_2)_+^{-\alpha}}{\Gamma(-\alpha + 1)} e^{2\pi i \xi_2} d\xi_2.$$

Since

$$\int \varphi\left(\frac{\xi_2}{x_2}\right) \frac{(\xi_2)_+^{-\alpha}}{\Gamma(-\alpha + 1)} e^{2\pi i \xi_2} d\xi_2 \rightarrow \mathcal{F}^{-1}\left(\frac{(\xi_2)_+^{-\alpha}}{\Gamma(-\alpha + 1)}\right)(1) \neq 0$$

as $x_2 \rightarrow \infty$ (see [10, p. 172]), we obtain (3.6).

Combining (3.5) and (3.6) gives that for sufficiently large R and small c ,

$$\iint |S^{-\alpha} b|^q dx dt \geq C \iint_{A_{R,c}} |x|^{(-3/2+\alpha)q} dx dt$$

where $A_{R,c} = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : |x_2| > R, |x_1| < C|x_2|, |t| \leq c\}$. If $1/q \geq 3/4 - \alpha/2$, then it follows that

$$\iint_{A_{R,c}} |x|^{(-3/2+\alpha)q} dx dt \geq C \int_{|x|>R} |x|^{(-3/2+\alpha)q} dx = \infty,$$

and hence the proof is complete.

We now turn to (3.4). Let $0 < c_1, c_2 \ll 1$ be constants to be chosen later. Let E_δ be the set defined by

$$E_\delta = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : 1 \leq \tau \leq 2, c_1 \delta \leq \xi_1/\tau \leq 2c_1 \delta, c_2 \delta^m \leq \xi_2 \leq 2c_2 \delta^m\}.$$

Since γ is of finite type m at zero, one can choose c_1, c_2 such that, for all $0 < \delta \ll 1$,

$$\text{dist}(E_\delta, \Gamma_\gamma) \sim \delta^m \quad \text{and} \quad E_\delta \subset \left\{ (\xi, \tau) : \xi_2 > \tau\gamma\left(\frac{\xi_1}{\tau}\right) \right\},$$

where $\Gamma_\gamma = \{(\xi, \tau) : \xi_2 = \tau\gamma(\xi_1/\tau)\}$. This means that E_δ is supported in the region away from the cone Γ_γ . Moreover, for $\varepsilon > 0$, we define a dual set E_δ^* of E_δ which is given by

$$E_\delta^* = \{(x, t) \in \mathbb{R}^2 \times \mathbb{R} : |x_1| \leq \varepsilon\delta^{-1}, |x_2| \leq \varepsilon\delta^{-m}, |t| \leq \varepsilon\}.$$

If $(\xi, \tau) \in E_\delta$ and $(x, t) \in E_\delta^*$, then it is obvious that

$$e^{2\pi i(\xi, \tau) \cdot (x, t)} = 1 + O(\varepsilon). \tag{3.7}$$

Now choose a positive function $\eta \in C_0^\infty(1, 2)$ and set

$$\hat{\chi}_\delta(\xi, \tau) = \phi(\tau)\eta\left(\frac{\xi_1/\tau}{c_1\delta}\right)\eta\left(\frac{\xi_2}{c_2\delta^m}\right).$$

Since $\hat{\chi}_\delta$ is supported in E_δ which is away from the cone Γ_γ , we obtain from (3.7) that, for $(x, t) \in E_\delta^*$,

$$\begin{aligned} S^{-\alpha}\chi_\delta(x, t) &= C_{-\alpha}(1 + O(\varepsilon)) \int_{\mathbb{R}} \int_{\mathbb{R}^2} \frac{\phi^2(\tau)}{\Gamma(-\alpha + 1)} \chi(\xi) \left(\xi_2 - \tau\gamma\left(\frac{\xi_1}{\tau}\right)\right)_+^{-\alpha} \\ &\quad \times \eta\left(\frac{\xi_1/\tau}{c_1\delta}\right)\eta\left(\frac{\xi_2}{c_2\delta^m}\right) d\xi d\tau. \end{aligned}$$

This is because the distribution is a function away from the cone Γ_γ if $\alpha \neq 1$. Then it is easy to see that for sufficiently small $\varepsilon > 0$ and $(x, t) \in E_\delta^*$,

$$|S^{-\alpha}\chi_\delta(x, t)| \geq C\delta \int_{c_2\delta^m}^{2c_2\delta^m} t^{-\alpha} dt,$$

which implies that

$$\|S^{-\alpha}\chi_\delta\|_q \geq C\delta\delta^{m(-\alpha+1)}|E_\delta^*|^{1/q} \geq C\delta\delta^{m(-\alpha+1)}\delta^{-(m+1)/q}.$$

However, we have that $\|\chi_\delta\|_p \leq C\delta^{m+1}\delta^{-(m+1)/p}$ by using a change of variables. This gives condition (3.4). □

4. Proof of Proposition 2.1

To prove Proposition 2.1, we need the bilinear restriction estimates for conic surfaces. This is a generalized version of the bilinear cone restriction estimates due to Wolff [28] and Tao [23].

Let $\Gamma_\gamma = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \xi_2 = \tau\gamma(\xi_1/\tau), 1 \leq \tau \leq 2\}$ and let V_1 and V_2 be closed subsets of $[-1, 1]$. We set

$$\Gamma_i = \{(\xi, \tau) \in \Gamma_\gamma : \xi_1/\tau \in V_i\}$$

for $i = 1, 2$. The following is a bilinear restriction theorem for the conic surfaces Γ_γ in \mathbb{R}^3 . It is a special case of Theorem 1.2 given in [18] (see also [17]).

THEOREM 4.1. *Let γ be a smooth function defined on $I = [-1, 1]$ with $|\gamma''| \geq c > 0$ on I . If $\text{dist}(V_1, V_2) \sim 1$, then for $r > 5/3$,*

$$\|(fd\mu_1)\hat{(}gd\mu_2)\|_{L^r} \leq C\|f\|_{L^2(d\mu_1)}\|g\|_{L^2(d\mu_2)} \tag{4.1}$$

where $d\mu_1, d\mu_2$ are the surface measures on Γ_1, Γ_2 , respectively, and the constant C is stable under small smooth perturbations of γ .

REMARK 4.2. (a) When γ is the quadratic function given by $\gamma(s) = s^2$, Γ_i is a subset of the light cone. For this case, the bilinear estimate (4.1) was obtained by Wolff [28] ($r > 5/3$) and Tao [23] ($r = 5/3$). Recently, Lee [18] and Lee [14] extended Wolff’s and Tao’s results, respectively, to oscillatory integral operators with cinematic curvature condition. These are related to the regularity problem of Fourier integral operators in [21].

To be more precise, for $i = 1, 2$, let an oscillatory integral operator be defined by

$$W_\lambda^i f(z, x_2) = \int e^{i\lambda\Psi_i(z, x_2, \eta)} a_i(z, x_2, \eta) f(\eta) d\eta, \quad (z, x_2) = (x_1, t, x_2) \in \mathbb{R}^2 \times \mathbb{R}.$$

Here, $a_i(z, x_2, \eta)$ is a compactly supported smooth function in $\mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2$ and $\eta = (\xi_1, \tau) \in \mathbb{R}^2$. The phase $\Psi_i(z, x_2, \eta)$ is a smooth function, homogeneous of degree one in η on the support of a_i . Let us consider the $L^2 \times L^2 \rightarrow L^r$ bilinear estimate

$$\|W_\lambda^1 f W_\lambda^2 g\|_{L^r(\mathbb{R}^3)} \leq C\lambda^{-3/r} \|f\|_2 \|g\|_2 \tag{4.2}$$

under the following conditions for the phase functions.

- (i) For $i = 1, 2$, $\text{rank } \partial_{z, \eta}^2 \Psi_i = 2$ and $\partial_{x_2} \Psi_i \neq 0$ on the support of a_i .
- (ii) From the above conditions and the implicit function theorem, we may assume that

$$\partial_{x_2} \Psi_i(z, x_2, \eta) = q_i(z, x_2, \partial_z \Psi_i(z, x_2, \eta))$$

for some $q_i(z, x_2, \eta)$. Then $\text{rank } \partial_\eta^2 q_i = 1$ on the support of a_i . (This is called the cinematic curvature condition.)

- (iii) For $i = 1, 2$,

$$\left\langle \frac{\partial_z \Psi_i(z, x_2, \eta_i)}{\partial_z \Psi_i(z, x_2, \eta_i)}, \partial_\eta q_2(z, x_2, \partial_z \Psi_2(z, x_2, \eta_2)) - \partial_\eta q_1(z, x_2, \partial_z \Psi_1(z, x_2, \eta_1)) \right\rangle \geq c > 0$$

for all $(z, x_2, \eta_1) \in \text{supp } a_1$ and $(z, x_2, \eta_2) \in \text{supp } a_2$.

Then Lee [18] and Lee [14] obtained the estimate (4.2) for $r > 5/3$ and $r = 5/3$, respectively.

(b) The bilinear estimate (4.1) is a special case of the $L^2 \times L^2 \rightarrow L^r$ bilinear estimate (4.2). To verify this, let us set $\Psi_i(z, x_2, \eta) = x_1 \xi_1 + t\tau + x_2 \tau \gamma(\xi_1/\tau)$. Then the adjoint Fourier restriction operator $(fd\mu_i)$ related to the conic surface Γ_i can be viewed

as an oscillatory integral operator with the phase function Ψ_i . Note that we have $\partial_{x_2} \Psi_i = \tau\gamma(\xi_1/\tau)$. Hence, it is easy to see that for $i = 1, 2$, the phase function Ψ_i satisfies conditions (i), (ii) because $\gamma'' \neq 0$. Moreover, the condition $\text{dist}(V_1, V_2) \sim 1$ guarantees that condition (iii) holds. Thus, (4.1) is a consequence of the bilinear estimate (4.2) in [18].

(c) The stability of the bilinear estimate (4.1) under small smooth perturbations of γ plays an important role in the proof of our result. This comes from the fact that the $L^2 \times L^2 \rightarrow L'$ bilinear estimate (4.2) is uniform for small smooth perturbations of the phase function Ψ in [18]. For more details on the $L^2 \times L^2 \rightarrow L'$ bilinear estimates for oscillatory integral operators with cinematic curvature condition, we refer readers to [14, 17, 18].

We actually use the following lemma, which is a ‘thickened’ version of Theorem 4.1, to prove Proposition 2.1, because the operator U_δ is essentially supported in a δ -neighborhood of the cone Γ_γ . For this reason, let us define T_δ^i by

$$(T_\delta^i f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau)}{\delta}\right)\chi_i(\xi_1/\tau)\hat{f}(\xi, \tau)$$

where χ_1, χ_2 are functions supported on $[-1, 1]$.

LEMMA 4.3. *Let $0 < \delta \ll 1$ and let $|\gamma''| \geq c > 0$ on $I = [-1, 1]$. Then if $\text{dist}(\text{supp}\chi_1, \text{supp}\chi_2) \sim 1$, then for $5/3 < q \leq 2$,*

$$\|T_\delta^1 f T_\delta^2 g\|_q \leq C\delta\|f\|_2\|g\|_2,$$

where the constant C is stable under small smooth perturbations of γ .

PROOF. Let $\Gamma_i(\delta) = \{(\xi, \tau) : \xi_1/\tau \in \text{supp } \chi_i, \text{dist}((\xi, \tau), \Gamma_\gamma) \leq \delta\}$. Then we decompose $\Gamma_i(\delta)$ into a family of conic surfaces $\Gamma_i^s = \{(\xi, \tau) \in \Gamma_i(\delta) : \xi_2 = \tau\gamma(\xi_1/\tau) + s\}$. That is, we write

$$\Gamma_i(\delta) = \bigcup_{|s| \leq C\delta} \Gamma_i^s \quad \text{for } i = 1, 2.$$

Let $d\mu_i^s$ be the surface measure of Γ_i^s for $i = 1, 2$. Since $\text{dist}(\text{supp } \chi_1, \text{supp } \chi_2) \sim 1$, Theorem 4.1 gives that, for all s, u and $q > 5/3$,

$$\|(f d\mu_1^u)^\wedge (g d\mu_2^s)^\wedge\|_{L^q} \leq C\|f\|_{L^2(\Gamma_1^u, d\mu_1^u)}\|g\|_{L^2(\Gamma_2^s, d\mu_2^s)} \tag{4.3}$$

with C independent of s and u . We notice that the constant C is stable under small smooth perturbations, since the same is true for the bilinear restriction estimate.

Set $\tilde{f} = \phi(\tau)\chi_1(\xi_1/\tau)\hat{f}$ and $\tilde{g} = \phi(\tau)\chi_2(\xi_1/\tau)\hat{g}$. Let $\tilde{f}_s = \tilde{f}|_{\Gamma_1^s}$ and $\tilde{g}_s = \tilde{g}|_{\Gamma_2^s}$. Since \tilde{f}, \tilde{g} are supported in $\bigcup_{|s| \leq C\delta} \Gamma_1^s, \bigcup_{|s| \leq C\delta} \Gamma_2^s$, respectively, by the change of variables $\xi_2 = s + \tau\gamma(\xi_1/\tau)$ we see that

$$\begin{aligned} T_\delta^1 f(x, t) &= \int e^{2\pi i(x_1 \xi_1 + t\tau + x_2(s + \tau\gamma(\xi_1/\tau)))} \psi(s/\delta) \tilde{f}_s(\xi, \tau) d\xi_1 d\tau ds \\ &= \int \psi(s/\delta) (\tilde{f}_s d\mu_1^s)^\wedge ds \end{aligned}$$

and

$$T_\delta^2 g(x, t) = \int \psi(s/\delta)(\tilde{g}_s d\mu_2^s)^\wedge ds.$$

Hence, it follows that

$$\|T_\delta^1 f T_\delta^2 g\|_q^q \leq \iint \left| \iint \psi(u/\delta)\psi(s/\delta)(\tilde{f}_u d\mu_1^u)^\wedge(\tilde{g}_s d\mu_2^s)^\wedge du ds \right|^q dx dt.$$

Applying Hölder’s inequality to $\int \psi(u/\delta)(\tilde{f}_u d\mu_1^u)^\wedge du$ and $\int \psi(s/\delta)(\tilde{g}_s d\mu_2^s)^\wedge ds$, respectively, and using (4.3) yield that

$$\begin{aligned} \|T_\delta^1 f T_\delta^2 g\|_q^q &\leq C\delta^{2q-2} \iiint |(\tilde{f}_u d\mu_1^u)^\wedge(\tilde{g}_s d\mu_2^s)^\wedge|^q dx dt |\psi(u/\delta)\psi(s/\delta)| du ds \\ &\leq C\delta^{2q-2} \iint \|\tilde{f}_u\|_{L^2(\Gamma_1^q)}^q \|\tilde{g}_s\|_{L^2(\Gamma_2^q)}^q |\psi(u/\delta)\psi(s/\delta)| du ds. \end{aligned}$$

Since $q \leq 2$, applying Hölder’s inequality again to the last inequality,

$$\|T_\delta^1 f T_\delta^2 g\|_q^q \leq C\delta^q \|f\|_2^q \|g\|_2^q.$$

This completes the proof of the lemma. □

4.1. Proof of Proposition 2.1. We begin by defining $I_{a,r}$ to be an interval centered at a with length r . Let us set

$$\Lambda_{a,r} = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \xi_1/\tau \in I_{a,r}, 1 \leq \tau \leq 2\}. \tag{4.4}$$

Decomposing $\Lambda_{0,1}$ into small cubes, we may assume that the support of \hat{f} is contained in $\Lambda_{a,\epsilon}$ with $0 < \epsilon \ll 1$.

Observe that the change of variables $\xi_1 \rightarrow \xi_1 + \tau a$ for $U_\delta f$ gives

$$U_\delta f(x, t) = \int e^{2\pi i(L(x,t) \cdot (\xi, \tau))} \phi(\tau) \psi\left(\frac{\xi_2 - \tau\gamma(\xi_1/\tau + a)}{\delta}\right) \chi_{I_{a,\epsilon}}(\xi_1/\tau + a) \hat{f}(\xi, \tau) d\xi d\tau$$

where $L(x, t) = (x, t + ax_1)$. Since the translation $(x, t) \rightarrow (x, t - ax_1)$ does not affect the boundedness of $U_\delta f$ and $|\gamma''| \geq c > 0$ on $I = [-1, 1]$, it is sufficient to consider the case $\Lambda_{0,\epsilon}$. Hence, we may assume that \hat{f} is supported in $\Lambda_{0,\epsilon}$.

We utilize a Whitney type decomposition (see [26]). Let j_0 be the integer satisfying $2^{-j_0} < 4\sqrt{\delta} \leq 2^{-j_0+1}$. For each $j, 0 \leq j \leq j_0$, we divide the interval $I = [-1, 1]$ into 2^{j+1} disjoint dyadic intervals I_k^j of length 2^{-j} . When $j < j_0$, we write $I_k^j \sim I_{k'}^j$ to mean that I_k^j and $I_{k'}^j$ are not adjacent but have adjacent parent intervals of length 2^{-j+1} . So $\text{dist}(I_k^j, I_{k'}^j) \sim 2^{-j}$. When $j = j_0$, we write $I_k^j \sim I_{k'}^j$ to mean $\text{dist}(I_k^j, I_{k'}^j) \leq 2^{-j_0} \sim \sqrt{\delta}$. By adapting the idea of Whitney type decomposition of $I \times I$ away from the diagonal $I \times I$, we may write

$$I \times I = \bigcup_{0 \leq j \leq j_0} \bigcup_{I_k^j \sim I_{k'}^j} I_k^j \times I_{k'}^j. \tag{4.5}$$

Let f_k^j be given by

$$\hat{f}_k^j(\xi, \tau) = \chi_{I_k^j}(\xi_1/\tau)\hat{f}(\xi, \tau),$$

where $\chi_I \in C_0^\infty$ with $\chi = 1$ on the interval I . From (4.5), it follows that

$$U_\delta f \cdot U_\delta f = \sum_{0 \leq j \leq j_0} \sum_{I_k^j \sim I_{k'}^j} U_\delta f_k^j(x, t) U_\delta f_{k'}^j(x, t).$$

For each j , we define a bilinear operator B_j by

$$B_j(f, g)(x, t) = \sum_{I_k^j \sim I_{k'}^j} U_\delta f_k^j(x, t) U_\delta g_{k'}^j(x, t).$$

Then,

$$(U_\delta f)^2 = \sum_{0 \leq j \leq j_0} B_j(f, f). \tag{4.6}$$

So, we want to obtain the operator norm of B_j from $L^p \times L^p$ to $L^{q/2}$.

First, we consider L^∞ estimates. More precisely, we claim that

$$\|B_j(f, g)\|_\infty \leq C 2^{-j} \delta^{-1} \|f\|_\infty \|g\|_\infty. \tag{4.7}$$

Let $\omega \in C_0^\infty$ be supported in $(-1, 1)$ so that $\sum_{\nu \in \mathbb{Z}} \omega(\cdot - \nu) \equiv 1$. We set

$$\hat{f}_{k,\nu}^j(\xi, \tau) = \omega\left(\frac{\xi_1/\tau - \nu}{\sqrt{\delta}}\right) \hat{f}_k^j(\xi, \tau)$$

where $\nu \in \sqrt{\delta}\mathbb{Z}$. Since $(U_\delta f_{k,\nu}^j)^\wedge$ is supported in a cube of size $1 \times \delta^{1/2} \times \delta$, from the kernel estimates, it is easy to see that for $p \leq q$,

$$\|U_\delta f_{k,\nu}^j\|_q \leq C \delta^{3/2(1/p-1/q)} \|f\|_p. \tag{4.8}$$

Moreover, the number of ν is about $2^{-j} \delta^{-1/2}$ for each j because $|I_k^j| \sim 2^{-j} \gtrsim \sqrt{\delta}$. Therefore, for $0 \leq j \leq j_0$,

$$\|U_\delta f_k^j\|_\infty \leq C \sum_\nu \|U_\delta f_{k,\nu}^j\|_\infty \leq C 2^{-j} \delta^{-1/2} \|f\|_\infty. \tag{4.9}$$

Since the number of k' associated with k is at most four, the number of pairs (k, k') is about 2^j . Then we obtain, for $0 \leq j \leq j_0$,

$$\begin{aligned} \|B_j(f, g)\|_\infty &\leq C \sum_{I_k^j \sim I_{k'}^j} \|U_\delta f_k^j U_\delta g_{k'}^j\|_\infty \\ &\leq C 2^j (2^{-j} \delta^{-1/2})^2 \|f\|_\infty \|g\|_\infty. \end{aligned}$$

This gives the desired estimates.

Next, we consider the following lemma which is the main estimate of this section.

LEMMA 4.4. *Suppose that $1/p + 1/q \geq 1/2$, $2p \geq q > 5p/3$, $p \geq 2$ and $q \geq 4$. Then there is a constant C , independent of j, δ , such that for $0 \leq j \leq j_0$,*

$$\|B_j(f, g)\|_{L^{q/2}} \leq C\delta^{4/p-1} 2^{2j(1/p+3/q-1)} \|f\|_{L^p} \|g\|_{L^p}. \tag{4.10}$$

Assuming this for the moment, we give the proof of Proposition 2.1. When $1/p + 3/q < 1$, interpolation between (4.7) and (4.10) gives

$$\|B_j(f, g)\|_{L^{q/2}} \leq C2^{-\epsilon_0 j} \delta^{4/p-1} \|f\|_p \|g\|_p$$

for some $\epsilon_0 > 0$. Summing in j and using (4.6), we have, for $1/p + 3/q < 1$ and $2p \geq q > 5p/3$,

$$\|U_\delta f\|_{L^q(\mathbb{R}^3)} \leq C\delta^{2/p-1/2} \|f\|_{L^p(\mathbb{R}^3)}. \tag{4.11}$$

When $1/p + 3/q = 1$, we use Lemma 2.5 as before. Indeed, choose $(1/p, 1/q_1)$ and $(1/p, 1/q_2)$ satisfying $1/p + 3/q_i = 1$ for $i = 1, 2$ and $1/p + 3/q_1 < 1 < 1/p + 3/q_2$. Then by (4.10), we obtain for $i = 1, 2$,

$$\|B_j(f, g)\|_{L^{q_i/2}} \leq C\delta^{4/p-1} 2^{2j(1/p+3/q_i-1)} \|f\|_{L^p} \|g\|_{L^p}.$$

Using Lemma 2.5 and (4.6), we see that, for $1/p + 3/q = 1$ and $2p > q > 5p/3$,

$$\|U_\delta f\|_{q,\infty} \leq C\delta^{2/p-1/2} \|f\|_{p,1}.$$

By real interpolation, these restricted weak-type estimates can be strengthened to strong type. From this and (4.11), we get, for $1/p + 3/q \leq 1$, $2p \geq q > 5p/3$ and $(p, q) \neq (5/2, 5)$,

$$\|U_\delta f\|_{L^q(\mathbb{R}^3)} \leq C\delta^{2/p-1/2} \|f\|_{L^p(\mathbb{R}^3)}. \tag{4.12}$$

On the other hand, it is well known that the $L^2 \rightarrow L^p$ adjoint restriction estimate holds for the cone-type operator in \mathbb{R}^3 . Since $\gamma'' \neq 0$, at least one of the principal curvatures is nonzero at each point of the cone Γ_γ . Thus, for $p \geq 6$,

$$\|(fd\mu)\|_{L^p(\mathbb{R}^3)} \leq C\|f\|_{L^2(d\mu)}$$

where $d\mu$ is the surface measure on Γ_γ (see [22, pp. 365–367]). From this and an argument similar to one used to prove Lemma 4.3, we obtain, for $p \geq 6$,

$$\|U_\delta f\|_p \leq C\delta^{1/2} \|f\|_2.$$

Interpolating this with (4.12) finishes the proof of Proposition 2.1.

LEMMA 4.5. *Suppose that, for $p \geq 2$, $q \geq 4$ and $1/p + 1/q \geq 1/2$, there is a constant B , independent of j, δ, I_k^j and $I_{k'}^j$, such that for $I_k^j \sim I_{k'}^j$,*

$$\|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_{L^{q/2}(\mathbb{R}^3)} \leq B\|f_k^j\|_{L^p(\mathbb{R}^3)} \|g_{k'}^j\|_{L^p(\mathbb{R}^3)}. \tag{4.13}$$

Then there is a constant C , independent of j and δ , such that

$$\|B_j(f, g)\|_{L^{q/2}(\mathbb{R}^3)} \leq CB\|f\|_{L^p(\mathbb{R}^3)} \|g\|_{L^p(\mathbb{R}^3)}.$$

PROOF. Note that the supports of \hat{f} and \hat{g} are contained in $\Lambda_{0,\epsilon}$ defined by (4.4). For fixed j , if $I_k^j \sim I_{k'}^j$, the supports of the Fourier transforms of $U_\delta f_k^j \cdot U_\delta g_{k'}^j$ are contained in the set

$$\{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \text{dist}(\xi_1/\tau, I_k^j) \leq C2^{-j}, |\xi_2| \leq C, \tau \sim 1\}.$$

From this, we can easily see that the Fourier transforms of $\{U_\delta f_k^j \cdot U_\delta g_{k'}^j\}_{I_k^j \sim I_{k'}^j}$ are supported in essentially disjoint cubes. By using Plancherel’s theorem and a standard argument (see [25, Lemma 7.1]), we have, for $q/2 \geq 2$,

$$\|B_j(f, g)\|_{L^{q/2}} \leq C \left(\sum_{I_k^j \sim I_{k'}^j} \|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_{L^{q/2}}^{(q/2)'} \right)^{1-2/q}.$$

By assumption (4.13),

$$\|B_j(f, g)\|_{L^{q/2}} \leq CB \left(\sum_{I_k^j \sim I_{k'}^j} \|f_k^j\|_p^{(q/2)'} \|g_{k'}^j\|_p^{(q/2)'} \right)^{1-2/q}. \tag{4.14}$$

Since the number of $I_{k'}^j$ satisfying $I_k^j \sim I_{k'}^j$ is at most four, using the Schwarz inequality,

$$\sum_{I_k^j \sim I_{k'}^j} \|f_k^j\|_p^{(q/2)'} \|g_{k'}^j\|_p^{(q/2)'} \leq C \left(\sum_k \|f_k^j\|_p^{2(q/2)'} \right)^{1/2} \left(\sum_k \|g_k^j\|_p^{2(q/2)'} \right)^{1/2}.$$

From the condition $1/p + 1/q \geq 1/2$ and $I^r \subset I^s$ for $r \leq s$, we see that the right-hand side of (4.14) is bounded by $C(\sum_k \|f_k^j\|_p^p)^{1/p} (\sum_k \|g_k^j\|_p^p)^{1/p}$.

Now it is sufficient to show that, for $p \geq 2$,

$$\left(\sum_k \|f_k^j\|_p^p \right)^{1/p} \leq C \|f\|_p. \tag{4.15}$$

By Plancherel’s theorem, it follows that $(\sum_k \|f_k^j\|_2^2)^{1/2} \leq C \|f\|_2$. From the fact that the support of \hat{f} is contained in $\Lambda_{0,\epsilon}$, for sufficiently small ϵ , we obtain $\sup_k \|f_k^j\|_\infty \leq C \|f\|_\infty$. Inequality (4.15) follows from interpolation between the above two estimates. \square

4.2. Proof of Lemma 4.4. From Lemma 4.5, it is sufficient to consider $U_\delta f_k^j \cdot U_\delta g_{k'}^j$ when $I_k^j \sim I_{k'}^j$. More precisely, we claim that for $1/p + 1/q \geq 1/2, 2p \geq q > 5p/3, p \geq 2$ and $q \geq 4$,

$$\|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_{L^{q/2}} \leq C \delta^{4/p-1} 2^{2j(1/p+3/q-1)} \|f_k^j\|_{L^p} \|g_{k'}^j\|_{L^p}$$

with C independent of j, δ, I_k^j and $I_{k'}^j$.

We first handle the case $j = j_0$. Since $2^{-j_0} \sim \sqrt{\delta}$, it is sufficient to show

$$\|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_{q/2} \leq C \delta^{3(1/p-1/q)} \|f_k^j\|_p \|g_{k'}^j\|_p.$$

However, this is an easy consequence of Hölder’s inequality and (4.8).

We now turn to the case $0 < j < j_0$. This case follows from interpolation between the following two estimates:

$$\|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_\infty \leq C2^{-2j}\delta^{-1}\|f_k^j\|_\infty\|g_{k'}^j\|_\infty, \tag{4.16}$$

$$\|U_\delta f_k^j \cdot U_\delta g_{k'}^j\|_r \leq C\delta 2^{j(3/r-1)}\|f_k^j\|_2\|g_{k'}^j\|_2 \tag{4.17}$$

for $5/3 < r \leq 2$. From (4.9), we obtain (4.16).

To show (4.17), we use the bilinear cone restriction estimates for conic surfaces. Let $a \in \mathbb{R}$ be the center of the smallest interval containing both I_k^j and $I_{k'}^j$. Observe that by translation $\xi_1 \rightarrow \xi_1 + \tau a$, the intervals I_k^j and $I_{k'}^j$ are moved to I_1 and I_2 , respectively. Here, I_1 and I_2 are intervals contained in $I_{0,2^{1-j}}$ with $|I_1| \sim 2^{-j} \sim |I_2|$ and $\text{dist}(I_1, I_2) \sim 2^{-j}$. Recall that $I_{a,r}$ is an interval centered at a with length r . Let us set

$$(U_{\delta,a}f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma(a, \xi_1/\tau)}{\delta}\right)\hat{f}(\xi, \tau).$$

And for $i = 1, 2$, we set $\Lambda_i = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \xi_1/\tau \in I_i, 1 \leq \tau \leq 2\}$. Then by the affine map $(\xi, \tau) \rightarrow L_a(\xi, \tau) = (\xi_1 + \tau a, \xi_2 + \gamma'(a)\xi_1 + \tau\gamma(a), \tau)$, if $\text{supp } \hat{f} \subset \Lambda_1$ and $\text{supp } \hat{g} \subset \Lambda_2$, we see that

$$U_\delta f_k^j \cdot U_\delta g_{k'}^j(x, t) = U_{\delta,a}f \cdot U_{\delta,a}g(x_1 + \gamma'(a)x_2, x_2, t + ax_1 + \gamma(a)x_2).$$

Since the change of variables $(x_1, x_2, t) \rightarrow (x_1 - \gamma'(a)x_2, x_2, t - ax_1 - \gamma(a)x_2)$ in (x, t) space does not affect the boundedness, it is sufficient to show that if $\text{supp } \hat{f} \subset \Lambda_1$ and $\text{supp } \hat{g} \subset \Lambda_2$, then for $5/3 < r \leq 2$,

$$\|U_{\delta,a}f \cdot U_{\delta,a}g\|_r \leq C\delta 2^{j(3/r-1)}\|f\|_2\|g\|_2 \tag{4.18}$$

with C independent of a, j, I_k^j and $I_{k'}^j$.

Set $f_j(x, t) = f(2^j x_1, 2^{2j} x_2, t)$ and $g_j(x, t) = g(2^j x_1, 2^{2j} x_2, t)$. Let \tilde{I}_1, \tilde{I}_2 be intervals contained in $I_{0,2}$ with $|\tilde{I}_1| \sim 1 \sim |\tilde{I}_2|$ and $\text{dist}(\tilde{I}_1, \tilde{I}_2) \sim 1$. Observe that \hat{f}_j, \hat{g}_j are supported in the sets $\tilde{\Lambda}_1, \tilde{\Lambda}_2$, respectively, where $\tilde{\Lambda}_i = \{(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R} : \xi_1/\tau \in \tilde{I}_i, 1 \leq \tau \leq 2\}$. Let χ_1, χ_2 be smooth functions supported in $I_{0,2}$ satisfying $\chi_i = 1$ on \tilde{I}_i for $i = 1, 2$ and $\text{dist}(\text{supp } \chi_1, \text{supp } \chi_2) \sim 1$. Define $U_{\lambda,a}^j f$ and $U_{\lambda,a}^j g$ by

$$(U_{\lambda,a}^j f)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma_{2^{-j}}(a, \xi_1/\tau)}{\lambda}\right)\chi_1(\xi_1/\tau)\hat{f}(\xi, \tau),$$

$$(U_{\lambda,a}^j g)^\wedge(\xi, \tau) = \phi(\tau)\psi\left(\frac{\xi_2 - \tau\gamma_{2^{-j}}(a, \xi_1/\tau)}{\lambda}\right)\chi_2(\xi_1/\tau)\hat{g}(\xi, \tau).$$

By rescaling $U_{\delta,a}f$ and $U_{\delta,a}g$ by $(\xi_1, \xi_2, \tau) \rightarrow (2^{-j}\xi_1, 2^{-2j}\xi_2, \tau)$ in frequency space, we see that

$$(U_{\delta,a}f \cdot U_{\delta,a}g)(x, t) = (U_{\delta 2^{2j},a}^j f_j \cdot U_{\delta 2^{2j},a}^j g_j)(2^{-j}x_1, 2^{-2j}x_2, t). \tag{4.19}$$

Recall that \hat{f} is supported in $\Lambda_{0,\epsilon}$ defined by (4.4). Then we may assume that $2^{-j} \ll 1$ and $|a| \ll 1$. Since γ is of finite type 2 at a , from Lemma 2.3, $|\gamma_{2^{-j}}''(a, \xi_1/\tau)| \geq c > 0$

uniformly in a , j on the support of χ_i for $i = 1, 2$. Therefore, applying Lemma 4.3 to $U_{\lambda,a}^j f \cdot U_{\lambda,a}^j g$, we see that there is a uniform constant C , independent of a, j , such that for $0 < \lambda \ll 1$ and $5/3 < r \leq 2$,

$$\|U_{\lambda,a}^j f \cdot U_{\lambda,a}^j g\|_r \leq C\lambda \|f\|_2 \|g\|_2.$$

Since $0 < \delta 2^{2j} \ll 1$ for $0 < j < j_0$, by applying the above inequality to (4.19) and rescaling, we get (4.18). More precisely,

$$\begin{aligned} \|U_{\delta,a} f \cdot U_{\delta,a} g\|_r &= 2^{3j/r} \|U_{\delta 2^{2j},a}^j f_j \cdot U_{\delta 2^{2j},a}^j g_j\|_r \\ &\leq C 2^{3j/r} \delta 2^{2j} \|f_j\|_2 \|g_j\|_2 \\ &\leq C 2^{3j/r} \delta 2^{2j} 2^{-3j} \|f\|_2 \|g\|_2. \end{aligned}$$

This completes the proof of Lemma 4.4. \square

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