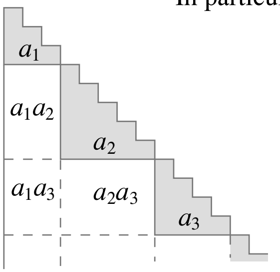
108.37 A parent figure for a family of triangle number identities

$$T_{a_1 + a_2 + a_3 + \dots + a_n} = T_{a_1} + T_{a_2} + T_{a_2} + \dots + T_{a_n} + \sum_{\substack{1 \leqslant i,j \leqslant n \\ i \neq j}} a_i a_j$$



In particular:

$$T_{ab} = aT_b + b^2 T_{a-1} = bT_a + a^2 T_{b-1} \quad [1, \text{p. } 102]$$

$$= aT_b + (T_{b-1} + T_b) T_{a-1}$$

$$= (a + T_{a-1}) T_b + T_{a-1} T_{b-1}$$

$$= T_{a-1} T_{b-1} + T_a T_b \quad [1, \text{p. } 101]$$

$$T_{a^2} = T_{a-1}^2 + T_a^2 \quad [1, \text{p. } 99]$$

Reference

1. R. B. Nelsen, *Proofs without Words II*, Mathematical Association of America (2000).

10.1017/mag.2024.121 © The Authors, 2024
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on behalf of The Mathematical Association

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108.38 A number whose square root is the sum of its digits

The number 81 has the property that its square root is equal to the sum of its digits: $\sqrt{81} = 8 + 1$.

Are there other numbers with the same property?

To answer this question, we, first of all, prove the following Lemma.

Lemma: If $n \geqslant 5$, then $10^{(n-1)/2} > 9n$.

Proof: We use mathematical induction.

For $n = 5$, (1) is true, since $10^{(5-1)/2} = 100 > 9 \times 5 = 45$.

Now we assume that (1) holds for $n = k$, so that $10^{(k-1)/2} > 9k$. Then

$$10^{k/2} = \sqrt{10} \times 10^{(k-1)/2} > 3 \times 9k = 27k = 9(k+1) + 9(2k-1). \qquad (1)$$

But $k \geqslant 5$, so $10^{k/2} > 9(k + 1)$, whence (1) holds for $n = k + 1$.

Now we prove the main Theorem.

Theorem: If A is a positive integer whose square root equals the sum of its digits, then $A = 1$ or $A = 81$.

Proof: Let A have n digits. Then $A \geqslant 10^{n-1}$, so $\sqrt{A} \geqslant 10^{(n-1)/2}$. If $n \geqslant 5$, then, by the Lemma, $\sqrt{A} > 9n$. But the sum of the digits of A is at most $9n$ (reached when each digit is 9). Thus if $n \geqslant 5$, $\sqrt{A}$ exceeds the sum of the digits of A.

If $n = 4$, the digit sum of A is at most $9 \times 4 = 36$, so if $A > 1296 = 36^2$, then $\sqrt{A}$ exceeds the sum of the digits of A. But if $A \leqslant 1296$, then the sum of its digits is less than $1 + 2 + 9 + 9 = 21$, yet $\sqrt{A} \geqslant \sqrt{1000} > 21$.

It remains to consider the case $n \leqslant 3$. Now, by direct verification, it is easy to find that there are only two numbers, 1 and 81, that satisfy the problem:

number	$(\text{number})^2$	sum of digits	number	$(\text{number})^2$	sum of digits
1	1	1	16	256	13
2	4	4	17	289	19
3	9	9	18	324	9
4	16	7	19	361	10
5	25	7	20	400	4
6	36	9	21	441	9
7	49	13	22	484	16
8	64	10	23	529	16
9	81	9	24	576	18
10	100	1	25	625	13
11	121	4	26	676	19
12	144	9	27	729	18
13	169	16	28	784	19
14	196	16	29	841	13
15	225	9	30	900	9
			31	961	16

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108.39 A quick proof that π is less than 2ϕ

The golden ratio ϕ is $\frac{1}{2}(1 + \sqrt{5})$ and $\tan\frac{1}{12}\pi = 2 - \sqrt{3}$. The aim of this Note is to give a quick proof of the well known inequality $\pi < 2\phi$. Our proof is more elementary than Nelsen ([1]).

This proof uses the familiar inequality $\sin x < x < \tan x$ for $0 < x < \frac{1}{2}\pi$. An alternative to the standard proof is given by the following diagram: