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INTERSECTION COHOMOLOGY OF RANK 2 CHARACTER VARIETIES OF SURFACE GROUPS

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Abstract For $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$ we compute the intersection E-polynomials and the intersection Poincaré polynomials of the G-character variety of a compact Riemann surface C and of the moduli space of G-Higgs bundles on C of degree zero. We derive several results concerning the P=W conjectures for these singular moduli spaces.

Keywords: intersection cohomology; Higgs bundles; P=W conjecture; isosingularity

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1. Introduction

Let C be a compact Riemann surface of genus $g \geq 2$ and G be a complex reductive algebraic group. The G-character variety of C, or **Betti moduli space**, is the affine GIT quotient

$$M_{\mathrm{B}}(C,G) := \mathrm{Hom}(\pi_1(C),G) /\!\!/ G$$
$$= \Big\{ (A_1,B_1,\dots,A_g,B_g) \in G^{2g} \,\Big|\, \prod_{j=1}^{g}[A_j,B_j] = 1_G \Big\} /\!\!/ G. \tag{1}$$

It parametrises isomorphism classes of semi-simple representations of the fundamental group of C with value in G. Remarkably, the space $M_{\mathrm{B}}(C,G)$ is homeomorphic to the **Dolbeault moduli space** $M_{\mathrm{Dol}}(C,G)$, parametrising instead isomorphism classes of semistable principal G-Higgs bundles on C of degree zero; see [66]. For example, a GL_n-Higgs bundle is a pair (E,ϕ) with E vector bundle of rank n and $\phi \in H^0(C,\mathrm{End}(E)\otimes K)$. Such a pair is an SL_n-Higgs bundle if in addition the determinant of E is trivial and the trace of ϕ vanishes.

Since the pioneering work of Hitchin [37], the (nonalgebraic) homeomorphism between $M_{\mathrm{B}}(C,G)$ and $M_{\mathrm{Dol}}(C,G)$ has been exploited to study the topology of character varieties. The main result of this article is the computation of some geometric invariants of $M_{\mathrm{Dol}}(C,G)$ and $M_{\mathrm{B}}(C,G)$, namely, the intersection E-polynomials and the intersection Poincaré polynomials of $M_{\mathrm{Dol}}(C,G)$ and $M_{\mathrm{B}}(C,G)$ for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$; see §1.2.

The motivation for this work stems from the newly stated P=W conjectures [26, Conjecture 1.2, 1.4, 1.5] for singular character varieties; see also [15, Question 4.1.7] and the seminal paper [13]. In fact, the explicit knowledge of intersection E-polynomials and intersection Poincaré polynomials is an essential ingredient in the proof of the P=W conjectures in rank 2 and genus 2 in [26, Main Theorem]. Here, as an applications of our computations, we collect in Theorem 1.1 several results related to the P=W conjectures in rank 2 and arbitrary genus. For brevity, we simply write M_{B} for $M_{\mathrm{B}}(C,G)$ and M_{Dol} for $M_{\mathrm{Dol}}(C,G)$ when $G = \mathrm{GL}_2$ or SL_2, and we suppress subscripts $_{\mathrm{B}}$ or $_{\mathrm{Dol}}$ when we refer indifferently to the Betti or the Dolbeault side.

Theorem 1.1 (Remarks on the P=W conjectures). *The following facts hold:*

A. *the intersection E-polynomial $IE(M_{\mathrm{B}})$ is palindromic;*
B. *the PI=WI conjecture for* SL_2 *is equivalent to the PI=WI conjecture for* GL_2*;*

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C. $IH^{<4g-6}(M)$ *is freely generated by tautological classes;*

D. *the mixed Hodge structure on* $H^*(M_{\mathrm{Dol}})$ *is not pure for* $g > 3$*;*

E. *the P=W conjecture for any resolution of* $M(C, \mathrm{GL}_n)$ *fails when* $M(C, \mathrm{GL}_n)$ *does not admit a symplectic resolution.*[1]

In Section 5 we recall the content of the P=W conjectures and we give a proof of Theorem 1.1. Here we briefly explain the relevance of the previous statements in view of the P=W conjectures.

A. Theorem 1.1.A (Theorem 5.4) provides numerical evidence for the PI=WI conjecture. Indeed, the PI=WI conjecture implies the palindromicity of $IE(M_B)$.

B. Theorem 1.1.B (Corollary 5.6) is a useful reduction statement. It says that it is enough to prove the PI=WI conjecture only for a portion of the intersection cohomology, namely, its Γ-invariant part; see (5) and (6).

C. The known proofs of the P=W conjecture for twisted character varieties [13] and [16] (cf. also Subsection 5.1) rely on the generation by tautological classes of the Γ-invariant part of $H^*(M)$. This is unknown for the intersection cohomology of the singular moduli spaces. Theorem 1.1.C (Theorem 5.7) provides a partial answer; that is, the tautological generation of the intersection cohomology in low degree.

D&E. Theorem 1.1.D and 1.1.E stress the difference between the P=W conjectures for character varieties with or without a symplectic resolution; see Subsection 5.5, Subsection 5.6, and also [26].

Our strategy to compute the intersection E-polynomials of M is to use the Kirwan–O'Grady desingularisation $\pi_T\colon T \to M$ (Subsection 3.2) and determine all of the summands of the decomposition theorem for π_T; cf. [44, Remark 2.28]. This is a subtle task that we can complete thanks to a tight control of the geometry of π_T.

Theorem 1.2 (Decomposition theorem for π_T). *There is an isomorphism in* $D^b MHM_{\mathrm{alg}}(M)$ *or in* $D^b(M)$ *(ignoring the Tate shifts)*

$$\begin{aligned} R\pi_{T,*}\mathbb{Q}[\dim T] = IC_M &\oplus \bigoplus_{i=-2g+4}^{2g-4} \mathbb{Q}_{\Sigma}^{\lceil \frac{2g-3-|i|}{2} \rceil}[\dim \Sigma - 2i](-2g+3-i) \\ &\oplus \bigoplus_{i=-2g+4}^{2g-4} i_{\Sigma^\circ,*}\mathscr{L}^{\lfloor \frac{2g-3-|i|}{2} \rfloor}[\dim \Sigma - 2i](-2g+3-i) \\ &\oplus \bigoplus_{j=-3g+4}^{3g-4} \mathbb{Q}_{\Omega}^{b(j)}[\dim \Omega - 2j](-3g+3-j), \end{aligned}$$

[1] A resolution of singularities $f\colon X \to Y$ is symplectic if a holomorphic symplectic form on the smooth locus of X extends to a symplectic form on Y.

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where

- Σ *is the singular locus of* M*;*
- Ω *is the singular locus of* Σ*;*
- $i_{\Sigma^\circ}\colon \Sigma^\circ := \Sigma \setminus \Omega \to \Sigma$ *is the natural inclusion;*
- $\mathscr{L}$ *is the rank-1 local system on* Σ° *corresponding to a quasi-étale double cover* $q\colon \Sigma_\iota \to \Sigma$ *branched along* Ω *(see Definition 3.3);*
- $b(-3g+3+j)$ *is the coefficient of the monomial* q^j *in the polynomial*

$$\frac{\left(1-q^{2g-2}\right)\left(1-q^{2g}\right)\left(1-q^4-q^{2g-3}-q^{2g-1}+2q^{2g}\right)}{(1-q)^3\left(1-q^2\right)}-\frac{1-q^{2g}}{1-q^2}$$
$$-\frac{q(1-q^{2g-3})(1-q^{2g-2})}{(1-q)(1-q^2)}.$$

Note in addition that the same decomposition holds for the Mukai moduli space of semistable sheaves on K3 or abelian surfaces with Mukai vector $v = 2w \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$, where w is primitive and $w^2 = 2(g-1)$, thus suggesting other potential applications of Theorem 1.2. This is indeed a consequence of the **stable isosingularity principle** (Theorem 2.11), which roughly says that Betti, Dolbeault and Mukai moduli spaces have the same type of singularities, in the sense of Definition 2.6.

It is conceivable that the computation of the intersection E-polynomials in rank 3 can be pursued with no substantial conceptual difference. In higher rank, however, this seems hard. In fact, closed formulas may be cumbersome and less enlightening. Notwithstanding, we believe that the rank 2 case can inspire the investigation of the higher rank case, especially in relation to the P=W conjectures [26] and the Hausel–Thaddeus topological mirror symmetry conjecture for singular character varieties [32, Remark 3.30].

1.1. Notation

The intersection cohomology of a complex variety X with middle perversity and rational coefficients is denoted by $IH^*(X)$. Ordinary singular cohomology with rational coefficients is denoted by $H^*(X)$. The subscript c stands for compactly supported intersection or ordinary cohomology, respectively $IH^*_c(X)$ and $H^*_c(X)$. Recall that they all carry mixed Hodge structures.

The Poincaré polynomial, the intersection Poincaré polynomial, the intersection Euler characteristic, the E-polynomial and the intersection E-polynomial are defined by

$$P_t(X) = \sum_d \dim H^d(X) t^d,$$

$$IP_t(X) = \sum_d \dim IH^d(X) t^d,$$

$$I\chi(X) = \sum_d (-1)^d \dim IH^d(X),$$

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$$E(X) = \sum_{r,s,d} (-1)^d \dim(\mathrm{Gr}^W_{r+s} H^d_c(X,\mathbb{C}))^{r,s} u^r v^s,$$
$$IE(X) = \sum_{r,s,d} (-1)^d \dim(\mathrm{Gr}^W_{r+s} IH^d_c(X,\mathbb{C}))^{r,s} u^r v^s.$$

We will often write $q := uv$.

The action of a finite group Γ on X induces the splitting

$$H^d(X) = H^d(X)^\Gamma \oplus H^*_{\mathrm{var}}(X),$$

where $H^d(X)^\Gamma$ is fixed by the action of Γ and $H^d_{\mathrm{var}}(X)$ is the variant part; that is, the unique Γ-invariant complement of $H^d(X)^\Gamma$ in $H^d(X)$. Analogous splittings hold for the Γ-modules $H^d_c(X)$, $IH^d(X)$ and $IH^d_c(X)$. The label $^\Gamma$ or $_{\mathrm{var}}$, written after the polynomials above, imposes to replace ordinary (or intersection) cohomology with its Γ-invariant or Γ-variant part respectively; for example, $IE(X)^\Gamma = \sum_{r,s,d}(-1)^d \dim(\mathrm{Gr}^W_{r+s} IH^d_c(X,\mathbb{C})^\Gamma)^{r,s} u^r v^s$.

If $\iota\colon X \to X$ is an involution, we simply use the superscript $+$ or $-$ to denote the ι-invariant and ι-variant part; for example, $P(X)^+ = \sum_d \dim H^d(X)^+ t^d := P(X)^{\langle \iota \rangle}$.

We always denote by C a complex projective curve of genus $g \geq 2$, unless differently stated. For notational convenience, we simply write M_{B} for $M_{\mathrm{B}}(C,G)$ and M_{Dol} for $M_{\mathrm{Dol}}(C,G)$ when $G = \mathrm{GL}_2$ or SL_2, and we suppress subscripts $_{\mathrm{B}}$ or $_{\mathrm{Dol}}$ when we refer indifferently to the Betti or the Dolbeault side. We adopt the same convention for the strata $\Sigma_{\mathrm{B}}(C,G)$, $\Sigma_{\mathrm{Dol}}(C,G)$, $\Omega_{\mathrm{B}}(C,G)$, $\Omega_{\mathrm{Dol}}(C,G)$.

1.2. Computations

As an application of the decomposition theorem (Theorem 1.2), we can express $IE(M)$ as a function of the E-polynomials of M, Σ_ι and Ω; see Proposition 3.2 for the definition of these strata and Subsection 4.1 for the proofs of the following expressions.

Theorem 1.3.

$$IE(M) = E(M) + (q^2 E(\Sigma_\iota)^+ + qE(\Sigma_\iota)^-) \cdot \frac{1-q^{2g-4}}{1-q^2} + E(\Omega) \cdot q^{2g-2}. \tag{2}$$

Theorem 1.4. *The intersection E-polynomials of M_{B} are*

$$\begin{aligned} IE(M_{\mathrm{B}}(C,\mathrm{SL}_2)) =& (q^{2g-2}+1)(q^2-1)^{2g-2} + \frac{1}{2}q^{2g-3}(q^2+1)((q+1)^{2g-2} \\ &- (q-1)^{2g-2}) + 2^{2g-1}q^{2g-2}((q+1)^{2g-2} + (q-1)^{2g-2}), \\ IE(M_{\mathrm{B}}(C,\mathrm{PGL}_2)) =& (q^{2g-2}+1)(q^2-1)^{2g-2} + \frac{1}{2}q^{2g-3}(q^2+q+1)(q+1)^{2g-2} \\ &- \frac{1}{2}q^{2g-3}(q^2-q+1)(q-1)^{2g-2}, \\ IE(M_{\mathrm{B}}(C,\mathrm{GL}_2)) =& (q-1)^{2g} \cdot IE(M_{\mathrm{B}}(C,\mathrm{PGL}_2)). \end{aligned}$$

Corollary 1.5. *The intersection E-polynomials $IE(M_{\mathrm{B}})$ are palindromic.*

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Corollary 1.6. *The intersection Euler characteristics of M are*

$$I\chi(M(C,\mathrm{SL}_2)) = 2^{2g-2}(2^{2g-1}+1),$$
$$I\chi(M(C,\mathrm{PGL}_2)) = 3\cdot 2^{2g-3}.$$

We list the intersection E-polynomial of $M_{\mathrm{B}}(C,\mathrm{SL}_2)$ in low genus (we truncate the polynomial at degree $3g-3$: the coefficients of the monomials of higher degree can be determined by symmetry, since $IE(M_{\mathrm{B}}(C,\mathrm{SL}_2))$ is palindromic of degree $6g-6$):

$$\begin{aligned}
g=2: &\quad 1+17q^2+\cdots\\
g=3: &\quad 1-4q^2+75q^4+384q^6+\cdots\\
g=4: &\quad 1-6q^2+15q^4+243q^6+3875q^8+\cdots\\
g=5: &\quad 1-8q^2+28q^4-56q^6+1103q^8+28672q^{10}+71848q^{12}+\cdots.
\end{aligned}$$

Remark 1.7. The intersection E-polynomial of $M_{\mathrm{B}}(C,\mathrm{SL}_2)$ is a polynomial in q^2. This fails for twisted SL_2-character varieties (cf. [52, (2)]; see also Subsection 5.1 for the definition of the twist), but it holds true for twisted PGL_2-character varieties, since their cohomology is generated by classes of weight 4; see [34, Proposition 4.1.8]. The E-polynomial of $M_{\mathrm{B}}(C,\mathrm{SL}_2)$ is a polynomial in q^2, too; see [48, Theorem 2] or [4, Theorem 1.3].

Theorem 1.8. *The intersection E-polynomial of $M_{\mathrm{Dol}}(C,\mathrm{SL}_2)$ is*

$$\begin{aligned}
IE(M_{\mathrm{Dol}}(C,\mathrm{SL}_2)) &= E(M_{\mathrm{Dol}}(C,\mathrm{SL}_2)^{\mathrm{sm}}) + \frac{1}{2}(uv)^g\left((1-u)^g(1-v)^g+(1+u)^g(1+v)^g\right)\\
&\quad + \frac{1}{2}(uv)^{g+1}(1-(uv)^{2g-4})\left(\frac{(1-u)^g(1-v)^g}{1-uv} - \frac{(1+u)^g(1+v)^g}{1+uv}\right)\\
&\quad + 2^{2g}(uv)^{2g-2}.
\end{aligned}$$

An explicit formula for the E-polynomial of the smooth locus of $M_{\mathrm{Dol}}(C,\mathrm{SL}_2)$ was computed in [41, Theorem 3.7]. Together with Theorem 1.8 and Proposition 2.4, this gives the intersection Poincaré polynomial of $M(C,\mathrm{SL}_2)$.

Theorem 1.9. *The intersection Poincaré polynomial of $M(C,\mathrm{SL}_2)$ is*

$$\begin{aligned}
IP_t(M(C,\mathrm{SL}_2)) &= \frac{(t^3+1)^{2g}}{(t^2-1)(t^4-1)} + (g-1)t^{4g-3}\frac{(t+1)^{2g-2}}{t-1}\\
&\quad - \frac{t^{4g-4}}{4(t^2-1)(t^4-1)}\left((t^2+1)^2(t+1)^{2g}-(t+1)^4(t-1)^{2g}\right)\\
&\quad + \frac{1}{2}t^{4g-4}((t+1)^{2g-2}-(t-1)^{2g-2}) - \frac{1}{2}t^{4g-6}((t+1)^{2g}-(t-1)^{2g})\\
&\quad + \frac{1}{2}(2^{2g}-1)t^{4g-4}((t+1)^{2g-2}+(t-1)^{2g-2}).
\end{aligned}$$

Therefore, we have

$$IP_t(M(C,\mathrm{SL}_2)) = \frac{(t^3+1)^{2g}}{(t^2-1)(t^4-1)} - 2g\cdot t^{4g-5} + O(t^{4g-4}). \tag{3}$$

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We list the intersection Poincaré polynomial of $M_{\mathrm{B}}(C,\mathrm{SL}_2)$ in low genus:

$$\begin{aligned}
g=2:&\quad 1+t^2+17t^4+17t^6\\
g=3:&\quad 1+t^2+6t^3+2t^4+6t^5+17t^6+6t^7+81t^8+12t^9+396t^{10}+6t^{11}+66t^{12}\\
g=4:&\quad 1+t^2+8t^3+2t^4+8t^5+30t^6+16t^7+31t^8+72t^9+59t^{10}+72t^{11}+385t^{12}\\
&\quad +80t^{13}+3955t^{14}+80t^{15}+3885t^{16}+16t^{17}+259t^{18}\\
g=5:&\quad 1+t^2+10t^3+2t^4+10t^5+47t^6+20t^7+48t^8+140t^9+93t^{10}+150t^{11}\\
&\quad +304t^{12}+270t^{13}+349t^{14}+522t^{15}+1583t^{16}+532t^{17}+29414t^{18}+532t^{19}\\
&\quad +72170t^{20}+280t^{21}+28784t^{22}+30t^{23}+1028t^{24}.
\end{aligned}$$

The Poincaré polynomial of $M_{\mathrm{B}}(C,\mathrm{SL}_2)$ was obtained in [12, Theorem 1.5], which, however, contains small transcription errors (cf. [7, Theorem 2.2] and [10, (47)]).

$$\begin{aligned}
P_t(M(C,\mathrm{SL}_2)) &= \frac{(t^3+1)^{2g}}{(t^2-1)(t^4-1)}+\frac{(t+1)^{2g}(t^2+1)+(t-1)^{2g}(t^2-1)}{2(t^4-1)}\\
&\quad +\sum_{k=2}^{g}\left\{\binom{2g}{k}-\binom{2g}{k-2}\right\}t^{k+2\mathrm{Mod}[k,2]}\frac{(t^{2k-2\mathrm{Mod}[k,2]}-1)(t^{2g-2k+2}-1)}{(t-1)(t^4-1)}\\
&\quad -\frac{1}{2}t((t+1)^{2g}+(t-1)^{2g})+\frac{t^{2g+2}-1}{t-1}-t^{4g-4}+\frac{(t-1)^{2g}t^{4g-4}}{4(t^2+1)}\\
&\quad -\frac{(t+1)^{2g}t^{4g-4}}{2(t^2-1)}\left(\frac{2g}{t+1}+\frac{1}{t^2-1}-\frac{1}{2}+3-2g\right)\\
&\quad +\frac{1}{2}(2^{2g}-1)t^{4g-4}((t+1)^{2g-2}+(t-1)^{2g-2}-2),
\end{aligned}$$

where $\mathrm{Mod}[k,l]$ is the remainder on division of k by l. We can then inspect the difference $IP_t(M(C,\mathrm{SL}_2))-P_t(M(C,\mathrm{SL}_2))$ in low genus:

$$\begin{aligned}
g=2:&\quad 16t^4\\
g=3:&\quad 6t^3+t^4+6t^5+t^6+6t^7+79t^8+t^{10}\\
g=4:&\quad 8t^3+t^4+8t^5-20t^6+16t^7-19t^8+22t^9+56t^{10}+56t^{11}+327t^{12}\\
&\quad +8t^{13}+28t^{14}+t^{16}\\
g=5:&\quad 10t^3+t^4+10t^5-65t^6+20t^7-196t^8-35t^9-20t^{10}-25t^{11}+124t^{12}\\
&\quad +240t^{13}+256t^{14}+262t^{15}+1279t^{16}+120t^{17}+211t^{18}+10t^{19}+45t^{20}+t^{22}.
\end{aligned}$$

Corollary 1.10. *Let C be a curve of genus $g\geq 6$. Then we have*

$$IP_t(M)-P_t(M)=2g\cdot t^3+t^4+2g\cdot t^5-\left\{\binom{2g}{3}-\binom{2g}{2}-2g\right\}t^6+O(t^7).$$

At this point it is worth recalling how the (intersection) cohomology of $M(C,G)$ with $G=\mathrm{GL}_n,\mathrm{SL}_n$ and PGL_n compares and how to extend the previous results for the SL_2 case to PGL_2 and GL_2.

The morphism

$$\mathrm{alb}\colon M(C,\mathrm{GL}_n)\to M(C,\mathrm{GL}_1) \tag{4}$$

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sends a representation or a Higgs bundle to its associated determinant representation or Higgs bundle. It is an étale locally trivial fibration with monodromy group $\Gamma \simeq (\mathbb{Z}/2\mathbb{Z})^{2g}$ and fibre isomorphic to $M(C,\mathrm{SL}_n)$. The quotient of $M(C,\mathrm{SL}_n)$ for the residual action of Γ is $M(C,\mathrm{PGL}_n)$. Hence, there exist morphisms of mixed Hodge structures

$$H^*(M(C,\mathrm{GL}_n)) \simeq H^*(M(C,\mathrm{SL}_n))^{\Gamma} \otimes H^*(M(C,\mathrm{GL}_1)), \tag{5}$$

$$H^*(M(C,\mathrm{PGL}_n)) \simeq H^*(M(C,\mathrm{SL}_n))^{\Gamma}. \tag{6}$$

Analogous splittings hold for IH^*, H^*_c, IH^*_c. A proof of these facts can be found, for instance, in [26, §3.2].

The analogues of Theorems 1.8 and 1.9 for PGL_2 can be obtained by substituting all of the occurrences of the coefficient 2^{2g} with 1 in the formulas of the theorems; see Remark 4.3. According to (5), the corresponding polynomials for GL_2 are the product of the E-polynomials or the (intersection) Poincaré polynomial for PGL_2 with $E(T^*\operatorname{Jac}(C)) = (uv)^g(1-u)^g(1-v)^g$ or $P_t(\operatorname{Jac}(C)) = (t+1)^{2g}$, respectively.

Here, as a corollary of Theorems 1.4, 1.9 and (6), we study the portion of $IH^*(M(C,\mathrm{SL}_2))$ on which Γ acts nontrivially. The following should be considered the untwisted analogue of [35, Proposition 8.2] in rank 2. This suggests that intersection cohomology may be the right cohomology theory to formulate a topological mirror symmetry conjecture for $M(C,\mathrm{SL}_n)$ and $M(C,\mathrm{PGL}_n)$; see [32, Remark 3.30] and [49].

Corollary 1.11. *The variant intersection E-polynomial and Poincaré polynomials for the action of Γ are*

$$\begin{aligned}
IE_{\mathrm{var}}(M_{\mathrm{B}}(C,\mathrm{SL}_2)) &= \frac{1}{2}(2^{2g}-1)q^{2g-2}((q+1)^{2g-2}+(q-1)^{2g-2}),\\
IE_{\mathrm{var}}(M_{\mathrm{Dol}}(C,\mathrm{SL}_2)) &= \frac{1}{2}(2^{2g}-1)(uv)^{3g-3}((u+1)^{g-1}(v+1)^{g-1}+(u-1)^{g-1}(v-1)^{g-1}),\\
IP_{t,\mathrm{var}}(M(C,\mathrm{SL}_2)) &= \frac{1}{2}(2^{2g}-1)t^{4g-4}((t+1)^{2g-2}+(t-1)^{2g-2}).
\end{aligned}$$

In particular, $q^{-2g+2}IE_{\mathrm{var}}(M_{\mathrm{B}}(C,\mathrm{SL}_2))$ and $q^{-6g+6}IE_{\mathrm{var}}(M_{\mathrm{Dol}}(C,\mathrm{SL}_2);q,q)$ are palindromic polynomials of degree $2g-2$.

As a byproduct, we also obtain the E-polynomials and the Poincaré polynomial of the Kirwan–O'Grady desingularisation $\pi_T\colon T \to M$. We write explicitly the E-polynomials of $M(C,\mathrm{SL}_2)$ and leave the straightforward computations of the other polynomials to the reader (cf. Subsection 4.2 and Subsection 2.3).

Theorem 1.12. *The E-polynomials of $T(C,\mathrm{SL}_2)$ are*

$$\begin{aligned}
E(T_{\mathrm{B}}(C,\mathrm{SL}_2)) = {} & (q^{2g-2}+1)(q^2-1)^{2g-2} + \frac{1}{2}q^{2g-3}(q^2+1)((q+1)^{2g-2}\\
& -(q-1)^{2g-2}) + 2^{2g-1}q^{2g-2}((q+1)^{2g-2}+(q-1)^{2g-2})\\
& + \frac{1}{2}q((1+q)^{2g-1}(1+q^{2g-3})+(1-q)^{2g-1}(1-q^{2g-3}))\frac{1-q^{2g-3}}{1-q}\\
& + 2^{2g}\frac{q}{(1-q)^3(1-q^2)}(2-q-q^3-q^{2g-4}-2q^{2g-2}+q^{2g-1}
\end{aligned}$$

$$
\begin{aligned}
&-2q^{2g}+4q^{2g+1}-q^{2g+2}+q^{2g+3}+q^{4g-6}-q^{4g-5}+4q^{4g-4}\\
&-2q^{4g-3}+q^{4g-2}-2q^{4g-1}-q^{4g+1}-q^{6g-6}-q^{6g-4}+2q^{6g-3}),
\end{aligned}
$$

$$
\begin{aligned}
E(T_{\mathrm{Dol}}(C,\mathrm{SL}_2)) = E(&M_{\mathrm{Dol}}(C,\mathrm{SL}_2)^{\mathrm{sm}})\\
&+\frac{1}{2}(uv)^g\frac{(1-(uv)^{2g-2})}{(1-uv)}\left(\frac{(1-u)^g(1-v)^g(1-(uv)^{2g-3})}{1-uv}\right.\\
&\left.+\frac{(1+u)^g(1+v)^g(1+(uv)^{2g-3})}{1+uv}\right)-2^{2g}\frac{(1-(uv)^{2g-2})^2}{(1-(uv)^2)(1-uv)}\\
&+2^{2g}\frac{(1-(uv)^{2g-2})\left(1-(uv)^{2g}\right)}{(1-uv)^3\left(1-(uv)^2\right)}(1-(uv)^4-(uv)^{2g-3}\\
&-(uv)^{2g-1}+2(uv)^{2g}).
\end{aligned}
$$

In particular, the E-polynomial $E(T_B)$ is palindromic.

1.3. Outline and relation with other work

- In Section 2 we collect some preliminary results: the intersection cohomology of an affine cone, the decomposition theorem, some properties of the mixed Hodge structures of singular semi-projective varieties (cf. [34]) and the stable isosingularity principle (implicitly used in [41, p. 834]). Analogous degeneration techniques employed to establish the stable isosingularity principle have been discussed in [16] and [17, §4] in relation to twisted character varieties.
- In Section 3 we describe the singularities of M and the geometry of the Kirwan–O'Grady desingularisation $\pi_T\colon T\to M$. This part highly relies on [56] and [41]. The computation of the Poincaré polynomial of the incidence variety I_{2g-3} in [41] contains a mistake, and we fix it in Subsection 3.3.
- In Subsection 3.4 we use several times the decomposition theorem to determine the intersection cohomology of the normal slice to strata of a Whitney stratification of M.

  The singularities of the Betti and Dolbeault moduli spaces are locally modelled on Nakajima quiver varieties which usually do not admit a symplectic resolution. Although a lot is known about the intersection cohomology of quiver varieties with symplectic resolutions (see, for instance, [54]), the local computations in Subsection 3.4 seem new.
- In Section 4 we complete the proof of Theorem 1.2. Then in Subsection 4.1 and Subsection 4.2 we argue how to compute the intersection E-polynomials and intersection Poincaré polynomial of M and we prove the results of Subsection 1.2. The E-polynomial of M is known thanks to [48], [4] and [41], while the ordinary Poincaré polynomial of M appears in [12]. Despite the active research in the field and the stimuli from the PI=WI conjecture, there are few previous works exhibiting explicit computations of the intersection cohomology of Dolbeault and Betti moduli spaces; see [25] and [26].

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In [40] Kiem studied the intersection cohomology of character varieties with coefficients in a compact Lie group. However, the methods in [40] do not extend to the complex reductive case, since for a general complex reductive group G the representation space $\mathrm{Hom}(\pi_1(C),G)$ is not smooth, and the quotient map $\mathrm{Hom}(\pi_1(C),G) \to \mathrm{Hom}(\pi_1(C),G) /\!\!/ G$ is not placid in the sense of [30].

We mention another remarkable precedent. The Dolbeault moduli space is a partial compactification of the cotangent bundle of the moduli space of stable vector bundles. The intersection cohomology of the moduli space of semistable vector bundles was determined in [43] for rank 2 (or, equivalently, in [40]) and in [53] in full generality. It is unclear how this result may imply Theorem 1.2.

- Section 5 explores many implications concerning the P=W conjectures for M stemming from the previous calculations.

2. Preliminaries

2.1. Intersection cohomology of affine cones

Let X be a complex projective variety of dimension $n-1$ with an ample line bundle L. The graded ring associated to L is the graded $\mathbb{C}$-algebra

$$R(X,L) := \bigoplus_{m \geq 0} H^0(X,L^m).$$

The affine cone over X with conormal bundle L is

$$C(X,L) := \operatorname{Spec} R(X,L).$$

Let $s_1,\dots,s_N$ be a set of generators for $R(X,L)$ of degree $m_1,\dots,m_N$. Then there exists an embedding $C(X,L) \subseteq \mathbb{C}^N$ such that $C(X,L)$ is invariant with respect to the $\mathbb{G}_m$-action

$$t \cdot (x_1,\dots,x_N) = (t^{m_1}x_1,\dots,t^{m_N}x_N). \tag{7}$$

Conversely, any affine variety with a $\mathbb{G}_m$-action and a fixed point which is attractive for $t \to 0$ is isomorphic to an affine cone; see, for instance, [21, §3.5].

All of the singularities of this article are locally modelled on affine cones, whose coordinate rings are not necessarily generated in degree 1. For this reason, here we compute their intersection cohomology, thus generalising [19, Example 2.2.1].

Proposition 2.1 (Intersection cohomology of an affine cone).

$$IH^d(C(X,L)) \simeq \begin{cases} IH^d_{\mathrm{prim}}(X) & \text{for } d < n \\ 0 & \text{for } d \geq n, \end{cases}$$

where $IH^d_{\mathrm{prim}}(X) := \ker(c_1(L)^{n-d}\cup \colon IH^d(X) \to IH^{2n-d}(X))$ *is the primitive intersection cohomology.*

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Proof. Denote by $C(X,L)^* := C(X,L) \setminus \{\text{vertex}\}$ the punctured affine cone. By [23, Lemma 1] or [45, Proposition 4.7.2], we can write

$$IH^d(C(X,L)) \simeq \begin{cases} IH^d(C(X,L)^*) & \text{for } d < n \\ 0 & \text{for } d \geq n. \end{cases}$$

Suppose now that $R(X,L)$ is generated in degree 1. Then the blow-up of the origin

$$p\colon BC(X,L) := \mathrm{Spec}_X \bigoplus_{m \geq 0} L^m \to C(X,L)$$

is the total space of the line bundle L^*. By the hard Lefschetz theorem, the relative long exact sequence of the inclusion $C(X,L)^* \hookrightarrow BC(X,L)$ splits into the short exact sequences

$$0 \to IH^{d-2}(X) \xrightarrow{c_1(L)\cup} IH^d(X) \to IH^d(C(X,L)^*) \to 0 \quad \text{for } d < n. \tag{8}$$

Therefore, we obtain that for $d < n$,

$$IH^d(C(X,L)) \simeq \mathrm{coker}(c_1(L)\cup\colon IH^{d-2}(X) \to IH^d(X)) \simeq IH^d_{\mathrm{prim}}(X).$$

If $R(X,L)$ is not generated in degree 1, then $BC(X,L)$ is the total space of a line bundle only up to a finite cover; see [57, §1.2]. More precisely, consider the finite morphism $g\colon \mathbb{C}^N \to \mathbb{C}^N$ defined by

$$g(x_1,\dots,x_N) = (x_1^{m_1},\dots,x_N^{m_N}).$$

Set $V' = g^{-1}C(X,L)$, where $C(X,L)$ is embedded in $\mathbb{C}^N$ as in (7). We see that $C(X,L)$ is the quotient of V' by the finite group $A = (\mathbb{Z}/m_1\mathbb{Z}) \times \dots \times (\mathbb{Z}/m_N\mathbb{Z})$ acting on V' by coordinatewise multiplication.

V' has a $\mathbb{G}_m$-action defined by $t\cdot(x_1,\dots,x_N) = (tx_1,\dots,tx_N)$ and covering the $\mathbb{G}_m$-action on $C(X,L)$ given by (7). Since the $\mathbb{G}_m$-action on V' has weight 1, X' is the spectrum of a graded algebra generated in degree 1, say, $V' = C(X',L')$ for some projective variety X' and ample line bundle L'. In particular, there exists a commutative diagram

$$\begin{tikzcd} C(X',L') \arrow[d, "g"'] & BC(X',L') \arrow[l, "p'"'] \arrow[d] & X' \arrow[l, "i'"'] \arrow[d] \\ C(X,L) & BC(X,L) \arrow[l, "p"'] & X \arrow[l, "i"'] \end{tikzcd} \tag{9}$$

where p and p' are blow-ups of the vertices of the cones, i and i' are the embedding of the exceptional divisors and the vertical arrows are quotients with respect to (the lift of) the action of A. Thus, we have $IH^*(C(X',L'))^A \simeq IH^*(C(X,L))$.

The discussion above shows that the sequences (8) are exact for $C(X',L')$ and it is A-equivariant by the commutativity of (9). Taking invariants, we show then that (8) holds for $C(X,L)$ unconditionally. $\square$

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2.2. The perverse Leray filtration

In this section we briefly recall the statement of the decomposition theorem and the definition of the perverse filtration.

For a complex algebraic variety X let $D^b(X,\mathbb{Q})$ be the bounded derived category of complexes of sheaves of $\mathbb{Q}$-vector spaces with algebraically constructible cohomology. Denote the full abelian subcategory of perverse sheaves by $\operatorname{Perv}(X)$ and the perverse cohomology functors by ${}^{\mathfrak{p}}\mathcal{H}^i\colon D^b(S,\mathbb{Q}) \to \operatorname{Perv}(X)$; see [5] or [19].

Let $MHM_{\mathrm{alg}}(X)$ be the category of algebraic mixed Hodge modules with rational coefficients and $D^bMHM_{\mathrm{alg}}(X)$ its bounded derived category. Let ${}^{\mathfrak{p}}\mathcal{H}^i\colon D^bMHM_{\mathrm{alg}}(X) \to MHM_{\mathrm{alg}}(X)$ be the cohomology functors; see [62] or [63].

The simple objects of $D^bMHM_{\mathrm{alg}}(X)$ (respectively $D^b(X,\mathbb{Q})$) are the intersection cohomology complexes $IC_X(L)$, where L is a polarisable variation of pure Hodge structures (respectively a local system) on a Zariski-open subset of the smooth locus of X. We denote simply by IC_X the complex $IC_X(\mathbb{Q}_{X\setminus \operatorname{Sing}(X)})$. In particular, $IH^d(X) \simeq H^d(X, IC_X(\mathbb{Q}_{X\setminus \operatorname{Sing}(X)})[-\dim X])$.

There is a forgetful functor $\operatorname{rat}\colon D^bMHM_{\mathrm{alg}}(X) \to D^b(X,\mathbb{Q})$ which commutes with ${}^{\mathfrak{p}}\mathcal{H}^i$ and pushforward Rf_* and maps $MHM_{\mathrm{alg}}(X)$ in $\operatorname{Perv}(X)$. We will make no notational distinction between $K \in D^bMHM_{\mathrm{alg}}(X)$ and $\operatorname{rat}(K)$.

Now let $f\colon X \to Y$ be a proper morphism of varieties with defect of semismallness

$$r := \dim X \times_Y X - \dim X.$$

The decomposition theorem of Beilinson–Bernstein–Deligne–Gabber, or its mixed Hodge module version by Saito, says that there is an isomorphism in $D^bMHM_{\mathrm{alg}}(X)$ (respectively in $D^b(X,\mathbb{Q})$)

$$Rf_*IC_X = \bigoplus_{i=-r}^{r} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)[-i] = \bigoplus_{i=-r}^{r} \bigoplus_{l} IC_{\overline{Y}_{i,l}}(L_{i,l})[-i],$$

where $L_{i,l}$ are polarisable variations of pure Hodge structures (respectively local systems) on the strata of a stratification $Y = \bigsqcup_l Y_{i,l}$; see [5] and [61].

The perverse (Leray) filtration is

$$P_k IH^d(X) = \operatorname{Im}\left\{ H^d\Big(\bigoplus_{i=-r}^{k-r} {}^{\mathfrak{p}}\mathcal{H}^i(Rf_*IC_X)[-i-\dim X]\Big) \to IH^d(X) \right\}.$$

When Y is affine, de Cataldo and Migliorini provided a simple geometric characterisation of the perverse filtration; see [20, Theorem 4.1.1]. Let $\Lambda^s \subset Y$ be a general s-dimensional affine section of $Y \subset \mathbb{A}^N$. Then

$$P_k IH^d(X) = \operatorname{Ker}\left\{ IH^d(X) \to IH^d(f^{-1}(\Lambda^{d-k-1})) \right\}. \tag{10}$$

This means that the cocycle $\eta \in IH^d(X)$ belongs to $P_k IH^d(X)$ if and only if its restriction to $f^{-1}(\Lambda^{d-k-1})$ vanishes; that is, $\eta|_{f^{-1}(\Lambda^{d-k-1})} = 0$.

2.3. Mixed Hodge structure of semi-projective varieties

In order to compute the intersection Poincaré polynomial of M, we observe that M_{Dol} and T_{Dol} are semi-projective.

Definition 2.2. (34, Definition 1.1.1). A **semiprojective** variety is a complex quasi-projective variety X with a $\mathbb{G}_m$-action such that

- the fixed point set $\mathrm{Fix}(X)$ is proper;
- for every $m \in X$ the limit $\lim_{\lambda \to 0} \lambda \cdot m$ exists.

The **core** of X, denoted $\mathrm{Core}(X)$, is the (proper) union of the repelling sets of $\mathrm{Fix}(X)$; see [34, Corollary 1.2.2].

Proposition 2.3. *Let X be a semi-projective variety. Then the inclusion $\mathrm{Core}(X) \subset X$ is a homotopy equivalence.*

Proof. The flow $\mathbb{R}^+ \times X \to X$, induced by the $\mathbb{G}_m$-action, defines a deformation retract of X onto a neighbourhood of $\mathrm{Core}(X)$, which in turn is homotopy equivalent to $\mathrm{Core}(X)$. □

Proposition 2.4. *If X is a semiprojective variety, then*

1. *the mixed Hodge structure on $IH^*(X)$ is pure;*
2. $W_{d-1}H^d(X) = \ker\{H^d(X) \to IH^d(X)\}$, *and* $W_d H^d(X) = H^d(X)$.

Proof. Let $f\colon \widetilde{X} \to X$ be a $\mathbb{G}_m$-equivariant resolution of singularities of X. Then $\widetilde{X}$ is smooth and semiprojective, and it has pure cohomology by [34, Corollary 1.3.2]. Via the decomposition theorem, the mixed Hodge structure on $IH^*(X) \subset H^*(\tilde{X})$ is pure, too.

As X retracts onto the proper algebraic variety $\mathrm{Core}(X)$, the weight filtration on $H^*(X) \simeq H^*(\mathrm{Core}(X))$ is concentrated in degree $[0,2d]$ by [60, Theorem 5.39]; that is, $W_d H^d(X) = H^d(X)$.

The resolution f induces a surjective morphism $\mathrm{Core}(\widetilde{X}) \to \mathrm{Core}(X)$ of proper algebraic varieties. Hence, by [60, Corollary 5.43] we have

$$W_{d-1}H^d(\mathrm{Core}(X)) = \ker\{f^*\colon H^d(\mathrm{Core}(X)) \to H^d(\mathrm{Core}(\tilde{X}))\},$$

and so $W_{d-1}H^d(X) = \ker\{f^*\colon H^d(X) \to H^d(\widetilde{X})\}$ by Proposition 2.3. Finally, since f^* factors as $H^d(X) \to IH^d(X) \hookrightarrow H^d(\widetilde{X})$, we conclude that

$$W_{d-1}H^d(X) = \ker\{H^d(X) \to IH^d(X)\}. \qquad \square$$

The multiplicative group $\mathbb{G}_m$ acts on $M_{\mathrm{Dol}}(C, \mathrm{SL}_n)$ by $\lambda \cdot (E,\phi) = (E, \lambda\phi)$. The **Hitchin fibration**

$$\chi\colon M_{\mathrm{Dol}}(C, \mathrm{SL}_n) \to \bigoplus_{i=2}^{n} H^0(C, K_C^{\otimes i}) \tag{11}$$

assigns to (E,ϕ) the characteristic polynomial of the Higgs field ϕ. By [66, Theorem 6.11] the map χ is a proper $\mathbb{G}_m$-equivariant map, where $\mathbb{G}_m$ acts linearly on $H^0(X, K_X^{\otimes i})$

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with weight i. In particular, $\mathrm{Fix}(M_{\mathrm{Dol}}(C,\mathrm{SL}_n))$ is contained in the nilpotent cone $\chi^{-1}(0)$. Therefore, $M_{\mathrm{Dol}}(C,\mathrm{SL}_n)$ is semiprojective.

The same argument works for $G = \mathrm{GL}_n, \mathrm{PGL}_n$ as well.

2.4. Stable isosingularity principle

Let S be a smooth projective K3 surface or an abelian surface. In this section we establish a stable isosingularity principle for the (nonproper) Dolbeault moduli spaces $M(C,\mathrm{GL}_n)$ and $M(C,\mathrm{SL}_n)$ and the (proper) Mukai moduli spaces $M(S,v)$ and $K(S,v)$. This means that these moduli spaces have the same analytic singularities, up to multiplication by a polydisk. The upshot is that

- the description of the local model of the singularities of $M(S,v)$ in [56] or [8] holds for $M(C,\mathrm{GL}_2)$ and $M(C,\mathrm{SL}_2)$ mutatis mutandis;
- the same sequence of blow-ups which desingularises $M(S,v)$ in [56] resolves the singularities of $M(C,\mathrm{GL}_2)$ and $M(C,\mathrm{SL}_2)$ mutatis mutandis;
- the description of the summands of the decomposition theorem in Theorem 1.2 holds for $M(C,\mathrm{GL}_2)$, $M(C,\mathrm{SL}_2)$, $M(S,v)$ and $K(S,v)$ with Mukai vector $v = 2w \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$, where w is primitive and $w^2 = 2(g-1)$.

We briefly recall the definition of Mukai moduli space. Fix an effective Mukai vector[2] $v \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$. Define $M(S,v)$ the moduli space of Gieseker H-semistable sheaves on S with Mukai vector v for a sufficiently general polarisation H (which we will typically omit in the notation); see [65, §1]. Further, if S is an abelian variety with dual $\hat{S}$, and $\dim M(S,v) \geq 6$, then the Albanese morphism $\mathrm{alb}\colon M(S,v) \to S \times \hat{S}$ is isotrivial, and we set $K(S,v) := \mathrm{alb}^{-1}(0_S,\mathcal{O}_S)$.

Remark 2.5 (Donagi–Ein–Lazarsfeld degeneration). Mukai moduli spaces should be thought of as locally trivial deformations of Dolbeault moduli spaces as follows. Fix an ample curve $C \subset S$ of genus $g \geq 2$. Donagi, Ein and Lazarsfeld showed in [22] that there exists a flat family $\pi_{\mathcal{W}}\colon \mathcal{W} \to \mathbb{P}^1$ such that

1. $\pi_{\mathcal{W}}^{-1}(\mathbb{P}^1 \setminus \{0\}) \simeq M(S,(0,nC,-nC^2/2)) \times \mathbb{A}^1$;
2. $\pi_{\mathcal{W}}^{-1}(0) \simeq M_{\mathrm{Dol}}(C,\mathrm{GL}_n)$.

Unless $g = 2$ and $n = 2$, $M(S,(0,nC,-nC^2/2))$ and $M_{\mathrm{Dol}}(C,\mathrm{GL}_n)$ have $\mathbb{Q}$-factorial terminal symplectic singularities; see [39, Theorem B] and [6, Theorem 1.2], together with Simpson's isosingularity principle [66, Theorem 10.6]. Hence, by [55, Theorem 17] the morphism $\pi_{\mathcal{W}}$ is locally analytically trivial.[3] Roughly, this means that the two moduli spaces have the same singularities. We make this statement precise in Proposition 2.10.

We start by stating the notion of stable isosingularity.

[2]That is, there exists a coherent sheaf $\mathcal{F}$ on S such that $v = (rk(\mathcal{F}), c_1(\mathcal{F}), \chi(\mathcal{F}) - \epsilon(S)rk(\mathcal{F}))$, with $\epsilon(S) := 1$ if S is K3 and 0 if S is abelian.

[3]The local triviality of $\pi_{\mathcal{W}}$ holds for $g = 2$ and $n = 2$, too, by [58, Proposition 2.16].

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Definition 2.6. Two varieties X and Y are **stably isosingular** if there exist complex Whitney stratifications[4] by (smooth nonnecessarily connected) locally Zariski-closed subsets X_i and Y_i such that

1. $X = \bigsqcup_i X_i$ and $Y = \bigsqcup_i Y_i$;
2. the posets of closed subsets $\{\overline{X}_i\}$ and $\{\overline{Y}_i\}$ ordered by inclusion are equal;
3. the normal slices through X_i and Y_i are locally analytically isomorphic.

If $\dim X = \dim Y$, then we say that X and Y are **isosingular**.

It is implicit in *3.* that the stratifications above are **analytically equisingular** along each stratum; that is, the analytic type of the normal slices through $x \in X_i$ (respectively $y \in Y_i$) is independent of x (respectively y). Not all algebraic variety admits such a stratification; see [68, Example 13.1]. However, the moduli spaces considered below will satisfy the following stronger condition of analytic normal triviality.

Definition 2.7. A Whitney stratification $X = \bigsqcup_i X_i$ is **analytically trivial in the normal direction** to each strata, if for any $x \in X_i$ there exists a normal slice N_x through X_i at x, and a neighbourhood of x in X which is locally analytically isomorphic to $N_x \times T_x X_i$ at $(x,0)$.

Note that if X and Y are stably isosingular via Whitney stratifications which are analytically trivial in the normal direction and a sequence of blow-ups along (the strict transforms of) some X_i gives a desingularisation of X, then the same sequence of blow-ups along the corresponding strata Y_i gives a desingularisation of Y. In addition, if X and Y are isosingular, then an analytic neighbourhood of any point of X is isomorphic to an analytic neighbourhood of some point in Y.

Example 2.8 (Analytically trivial fibrations). Let $f\colon X \to Y$ be an analytic locally trivial fibration, and suppose that $F := f^{-1}(y)$, with $y \in Y$, admits an analytically equisingular Whitney stratification. Then X and F are stably isosingular. Indeed, by the local triviality, any Whitney stratification of $f^{-1}(y)$ can be lifted to a Whitney stratification on X with the same normal slices. In particular, if W is a smooth algebraic variety, F and $F \times W$ are stably isosingular.

Lemma 2.9 (Quadraticity of deformation spaces). *Let $[F] \in M_{\mathrm{B}}(C, \mathrm{GL}_n)$ or $M(S,v)$ be a singular point corresponding to the polystable representation or polystable sheaf F. Then the representation space $\mathrm{Hom}(\pi_1(C), \mathrm{GL}_n)$ at F or the deformation space Def_F (cf. [38, §2.A.6]) is quadratic; that is, it is locally isomorphic to a (reduced) complete intersection of homogeneous quadrics.*

Proof. This follows from the Goldaman-Millson theory [28] if $[F] \in M_{\mathrm{B}}(C, \mathrm{GL}_n)$ or [2, Theorem 1.2] if $[F] \in M(S,v)$, with S K3 surface. Looking into the proof of [2, Theorem 3.7 and 3.8] and [69], one can see that the same proof holds for $[F] \in M(S,v)$ with S abelian surface. □

[4]See [31, Chapter 1] for a definition.

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Proposition 2.10. *Let S be a K3 or an abelian surface with* $\mathrm{Pic}(S) \simeq \mathbb{Z}$ *generated by the class of a curve C of genus $g \geq 2$. Then $M_{\mathrm{B}}(C, \mathrm{GL}_n)$ and $M(S,(0,nC,-nC^2/2))$ are isosingular.*

Proof. Given a (quasi-projective) variety X equipped with the action of a reductive group G, let $\xi\colon X \to Y := X /\!\!/ G$ be the quotient map. Any fibre $\xi^{-1}(y)$, with $y \in Y$, contains a closed G-orbit $T(y)$. Denote the conjugacy class of a closed subgroup H of G by (H). Then $Y_{(H)}$ is the set of points $y \in Y$ such that the stabiliser of $x \in T(y)$ is in (H). The loci $Y_{(H)}$ are the strata of the **stratification by orbit type** of Y.

If Y is a Nakajima quiver variety,[5] then the stratification by orbit type is a complex Whitney stratification, which is analytically trivial in the normal direction to each stratum, due to [50, Proposition 4.2].

$M_{\mathrm{B}}(C, \mathrm{GL}_n)$ and $M(S,(0,nC,-nC^2/2))$ are PGL_N-quotients, and the quadraticity of the deformation spaces implies that they are locally modelled on Nakajima quiver varieties; see [6, Theorem 2.5] and [2, Proposition 6.1]. By construction, the stratifications by orbit type of $M_{\mathrm{B}}(C, \mathrm{GL}_n)$ and $M(S,(0,nC,-nC^2/2))$ are locally isomorphic to stratification by orbit type of quiver varieties, and so they are complex Whitney stratifications, analytically trivial in the normal direction to each stratum.

A singular point of either moduli space is a polystable objects

$$F = F_1^{l_1} \oplus \ldots \oplus F_s^{l_s},$$

where F_i are distinct stable factors. The automorphism group of F is

$$\prod_{i=1}^{s} \mathrm{GL}_{l_i} \subset \mathrm{GL}_n,$$

which can be identified up to constants with the stabiliser of a point in $T(F)$ under the PGL_N-action; see, for instance, [39, §2.5].[6]

The poset of inclusions of the orbit type strata for both the Dolbeault and Mukai moduli spaces is isomorphic to the poset of inclusion of the stabilisers of $T(F)$, and the analytic type of the normal slice through an orbit type strata is prescribed by the (abstract) isomorphism class of the stabiliser; see again [6, Theorem 2.5] and [39, §2.7]. This gives *2*. and *3*. of Definition 2.6. The isosingularity follows from

$$\dim M_{\mathrm{B}}(C, \mathrm{GL}_n) = 2(g-1)n^2 + 2 = v^2 + 2 = \dim M(S,(0,nC,-nC^2/2)). \qquad \square$$

There exists a clear geometric argument for Proposition 2.10, sketched below.

[5] We use Nakajima quiver varieties only tangentially in this place. For brevity we omit the definition, and we refer the reader, for instance, to [27].

[6] Note that this is the only place where we use the assumption that S has Picard number one. Otherwise, if $[C]$ can be decomposed in the sum of effective classes, then the rank of the automorphism group of $F \in M(S,(0,nC,-nC^2/2))$ may be greater than n, and the stratification by orbit type of $M(S,(0,nC,-nC^2/2))$ would have more strata than that of $M_{\mathrm{B}}(C, \mathrm{GL}_n)$. Observe, however, that a general result is achieved in Theorem 2.11.

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Sketch of the proof of Proposition 2.10 via a degeneration argument. Via the Donagi–Ein–Lazarsfeld degeneration one can actually prove that $M_{\mathrm{Dol}}(C,\mathrm{GL}_n)$ is isosingular to a neighbourhood of a nilpotent cone of $M(S,(0,nC,-nC^2/2))$ as defined in [22, §2]. This is an analytic open set of the Mukai moduli space that intersects all of the orbit type strata, if C generates the Picard group of S. In order to extend the result to the whole Mukai moduli space, it is sufficient to invoke the analytic triviality in the normal direction of the stratification by orbit type of $M(S,(0,nC,-nC^2/2))$, which follows from the quadraticity of the deformation spaces Lemma 2.9. □

Theorem 2.11 (Stable isosingularity principle). *Let C be a curve of genus $g \geq 2$, and let S be a K3 or an abelian surface. Fix a Mukai vector $v = nw \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$, where w is primitive and $w^2 = 2(g-1)$.*

Then $M(C,\mathrm{GL}_n)$, $M(C,\mathrm{SL}_n)$, $M(S,v)$ and $K(S,v)$ are stably isosingular.

Proof.

- $M_{\mathrm{Dol}}(C,G)$ and $M_{\mathrm{B}}(C,G)$ are isosingular by [66, Theorem 10.6], independently on the complex structure of C.
- Now let S' be a K3 or an abelian surface such that C embeds in S' and generates its Picard group. Then $M(S',(0,nC,-nC^2/2))$ and $M(S,v)$ (respectively $K(S',(0,nC,-nC^2/2))$ and $K(S,v)$) are isosingular by [59, Theorem 1.17], independently on the complex structure of S.
- $M_{\mathrm{B}}(C,\mathrm{GL}_n)$ and $M(S',(0,nC,-nC^2/2))$ are isosingular by Proposition 2.10.
- Let S be an abelian surface. The morphisms $\mathrm{alb}\colon M(S,v) \to S \times \hat{S}$ and $\mathrm{alb}\colon M_{\mathrm{Dol}}(C,\mathrm{GL}_n) \to M(C,\mathrm{GL}_1)$, given by $\mathrm{alb}((E,\phi)) = (\det E, \mathrm{tr}\phi)$, are étale locally trivial fibrations with fibers $M_{\mathrm{Dol}}(C,\mathrm{SL}_n)$ and $K(S,v)$, respectively. The restriction of alb to the orbit type strata is étale locally trivial, too. This means that there exists a neighbourhood of $[F] \in M(S,v)_{(H)}$ locally analytically isomorphic to

$$N_{[F]} \times T_{[F]}M(S,v)_{(H)} \simeq N_{[F]} \times T_{[F]}K(S,v)_{(H)} \times T_{\mathrm{alb}([F])}(S \times \hat{S}) \tag{12}$$

at $([F],0)$, where $N_{[F]}$ is a normal slice through $K(S,v)_{(H)}$ at $[F]$. Further, the morphism alb is locally given by the linear projection onto the last factor of (12) by Lemma 2.9. The same argument works for $M_{\mathrm{Dol}}(C,\mathrm{SL}_n)$, too. As in Example 2.8, we conclude that $M_{\mathrm{Dol}}(C,\mathrm{GL}_n)$ and $M_{\mathrm{Dol}}(C,\mathrm{SL}_n)$ (respectively $M(S,v)$ and $K(S,v)$) are stably isosingular. □

3. Kirwan–O'Grady desingularisation

3.1. Singularities of M

Recall that M denotes indifferently the moduli spaces $M_{\mathrm{B}}(C,G)$ or $M_{\mathrm{Dol}}(C,G)$ with $G = \mathrm{GL}_2$ or SL_2. The stratification by orbit type of M (cf. Subsection 2.4) determines a filtration by closed subsets

$$M \supset \Sigma := \mathrm{Sing}\, M \supset \Omega := \mathrm{Sing}\, \Sigma.$$

In this section we characterise Σ, Ω and their normal slices, mainly appealing to [56].

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Proposition 3.1.

1. *$M(C,G)$ is an algebraic variety of dimension $6g-6$ if $G = \mathrm{SL}_2$ or $8g-6$ if $G = \mathrm{GL}_2$.*
2. *The singular locus of M is the subvariety of strictly semi-simple Higgs bundles or representations.*
3. *If $g \geq 3$, M is factorial with terminal symplectic singularities. If $g = 2$, M admits a symplectic resolution.*

Proof. The statements have been proved for $M_{\mathrm{B}}(C, \mathrm{GL}_2)$ in [6, Theorem 1.1, 1.2, 1.5, Lemma 2.8]. The same holds for M by the stable isosingularity principle (Theorem 2.11), possibly with the exception of the factoriality. However, to show that M is factorial, one can repeat the argument of [6, Theorem 1.2] word for word. □

Proposition 3.2 (Singularities of M).

1. *The singular locus of $M_{\mathrm{B}}(C, \mathrm{SL}_2)$, denoted $\Sigma_{\mathrm{B}}(C, \mathrm{SL}_2)$, is*

$$\left\{(A_1, B_1, \dots, A_g, B_g) \in (\mathbb{C}^*)^{2g} \subset \mathrm{SL}_2^{2g}\right\} /\!\!/ \mathrm{SL}_2 \simeq (\mathbb{C}^*)^{2g}/(\mathbb{Z}/2\mathbb{Z}),$$

   where $\mathbb{C}^* \subset \mathrm{SL}_2$ is the torus of diagonal matrices, and $\mathbb{Z}/2\mathbb{Z}$ acts on $(\mathbb{C}^*)^{2g}$ by $v \mapsto v^{-1}$. Set $\Sigma_{\iota,\mathrm{B}}(C, \mathrm{SL}_2) := (\mathbb{C}^*)^{2g}$.
2. *The singular locus of $M_{\mathrm{B}}(C, \mathrm{GL}_2)$, denoted $\Sigma_{\mathrm{B}}(C, \mathrm{GL}_2)$, is*

$$\left\{(A_1, B_1, \dots, A_g, B_g) \in (\mathbb{C}^*)^{2g} \times (\mathbb{C}^*)^{2g} \subset \mathrm{GL}_2^{2g}\right\} /\!\!/ \mathrm{GL}_2,$$

   which is isomorphic to the second symmetric product of $(\mathbb{C}^*)^{2g}$. Set $\Sigma_{\iota,B}(C, \mathrm{GL}_2) := (\mathbb{C}^*)^{2g} \times (\mathbb{C}^*)^{2g}$.
3. *The singular locus of $M_{\mathrm{Dol}}(C, \mathrm{SL}_2)$, denoted $\Sigma_{\mathrm{Dol}}(C, \mathrm{SL}_2)$, is*

$$\{(E,\Phi) \,|\, (E,\Phi) \simeq (L,\phi) \oplus (L^{-1}, -\phi), L \in \mathrm{Jac}(C), \phi \in H^0(C, K_C)\},$$

   which is isomorphic to

$$(\mathrm{Jac}(C) \times H^0(C, K_C))/(\mathbb{Z}/2\mathbb{Z}) \simeq T^* \mathrm{Jac}(C)/(\mathbb{Z}/2\mathbb{Z}),$$

   where $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{Jac}(C)$ by $L \mapsto L^{-1}$ and on $H^0(C, K_C)$ by $\phi \mapsto -\phi$. Set $\Sigma_{\iota,Dol}(C, \mathrm{SL}_2) := T^* \mathrm{Jac}(C)$.
4. *The singular locus of $M_{\mathrm{Dol}}(C, \mathrm{GL}_2)$, denoted $\Sigma_{\mathrm{Dol}}(C, \mathrm{GL}_2)$, is*

$$\{(E,\Phi) \,|\, (E,\Phi) \simeq (L,\phi) \oplus (L', \phi'), L, L' \in \mathrm{Jac}(C), \phi, \phi' \in H^0(C, K_C)\},$$

   which is isomorphic to the second symmetric product of $T^* \mathrm{Jac}(C)$. Set $\Sigma_{\iota,Dol}(C, \mathrm{GL}_2) := T^* \mathrm{Jac}(C) \times T^* \mathrm{Jac}(C)$.
5. *The singular locus of $\Sigma(C, \mathrm{SL}_2)$, denoted $\Omega(C, \mathrm{SL}_2)$, is a set of 2^{2g} points.*
6. *The singular locus of $\Sigma(C, \mathrm{GL}_2)$, denoted $\Omega(C, \mathrm{GL}_2)$, is isomorphic to $M(C, \mathrm{GL}_1)$.*

Proof. The result follows easily from Proposition 3.1.(2). □

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Definition 3.3.

1. By Proposition 3.2, there exists a double cover $q\colon \Sigma_\iota \to \Sigma$ branched along Ω.
2. The involution $\iota\colon \Sigma_\iota \to \Sigma_\iota$ is the deck transformation of q.
3. The largest open subset of Σ_ι where q is étale is denoted $\Sigma_\iota^\circ := q^{-1}(\Sigma \setminus \Omega)$.
4. There exists a rank-1 local system $\mathscr{L}$ on $\Sigma^\circ := \Sigma \setminus \Omega$ such that

$$q_*\mathbb{Q}_{\Sigma_\iota^\circ} = \mathbb{Q}_{\Sigma^\circ} \oplus \mathscr{L}.$$

Proposition 3.4 (Normal slices).

1. *A slice N_Σ normal to Σ at a point in $\Sigma\setminus\Omega$ is locally analytically isomorphic to an affine cone over the incidence variety*

$$I_{2g-3} := \left\{([x_i],[y_j]) \in \mathbb{P}^{2g-3}\times\mathbb{P}^{2g-3} \mid \sum_{k=0}^{2g-3} x_k y_k = 0\right\}$$

with conormal bundle $\mathcal{O}(1,1) := (\mathcal{O}_{\mathbb{P}^{2g-3}}(1)\boxtimes\mathcal{O}_{\mathbb{P}^{2g-3}}(1))|_{I_{2g-3}}$.

2. *Let (W,q) be a vector space of dimension 3 endowed with a quadratic form q of maximal rank and (V,ω) be a symplectic vector space of dimension $2g$. Let $\mathrm{Hom}^\omega(W,V)$ be the cone of linear maps from W to V whose image is isotropic. Note that the group $SO(W)$ acts on $\mathrm{Hom}^\omega(W,V)$ by precomposition.*

   Then a normal slice N_Ω through Ω is isomorphic to an affine cone over $\mathbb{P}\mathrm{Hom}^\omega(W,V)/\!\!/SO(W)$.

Proof. The local models have been described in [56, (3.3.2)] (see also [8, Proposition 3.2.(2)]) and in [56, (1.5.1)] (together with Lemma 2.9) for $M(S,v)$ with $v = (2,0,-2c)$. The description holds for M, too, by the stable isosingularity principle (Theorem 2.11). □

3.2. Geometry of the desingularisation

Inspired by [42], O'Grady exhibits a desingularisation of the Mukai moduli spaces $M(S,v)$ of semistable sheaves on a projective K3 surface S with Mukai vector $v = (2,0,-2c) \in H^*_{\mathrm{alg}}(S,\mathbb{Z})$. By the stable isosingularity principle (cf. Subsection 2.4), the same sequence of blow-ups gives a desingularisation of M. In this section, we recall the geometry of the exceptional locus, and we compute the E-polynomials of its strata.

Proposition 3.5 (Kirwan–O'Grady desingularisation). *Let*

- $\pi_R\colon R \to M$ *be the blow-up of M along Ω;*
- $\pi_S\colon S \to R$ *be the blow-up of R along $\Sigma_R := \pi_{R,*}^{-1}\Sigma$;*
- $\pi_T\colon T \to S$ *be the blow-up of S along its singular locus.*

Then the composition $\pi := \pi_R \circ \pi_S \circ \pi_T\colon T \to M$ is a log resolution of M.[7]

[7] In genus 2 the (unique) symplectic resolution of M can be obtained by contracting a $\mathbb{P}^2$-bundle in S. See [26, Proposition 8.6] where π_R and π_S are denoted η and ζ, respectively. In particular, the resolution π is not symplectic.

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Proof. It follows from [56, 1.8.3] and Theorem 2.11. □

Notation 3.6.

- D_1, D_2 and D_3 are (the strict transform of) the exceptional divisors in T of the blow-ups π_R, π_S and π_T, respectively.
- $D_{ij} := D_i \cap D_j$ and $D_{123} := D_1 \cap D_2 \cap D_3$ are (smooth closed) strata of the exceptional locus of π.
- $I_k := \{([x_0 : \dots : x_k],[y_0 : \dots : y_k]) \in \mathbb{P}^k \times \mathbb{P}^k \,|\, \sum_{i=0}^{k} x_i y_i = 0\}$.
- $\mathrm{Hom}^{\omega}_k(W,V)$ is the subspace of linear maps in $\mathrm{Hom}^{\omega}(W,V)$ of rank $\leq k$.
- $\mathrm{Gr}^{\omega}(k,V)$ is the Grassmanian of k-dimensional linear subspaces of V, isotropic with respect to the symplectic form ω.
- $\hat{\mathbb{P}}^5$ is the blow-up of $\mathbb{P}^5 \simeq \mathbb{P}(S^2(W))$ (space of quadratic forms on W) along $\mathbb{P}^2$ (locus of quadratic form of rank 1).
- $\hat{Q}$ is the blow-up of $Q \subset \mathbb{P}(S^2(W))$ (space of degenerate quadratic forms on W) along $\mathbb{P}^2$ (locus of quadratic form of rank 1).

Proposition 3.7 (Geometry of the blow-ups π_R, π_S and π_T).

1. *The preimages $\pi_R^{-1}(\Omega)$, $(\pi_R \circ \pi_S)^{-1}(\Omega)$ and $\pi^{-1}(\Omega)$ are trivial fibrations over Ω;*
2. *The exceptional locus Ω_R of π_R is isomorphic to $(\mathbb{P}\mathrm{Hom}^{\omega}(W,V) /\!\!/ SO(W)) \times \Omega$.*
3. *Let I'_{2g-3} be the quotient of I_{2g-3} by the involution which exchanges the coordinates x_i and y_i. A slice normal to $\Sigma_R \cap \Omega_R \simeq (\mathbb{P}\mathrm{Hom}^{\omega}_1(W,V) /\!\!/ SO(W)) \times \Omega \simeq \mathbb{P}^{2g-1} \times \Omega$ in Ω_R is locally analytically isomorphic to an affine cone over I'_{2g-3}.*
4. *The singular locus Δ_S of S is the strict transform of $(\mathbb{P}\mathrm{Hom}^{\omega}_2(W,V) /\!\!/ SO(W)) \times \Omega \subseteq \Omega_R$ via π_R, which is isomorphic to a $\mathbb{P}^2$-bundle over $\mathrm{Gr}^{\omega}(2,V) \times \Omega$.*
5. *A slice normal to Δ_S in S is locally analytically isomorphic to the quotient $\mathbb{C}^{2g-3}/\pm 1$.*

Proof. Since $\Omega(C,\mathrm{SL}_2)$ is a collection of 2^{2g} points, (1) obviously holds. Consider now the étale cover $\tau\colon M(C,\mathrm{SL}_n) \times M(C,\mathrm{GL}_1) \to M(C,\mathrm{GL}_n)$ trivialising (4). For some (or any) $x \in \Omega(C,\mathrm{SL}_n)$ we have

$$\pi^{-1}_{T(C,\,\mathrm{SL}_n)}(x) \times \Omega(C,\mathrm{GL}_n) \simeq \pi^{-1}_{T(C,\,\mathrm{SL}_n)}(x) \times M(C,\mathrm{GL}_1) \simeq \pi^{-1}_{T(C,\,\mathrm{GL}_n)}(\Omega(C,\mathrm{GL}_n)),$$

since τ is étale. Thus, (1) holds for $G = \mathrm{GL}_2$, too.

(2), (3), (4), (5) follow instead from Proposition 3.4.(2), [56, (1.7.12) and (1.7.16)], [56, (3.5.1)] and [56, (3.5.1)] respectively; alternatively, see the proof of [8, Proposition 3.2]. □

Proposition 3.8 (The exceptional divisors of π_T).

1. *D_1 is a $\hat{\mathbb{P}}^5$-bundle over $\mathrm{Gr}^{\omega}(3,V) \times \Omega$.*

2. *There exists a vector bundle $\mathcal{E}$ on Σ_ι° such that D_2° is the quotient of the I_{2g-3}-bundle $\mathbb{I}_{2g-3}$ in $\mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E}^*)$ by the involution ι'*
$$\begin{aligned}\iota'\colon \mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E}^*) &\to \mathbb{P}(\mathcal{E})\times\mathbb{P}(\mathcal{E}^*),\\ (v,[x_i],[y_j]) &\mapsto (\iota(v),[y_j],[x_i]),\end{aligned}$$
extending the involution ι on Σ_ι° defined in Definition 3.3.
3. *D_3 is a $\mathbb{P}^{2g-4}$-bundle over a (Zariski locally trivial) $\mathbb{P}^2$-bundle over $\mathrm{Gr}^\omega(2,V)\times\Omega$.*
4. *D_{13} is a $\hat{Q}$-bundle over $\mathrm{Gr}^\omega(3,V)\times\Omega$.*

Proof. Let $\mathcal{U}_m$ be the universal bundle over $\mathrm{Gr}^\omega(m,V)$, with $m=2,3$, and $\mathrm{Hom}_k(W,\mathcal{U}_m)$ be the subbundle of $\mathrm{Hom}(W,\mathcal{U}_m)$ of rank $\leq k$. The quotient space $\mathbb{P}\mathrm{Hom}_k(W,\mathcal{U}_m)/\!\!/SO(W)$ is isomorphic to the space of quadrics $\mathbb{P}(S^2_k\mathcal{U}_m)$ of rank $\leq k$. There are obvious forgetful maps
$$\begin{aligned}f_3\colon \mathbb{P}\mathrm{Hom}(W,\mathcal{U}_3) &\to \mathbb{P}\mathrm{Hom}^\omega(W,V)\\ f_2\colon \mathbb{P}\mathrm{Hom}_2(W,\mathcal{U}_2) &\to \mathbb{P}\mathrm{Hom}^\omega_2(W,V),\end{aligned}$$
which induces the following diagrams

$$\begin{tikzcd}
\mathrm{Bl}_{\mathbb{P}(S^2_1\mathcal{U}_3)}\mathbb{P}(S^2_2\mathcal{U}_3)\times\Omega \arrow[r, hook] \arrow[rrr, bend left, "\simeq"] \arrow[ddr, "\hat{Q}-\text{bundle}"'] & \mathrm{Bl}_{\mathbb{P}(S^2_1\mathcal{U}_3)}\mathbb{P}(S^2\mathcal{U}_3)\times\Omega \arrow[r, "\simeq"] \arrow[d, "birat."'] & D_1 \arrow[d, "birat."] & D_{13}=\mathrm{Exc}(\pi_T|_{D_1}) \arrow[l, hook'] \arrow[d, "\mathbb{P}^{2g-5}-\text{bundle}"] \\
 & \mathbb{P}(S^2\mathcal{U}_3)\times\Omega \arrow[r, "f_3/\!\!/SO(W)"] \arrow[d, "\mathbb{P}^5-\text{bundle}"] & \Omega_S \arrow[d, "birat."] & \Delta_S \arrow[l, hook'] \\
 & \mathrm{Gr}^\omega(3,V)\times\Omega & \Omega_R. &
\end{tikzcd}$$

$$\begin{tikzcd}
 & D_3 \arrow[d, "\mathbb{P}^{2g-4}-\text{bundle}"] \\
\mathbb{P}(S^2\mathcal{U}_2)\times\Omega \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \Delta_S \arrow[d, "birat."] \\
(\mathbb{P}\mathrm{Hom}(W,\mathcal{U}_2)/\!\!/SO(W))\times\Omega \arrow[r, "f_2/\!\!/SO(W)"] \arrow[d, "\mathbb{P}^2-\text{bundle}"] & (\mathbb{P}\mathrm{Hom}^\omega_2(W,V)/\!\!/SO(W))\times\Omega \arrow[d, "birat."] \\
\mathrm{Gr}^\omega(2,V)\times\Omega & \Omega_R.
\end{tikzcd}$$

A proof of the isomorphisms above is provided in [56, (3.1.1) and (3.5.1)]; alternatively, see [8, Proposition 3.2]. This shows (1), (3), (4). To show (2), one can repeat the argument of [8, Proposition 3.2.(2)] verbatim. □

Proposition 3.9. $\mathrm{Gr}^\omega(m,V)$ *and the fibres of Δ_S, D_1, D_3, D_{13} and Ω_S over Ω have pure cohomology of Hodge–Tate type. In particular, they do not have odd cohomology.*

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Their E-polynomials are

$$
\begin{aligned}
E(\mathrm{Gr}^{\omega}(2,V)) &= \frac{(1-q^{2g-2})(1-q^{2g})}{(1-q)(1-q^2)}\\
E(\mathrm{Gr}^{\omega}(3,V)) &= \frac{(1-q^{2g-4})(1-q^{2g-2})(1-q^{2g})}{(1-q)(1-q^2)(1-q^3)}\\
E(\Delta_S) &= \frac{(1-q^3)(1-q^{2g-2})(1-q^{2g})}{(1-q)^2(1-q^2)}\cdot E(\Omega)\\
E(D_1) &= \frac{(1-q^4)(1-q^{2g-4})(1-q^{2g-2})(1-q^{2g})}{(1-q)^3(1-q^2)}\cdot E(\Omega)\\
E(D_3) &= \frac{(1-q^3)(1-q^{2g-3})(1-q^{2g-2})(1-q^{2g})}{(1-q)^3(1-q^2)}\cdot E(\Omega)\\
E(D_{13}) &= \frac{(1-q^3)(1-q^{2g-4})(1-q^{2g-2})(1-q^{2g})}{(1-q)^3(1-q^2)}\cdot E(\Omega)\\
E(\Omega_S) &= E(D_1)-E(\Delta_S)\cdot\sum_{i=0}^{2g-6} q^{i+1}\\
&= \frac{(1-q^{2g-2})(1-q^{2g-1})(1-q^{2g})}{(1-q)^2(1-q^2)}\cdot E(\Omega).
\end{aligned}
$$

Proof. Note that $\mathrm{Gr}^{\omega}(m,V)$ is a smooth projective variety whose cohomology is of Hodge–Tate type; see, for instance, [8, Lemma 3.1]. Hence, the fibres of Δ_S, D_1, D_3 and $D_1\cap D_3$ over Ω have pure cohomology of Hodge–Tate type by Proposition 3.7 and 3.8. Since Ω_S has only quotient singularities, the natural inclusion $H^*(\Omega_S)\simeq IH^*(\Omega_S)\hookrightarrow H^*(D_1)$ implies that the fibres of Ω_S over Ω have pure cohomology of Hodge–Tate type. The computation of the E-polynomials follows immediately from [8, Lemma 3.1], Proposition 3.7 and 3.8, except maybe for Ω_S. In that case, we use the decomposition theorem for the blow-up map $\pi_T|_{D_1}$, which actually reduces to [67, Theorem 7.31]. □

3.3. The incidence variety I_{2g-3}

The incidence variety $I_{2g-3}\subset\mathbb{P}^{2g-3}\times\mathbb{P}^{2g-3}$ is the projectivisation of the vector bundle $\Omega^1_{\mathbb{P}^{2g-3}}(1)$ over $\mathbb{P}^{2g-3}$. Hence, we can write

$$
H^*(I_{2g-3})=\mathbb{Q}[a,b]/(a^{2g-2},b^{2g-2},\sum_{i=0}^{2g-3}(-1)^i a^{2g-3-i}b^i), \tag{13}
$$

where a and b have degree 2, and they are pullback of the first Chern classes of the tautological line bundle of $\mathbb{P}^{2g-3}$ via the two projections $I_{2g-3}\subset\mathbb{P}^{2g-3}\times\mathbb{P}^{2g-3}\to\mathbb{P}^{2g-3}$. Note that I_{2g-3} has no odd cohomology.

The involution which exchanges the factors of the product $\mathbb{P}^{2g-3}\times\mathbb{P}^{2g-3}$ leaves I_{2g-3} invariant and in cohomology exchanges the classes a and b. Consider the decomposition into eigenspaces for the involution (relative to eigenvalues ±1 respectively)

$$
H^*(I_{2g-3})=H^*(I_{2g-3})^+\oplus H^*(I_{2g-3})^-.
$$

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For $d=2k<4g-7$, we have

$$H^d(I_{2g-3})^+ = H^d(\mathbb{P}^{2g-3}\times\mathbb{P}^{2g-3})^+ = \langle a^ib^j | i+j=d\rangle^+ = \langle a^ib^j + a^jb^i | i+j=d\rangle.$$

Therefore, we obtain that

$$\dim H^{2k}(I_{2g-3}) = k+1,\ \dim H^{2k}(I_{2g-3})^+ = \left\lceil \frac{k+1}{2}\right\rceil,\ \dim H^{2k}(I_{2g-3})^- = \left\lceil \frac{k}{2}\right\rceil.$$

Proposition 3.10. *Setting $q:=t^2=uv$, the Poincaré polynomials (equivalently E-polynomials) of I_{2g-3} of the invariant and variant parts of its cohomology are*

$$P_t(I_{2g-3}) = E(I_{2g-3}) = \frac{(1-q^{2g-2})(1-q^{2g-3})}{(1-q)^2} \tag{14}$$

$$P_t(I_{2g-3})^+ = E(I_{2g-3})^+ = \frac{(1-q^{2g-2})^2}{(1-q^2)(1-q)} \tag{15}$$

$$P_t(I_{2g-3})^- = E(I_{2g-3})^- = q\frac{(1-q^{2g-2})(1-q^{2g-4})}{(1-q^2)(1-q)}. \tag{16}$$

Proof. I_{2g-3} is a $\mathbb{P}^{2g-4}$-bundle over $\mathbb{P}^{2g-3}$, and this gives (14). We now estimate $P_t(I_{2g-3})^+ - P_t(I_{2g-3})^-$. For $d=2k<4g-7$, we have

$$\dim H^{2k}(I_{2g-3})^+ - \dim H^{2k}(I_{2g-3})^- = \left\lceil \frac{k+1}{2}\right\rceil - \left\lceil \frac{k}{2}\right\rceil = \begin{cases} 1 & \text{for } k=2l, \\ 0 & \text{for } k=2l+1. \end{cases}$$

Since the polarisation $\mathcal{O}(1,1)$ is ι-invariant, the hard Lefschetz theorem gives $\dim H^d(I_{2g-3})^{\pm} = \dim H^{8g-14-d}(I_{2g-3})^{\pm}$. Thus, we can write

$$P_t(I_{2g-3})^+ - P_t(I_{2g-3})^- = \sum_{l=0}^{g-2} q^{2l} + q^{2g-3}\sum_{l=0}^{g-2} q^{2l} = \frac{(1-q^{2g-2})(1+q^{2g-3})}{(1-q^2)}. \tag{17}$$

Finally, substituting (14) and (17) in

$$P_t(I_{2g-3})^+ = \frac{1}{2}(P_t(I_{2g-3}) + (P_t(I_{2g-3})^+ - P_t(I_{2g-3})^-))$$
$$P_t(I_{2g-3})^- = P_t(I_{2g-3}) - P_t(I_{2g-3})^+,$$

we obtain (15) and (16). □

3.4. Intersection cohomology of local models

The goal of this section is to compute the intersection cohomology of the normal slices N_Σ and N_Ω. This is an important step to determine the summands of the decomposition theorem in Theorem 1.2.

Proposition 3.11. *Let N_Σ be a slice normal to Σ at a point in $\Sigma\setminus\Omega$. Then*

$$IH^d(N_\Sigma) \simeq \begin{cases} H^d_{\mathrm{prim}}(I_{2g-3}) \simeq \mathbb{Q} & \textit{for } d=2k<4g-6, \\ 0 & \textit{otherwise.} \end{cases} \tag{18}$$

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$$IH^{2k}(N_\Sigma) \subseteq \begin{cases} H^{2k}(I_{2g-3})^+ & \textit{for } k=2l, \\ H^{2k}(I_{2g-3})^- & \textit{for } k=2l+1. \end{cases} \tag{19}$$

Proof. (18) follows from Proposition 2.1, since N_Σ is locally isomorphic to an affine cone over the smooth variety I_{2g-3} (with ι-invariant conormal bundle $\mathcal{O}(1,1)$) by Proposition 3.4.(1).

Since $\mathcal{O}(1,1)$ is ι-invariant, $H^d_{\mathrm{prim}}(I_{2g-3})$ is ι-invariant, too. Hence, (19) follows from the following dimensional argument:

$$\begin{aligned} 1 \geq \dim IH^{2k}(N_\Sigma) \geq \dim IH^{2k}(N_\Sigma)^+ &= \dim H^{2k}(I_{2g-3})^+ - \dim H^{2k-2}(I_{2g-3})^+ \\ &= \left\lceil \frac{k+1}{2} \right\rceil - \left\lceil \frac{k}{2} \right\rceil = \begin{cases} 1 & \text{for } k=2l, \\ 0 & \text{for } k=2l+1. \end{cases} \end{aligned}$$ □

Proposition 3.12. *Let $N_{\Sigma_R\cap\Omega_R}$ be a slice normal to $\Sigma_R\cap\Omega_R$ in Ω_R. Then*

$$IH^d(N_{\Sigma_R\cap\Omega_R}) \simeq \begin{cases} H^d_{\mathrm{prim}}(I'_{2g-3}) \simeq \mathbb{Q} & \textit{for } d=4k<4g-6, \\ 0 & \textit{otherwise.} \end{cases} \tag{20}$$

In particular,

$$\dim H^{4g-6+2i}(I'_{2g-3}) - \dim IH^{4g-6+2i}(N_{\Sigma_R\cap\Omega_R}) = \left\lceil \frac{2g-3-|i|}{2} \right\rceil. \tag{21}$$

Proof. Proposition 2.1 and 3.7.(3) give (20), while (21) follows immediately from $H^{2k}(I'_{2g-3}) = H^{2k}(I_{2g-3})^+$. □

Proposition 3.13. *The intersection E-polynomial of Ω_R is*

$$IE(\Omega_R) = \frac{(1-q^{4g-4})(1-q^{2g})}{(1-q)(1-q^2)} \cdot E(\Omega).$$

Proof. We apply the decomposition theorem to the restriction of π_S to the strict transform $\Omega_S := \pi^{-1}_{S,*}\Omega_R$.

By Proposition 3.7.(3), the defect of semismallness of $\pi_S|_{\Omega_S}$ is

$$\begin{aligned} r(\pi_S|_{\Omega_S}) &:= \dim \Omega_S \times_{\Omega_R} \Omega_S - \dim \Omega_R \\ &= 2\dim I'_{2g-3} + \dim \Sigma_R \cap \Omega_R - \dim \Omega_R = 4g-8, \end{aligned}$$

and $\Sigma_R \cap \Omega_R$ is the only support of the decomposition theorem for $\pi_S|_{\Omega_S}$. Note that $R^i\pi_{S,*}\mathbb{Q}_{\pi_S^{-1}(\Sigma_R\cap\Omega_R)}$ are trivial local systems over $\Sigma_R \cap \Omega_R \simeq \mathbb{P}^{2g-1} \times \Omega$, because of Proposition 3.7.(1) and the simple connectedness of $\mathbb{P}^{2g-1}$. Hence, there exist integers $a(i)$ such that

$$\begin{aligned} R(\pi_S|_{\Omega_S})_*\mathbb{Q}[\dim \Omega_S] &= \bigoplus_{i=-4g+8}^{4g-8} {}^{\mathfrak{p}}\mathcal{H}^i((\pi_S|_{\Omega_S})_*\mathbb{Q}[\dim \Omega_S])[-i] \\ &= IC_{\Omega_R} \oplus \bigoplus\nolimits_{i=-2g+4}^{2g-4} \mathbb{Q}^{a(i)}_{\Sigma_R\cap\Omega_R}[\dim \Sigma_R \cap \Omega_R - 2i](-2g+3-i). \end{aligned}$$

At the stalk level, at $x \in \Sigma_R \cap \Omega_R$, we obtain by (21)

$$a(i) = \dim H^{4g-6+2i}(I'_{2g-3}) - \dim IH^{4g-6+2i}(N_{\Sigma_R \cap \Omega_R}) = \left\lceil \frac{2g-3-|i|}{2} \right\rceil.$$

Together with Proposition 3.9, we get

$$\begin{aligned}
IE(\Omega_R) &= E(\Omega_S) - E(\Omega)\cdot E(\mathbb{P}^{2g-1})\cdot \sum_{i=-2g+4}^{2g-4} q^{2g-3+i}\left\lceil \frac{2g-3-|i|}{2} \right\rceil \\
&= \left(\frac{(1-q^{2g-2})(1-q^{2g-1})(1-q^{2g})}{(1-q)^2(1-q^2)} - \frac{q(1-q^{2g-3})(1-q^{2g-2})(1-q^{2g})}{(1-q)^2(1-q^2)}\right)\cdot E(\Omega) \\
&= \frac{(1-q^{4g-4})(1-q^{2g})}{(1-q)(1-q^2)}\cdot E(\Omega).
\end{aligned}$$

□

Proposition 3.14. *Let N_Ω be a slice normal to Ω. Then $IH^*(N_\Omega)$ is pure of Hodge–Tate type with intersection Poincaré polynomial (equivalently, intersection E-polynomials)*

$$IP_t(N_\Omega) = IE(N_\Omega) = \frac{1-q^{2g}}{1-q^2}.$$

Proof. Since $IH^*(N_\Omega) \hookrightarrow IH^*(\pi_R^{-1}(x))$ for some $x \in \Omega$, $IH^*(N_\Omega)$ is pure of Hodge–Tate type by Proposition 3.9.

Recall now that N_Ω is an affine cone over Ω_R by Proposition 3.2.(2). Hence, Proposition 2.1 implies that the intersection Poincaré polynomial $IP_t(N_\Omega)$ is a polynomial in the variable $q = t^2$ of degree at most $3g-4$, given by

$$\begin{aligned}
IE(N_\Omega) &= [IE(\pi_R^{-1}(x)) - qIE(\pi_R^{-1}(x))]_{\leq 3g-4} = \frac{1}{E(\Omega)}[IE(\Omega_R) - qIE(\Omega_R)]_{\leq 3g-4} \\
&= \left[\frac{(1-q^{4g-4})(1-q^{2g})}{(1-q^2)}\right]_{\leq 3g-4} = \frac{1-q^{2g}}{1-q^2}.
\end{aligned}$$

□

4. Decomposition theorem

Proof of Theorem 1.2. Let π_T° be the restriction of π_T over $M^\circ := M \setminus \Omega$. By Proposition 3.2.(1) the defect of semismallness of π_T° is

$$\begin{aligned}
r(\pi_T^\circ) &:= \dim \pi_T^{-1}(M^\circ) \times_{M^\circ} \pi_T^{-1}(M^\circ) - \dim M \\
&= 2\dim I_{2g-3} + \dim \Sigma - \dim M = 4g-8,
\end{aligned}$$

and Σ° is the only support of the decomposition theorem for π_T°. Hence, there exists a splitting

$$R\pi^\circ_{T,*}\mathbb{Q}[\dim T] = IC_{M^\circ} \oplus \bigoplus_{i=-2g+4}^{2g-4} \mathscr{L}_i[\dim \Sigma - 2i]$$

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for some semisimple local systems $\mathscr{L}_i$ supported on Σ°. Restricting to $D_2^\circ = \pi_T^{-1}(\Sigma^\circ)$, we obtain

$$R(\pi^\circ_{T|_{D_2^\circ}})_*\mathbb{Q}[\dim T] = IC_{M^\circ}|_{D_2^\circ} \oplus \bigoplus_{i=-2g+4}^{2g-4} \mathscr{L}_i[\dim\Sigma - 2i].$$

By Proposition 3.8.(2) there exists a commutative square

$$\begin{tikzcd}
\mathbb{I}_{2g-3} \arrow[r, "q'"] \arrow[d, "p_2"'] & D_2^\circ \arrow[d, "\pi^\circ_{T|D_2^\circ}"] \\
(\mathbb{C}^*)^{2g,\circ} \arrow[r, "q"] & \Sigma^\circ,
\end{tikzcd}$$

where the horizontal arrows are étale double covers, p_2 is a Zariski locally trivial fibration with fibre I_{2g-3} and $(\mathbb{C}^*)^{2g,\circ}$ is the complement in $(\mathbb{C}^*)^{2g}$ of the locus fixed by the involution $v \mapsto v^{-1}$. Taking cohomology, we write

$$\begin{aligned}
R(\pi^\circ_{T|_{D_2^\circ}})_*\mathbb{Q} &= R(\pi^\circ_{T|_{D_2^\circ}})_* \circ (Rq'_*\mathbb{Q})^+ = (R(\pi^\circ_{T|_{D_2^\circ}} \circ q')_*\mathbb{Q})^+ = (R(q\circ p_2)_*\mathbb{Q})^+ \\
&= \left(q_* \bigoplus_{i=0}^{4g-7} (\mathbb{Q}_{(\mathbb{C}^*)^{2g,\circ}} \otimes H^{2i}(I_{2g-3}))[-2i] \right)^+ \\
&= \bigoplus_{i=0}^{2g-4} \mathbb{Q}_{\Sigma^\circ}^{\lceil \frac{i+1}{2} \rceil}[-2i](-i) \oplus \bigoplus_{i=0}^{2g-4} \mathscr{L}^{\lceil \frac{i}{2} \rceil}[-2i](-i) \\
&\quad \oplus \bigoplus_{i=2g-3}^{4g-7} \mathbb{Q}_{\Sigma^\circ}^{\lceil 2g-3-\frac{i}{2} \rceil}[-2i](-i) \oplus \bigoplus_{i=2g-3}^{4g-7} \mathscr{L}^{\lceil 2g-3-\frac{i+1}{2} \rceil}[-2i](-i),
\end{aligned}$$

where $\mathscr{L}$ is the rank-1 local system defined in Definition 3.3. Together with Proposition 3.11, we obtain

$$R\pi^\circ_{T,*}\mathbb{Q}[\dim T] = IC_{M^\circ} \oplus \bigoplus_{i=-2g+4}^{2g-4} (\mathbb{Q}_{\Sigma^\circ}^{\lceil \frac{2g-3-|i|}{2} \rceil} \oplus \mathscr{L}^{\lfloor \frac{2g-3-|i|}{2} \rfloor})[\dim\Sigma - 2i](-2g+3-i).$$

This splitting holds on M°, and we now extend it through Ω. Note that the defect of semismallness of π_T is

$$\begin{aligned}
r(\pi_T) &= \dim \pi_T^{-1}(M) \times_M \pi_T^{-1}(M) - \dim M \\
&= 2\dim D_1 - \dim\Omega - \dim M = 6g-8.
\end{aligned}$$

Since Σ is a rational homology manifold, $IC_\Sigma(\mathbb{Q}_{\Sigma^\circ}) \simeq \mathbb{Q}_\Sigma[\dim\Sigma]$. Further, the definition of $\mathscr{L}$ yields $IC_\Sigma(\mathscr{L}) = i_{\Sigma^\circ,*}\mathscr{L}[\dim\Sigma]$. Therefore, there exists integers $b(j)$ such that

$$\begin{aligned}R\pi_{T,*}\mathbb{Q}[\dim T] = IC_M &\oplus \bigoplus_{i=-2g+4}^{2g-4} \mathbb{Q}_{\Sigma}^{\lceil \frac{2g-3-|i|}{2} \rceil}[\dim\Sigma - 2i](-2g+3-i)\\ &\oplus \bigoplus_{i=-2g+4}^{2g-4} i_{\Sigma^\circ,*}\mathscr{L}^{\lfloor \frac{2g-3-|i|}{2} \rfloor}[\dim\Sigma - 2i](-2g+3-i)\\ &\oplus \bigoplus_{j=-3g+4}^{3g-4} \mathbb{Q}_{\Omega}^{b(j)}[\dim\Omega - 2j](-3g+3-j).\end{aligned}$$

Localising at $x \in \Omega$, we obtain

$$\dim H^{2d}(\pi_T^{-1}(x)) = \begin{cases} \dim IH^{2d}(N_\Omega) + \left\lceil \frac{2g-3-|d-2g+3|}{2} \right\rceil + b(d-3g+3) & \text{for } 0 \le d < 4g-6,\\ \dim IH^{2d}(N_\Omega) + b(d-3g+3) & \text{otherwise.}\end{cases}$$

Therefore, $b(j)$ is the coefficient of q^{d-3g+3} of the polynomial

$$\begin{aligned}&E(\pi_T^{-1}(x)) - IE(N_\Omega) - \sum_{i=-2g+3}^{2g-3} \left\lceil \frac{2g-3-|i|}{2} \right\rceil q^{2g-3+i}\\ &= \frac{1}{E(\Omega)}(E(D_1)+E(D_3)-E(D_{13})) - IE(N_\Omega) - \frac{q(1-q^{2g-3})(1-q^{2g-2})}{(1-q)(1-q^2)}\\ &= \frac{\left(1-q^{2g-2}\right)\left(1-q^{2g}\right)\left(1-q^4-q^{2g-3}-q^{2g-1}+2q^{2g}\right)}{(1-q)^3(1-q^2)} - \frac{1-q^{2g}}{1-q^2}\\ &\quad - \frac{q(1-q^{2g-3})(1-q^{2g-2})}{(1-q)(1-q^2)}.\end{aligned}$$

4.1. Applications of the decomposition theorem

Proof of Theorem 1.3. Taking cohomology with compact support, Theorem 1.2 gives

$$\begin{aligned}IE(M) = E(T) &- E(\Sigma_\iota)^+ \cdot \left(\sum_{i=-2g+3}^{2g-3} \left\lceil \frac{2g-3-|i|}{2} \right\rceil q^{2g-3+i} \right)\\ &- E(\Sigma_\iota)^- \cdot \left(\sum_{i=-2g+3}^{2g-3} \left\lfloor \frac{2g-3-|i|}{2} \right\rfloor q^{2g-3+i} \right)\\ &- E(\pi_T^{-1}(\Omega)) + E(\Omega)\cdot IE(N_\Omega) + E(\Omega)\cdot \sum_{i=-2g+3}^{2g-3} \left\lceil \frac{2g-3-|i|}{2} \right\rceil q^{2g-3+i}\\ &= E(M) - E(\Sigma) + E(D_2^\circ)\\ &\quad - (E(\Sigma_\iota)^+ - E(\Omega)) \cdot q\frac{(1-q^{2g-3})(1-q^{2g-2})}{(1-q)(1-q^2)}\end{aligned}$$

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$$- E(\Sigma_\iota)^- \cdot q^2 \frac{(1-q^{2g-4})(1-q^{2g-3})}{(1-q)(1-q^2)} + E(\Omega) \cdot IE(N_\Omega).$$

Now Proposition 3.8.(2) gives

$$E(D_2^\circ) = E(I_{2g-3})^+ \cdot (E(\Sigma_\iota)^+ - E(\Omega)) + E(I_{2g-3})^- \cdot E(\Sigma_\iota)^-. \tag{22}$$

Therefore, we obtain

$$\begin{aligned} IE(M) &= E(M) - E(\Sigma) + (E(\Sigma_\iota)^+ - E(\Omega)) \cdot \frac{1-q^{2g-2}}{1-q^2} \\ &\quad + E(\Sigma_\iota)^- \cdot q \frac{1-q^{2g-4}}{1-q^2} + E(\Omega) \cdot \frac{1-q^{2g}}{1-q^2} \\ &= E(M) + (q^2 E(\Sigma_\iota)^+ + q E(\Sigma_\iota)^-) \cdot \frac{1-q^{2g-4}}{1-q^2} + E(\Omega) \cdot q^{2g-2}. \end{aligned} \qquad \square$$

The variant and anti-invariant E-polynomials of $(\mathbb{C}^*)^{2g}$ and $T^* \operatorname{Jac}(C)$ with respect to the involution ι defined in Definition 3.3 are

$$E((\mathbb{C}^*)^{2g})^+ = \sum_{d=0}^{g} \dim(\Lambda^{2d} V) q^{2d} = \frac{1}{2}((1-q)^{2g} + (1+q)^{2g}), \tag{23}$$

$$E((\mathbb{C}^*)^{2g})^- = -\sum_{d=0}^{g-1} \dim(\Lambda^{2d+1} V) q^{2d+1} = \frac{1}{2}((1-q)^{2g} - (1+q)^{2g}), \tag{24}$$

$$E(T^* \operatorname{Jac}(C))^+ = \frac{1}{2}(uv)^g \left((1-u)^g (1-v)^g + (1+u)^g (1+v)^g\right), \tag{25}$$

$$E(T^* \operatorname{Jac}(C))^- = \frac{1}{2}(uv)^g \left((1-u)^g (1-v)^g - (1+u)^g (1+v)^g\right). \tag{26}$$

Proof of Theorem 1.4. We compute $IE(M_{\mathrm{B}}(C, \mathrm{SL}_2))$ and $IE(M_{\mathrm{B}}(C, \mathrm{GL}_2))$ from (2) by substitution. To this end, recall that $E(M_{\mathrm{B}}(C, \mathrm{SL}_2))$ and $E(M_{\mathrm{B}}(C, \mathrm{GL}_2))$ have been computed in [4, Theorem 1.3] (equivalently, [48, Theorem 2]). The E-polynomials of $\Sigma_{\iota,B}$ and Ω_B instead can be determined by using the description of these loci in Proposition 3.2, together with (23) and (24). Further, by (6) we have $IE(M_{\mathrm{B}}(C, \mathrm{PGL}_2)) = (1-q)^{-2g} \cdot IE(M_{\mathrm{B}}(C, \mathrm{GL}_2))$. $\square$

Proof of Theorem 1.8. In view of

$$E(M) = E(M^{\mathrm{sm}}) + E(\Sigma) = E(M^{\mathrm{sm}}) + E(\Sigma_\iota)^+,$$

we obtain Theorem 1.8 simply by substituting (25) and (26) in (2). $\square$

Proof of Theorem 1.9. The purity of $IH^*(M_{\mathrm{Dol}})$ (Proposition 2.4) and the Poincaré duality give

$$IP_t(M) = t^{2\dim M} IE(M_{\mathrm{Dol}}; -t^{-1}, -t^{-1}).$$

Theorem 1.9 then follows from Theorem 1.8 and elementary algebraic manipulations. $\square$

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Proof of Theorem 1.12. By the additivity of the E-polynomial, we have

$$E(T)=E(M^{\mathrm{sm}})+E(D_1)+E(D_2^{\circ})+E(D_3)-E(D_{13}).$$

The formula (22), together with (23), (24), (25) and (26), yields

$$\begin{aligned}
E(D^{\circ}_{2,\mathrm{B}}) &= \frac{1-q^{2g-2}}{2(1-q)}\left((1-q^{2g-3})(1-q)^{2g-1}+(1+q^{2g-3})(1+q)^{2g-1}\right)\\
&\quad -2^{2g}\frac{(1-q^{2g-2})^2}{(1-q^2)(1-q)},\\
E(D^{\circ}_{2,\mathrm{Dol}}) &= \frac{1}{2}(uv)^g\frac{1-(uv)^{2g-2}}{1-uv}\left(\frac{(1-u)^g(1-v)^g(1-(uv)^{2g-3})}{1-uv}\right.\\
&\quad \left.+\frac{(1+u)^g(1+v)^g(1+(uv)^{2g-3})}{1+uv}\right)-2^{2g}\frac{(1-(uv)^{2g-2})^2}{(1-(uv)^2)(1-uv)},
\end{aligned}$$

and Proposition 3.9 gives

$$E(D_1)+E(D_3)-E(D_{13})=2^{2g}\frac{\left(1-q^{2g-2}\right)\left(1-q^{2g}\right)\left(1-q^4-q^{2g-3}-q^{2g-1}+2q^{2g}\right)}{(1-q)^3\left(1-q^2\right)}. \qquad \square$$

4.2. From SL_2 to PGL_2 or GL_2

In order to compute $IE(M_{\mathrm{Dol}}(C,G))$ or $IP_t(M(C,G))$ for $G=\mathrm{PGL}_2,\mathrm{GL}_2$, one can repeat the arguments for SL_2 and realise that in practise one can obtain the polynomials for PGL_2 by replacing the coefficients 2^{2g} with 1 in the corresponding polynomials for SL_2, as explained below. Further, one can use (5) and (6) to write the polynomials for GL_2 from the PGL_2 counterparts.

Definition 4.1. Var^{Γ} is the category of algebraic varieties endowed with a Γ-action, with Γ-equivariant morphisms as morphisms.

$\Gamma-$mHs is the abelian category whose objects are Γ-modules over $\mathbb{Q}$ endowed with a mixed Hodge structure and whose morphisms are Γ-equivariant morphisms of mixed Hodge structures.

Definition 4.2. Let X be an algebraic variety endowed with an algebraic Γ-action. The **virtual Hodge realisation** of $(X;\Gamma\curvearrowright X)$ is the element in the Groethendieck ring $K_0(\Gamma-\mathrm{mHs})$ defined by the formula

$$\chi_{\mathrm{Hdg};\Gamma}(X)=\sum_k(-1)^k[H_c^k(T(C,\mathrm{SL}_2));\rho_M\colon \Gamma\to\mathrm{Aut}(H_c^k(X))].$$

The morphism $\chi_{\mathrm{Hdg};\Gamma}(\cdot)\colon K_0(\mathrm{Var}^{\Gamma})\to K_0(\Gamma-\mathrm{mHs})$ is additive.

The same Hodge realisation was considered in [35, §4], when Hausel and Thaddeus defined E-polynomials with coefficient in the characters of the finite abelian group Γ.

Now consider the Γ-invariant stratification of $T_{\mathrm{Dol}}(C,\mathrm{SL}_2)$ whose strata are

1. $S_0 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \text{ is stable}\}$;
2. $S_1 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \simeq L \oplus L^{-1}, L \in \mathrm{Jac}(C), L \neq L^{-1}\}$;
3. $S_2 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \text{ is a nontrivial extension of } L^{-1} \text{ by } L \text{ for } L \in \mathrm{Jac}(C) \text{ with } L \neq L^{-1}\}$;
4. $S_3 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \simeq L \oplus L, L \in \mathrm{Jac}(C)\}$;
5. $S_4 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \text{ is a nontrivial extension of } L \text{ by } L \text{ for } L \in \mathrm{Jac}(C)\}$;
6. $S_5 \simeq \{(E,\phi) \in M_{\mathrm{Dol}}(C,\mathrm{SL}_2) | E \text{ is unstable}\}$;
7. $S_6 = D_2^{\circ}$, $S_7 = D_3 \setminus D_{13}$ and $S_8 = D_1$.

This is indeed a stratification of $T_{\mathrm{Dol}}(C,\mathrm{SL}_2)$, since $M_{\mathrm{Dol}}(C,\mathrm{SL}_2)^{\mathrm{sm}} = \bigsqcup_{i=0}^{5} S_i \simeq T_{\mathrm{Dol}}(C,\mathrm{SL}_2) \setminus (D_1 \cup D_2 \cup D_3)$ by [37, Example 3.13] and Proposition 3.2. The additivity of the virtual realisation implies that

$$\chi_{\mathrm{Hdg};\Gamma}(T_{\mathrm{Dol}}(C,\mathrm{SL}_2)) = \sum_{i=0}^{8} \chi_{\mathrm{Hdg};\Gamma}(S_i).$$

By direct inspection (see [41, §3]) one can check that there exist algebraic varieties Z_{ij} endowed with a Γ-action such that

1. the Γ-module $H_c^k(Z_{ij})$ is isomorphic to the direct sum of copies of the trivial and of the regular representation of Γ; that is, there exist integers l_{ijk} and m_{ijk} such that there exists a Γ-equivariant isomorphism
$$H_c^k(Z_{ij}) \simeq V_{\mathrm{tr}}^{\oplus n_{ijk}} \oplus V_{\mathrm{reg}}^{\oplus m_{ijk}},$$
where Γ acts trivially on $V_{\mathrm{tr}} \simeq \mathbb{Q}$ and via the regular representation on $V_{\mathrm{reg}} \simeq \mathbb{Q}^{2^{2g}}$. We call $V_{\mathrm{reg}}^{\oplus m_{ijk}}$ the **regular part** of $H_c^k(Z_{ij})$.
2. we have
$$\chi_{\mathrm{Hdg};\Gamma}(S_i) = \sum_{j,k} \epsilon_{ij}(-1)^k [H_c^k(Z_{ij}); \rho_{ij} : \Gamma \to \mathrm{Aut}(H_c^k(Z_{ij}))],$$
where ϵ_{ij} is ± 1, and ρ_{ij} is a direct sum of copies of the trivial and/or of the regular representation.

Denote by $E_{\mathrm{reg}}(Z_{ij})$ the E-polynomial of the regular part of $H_c^*(Z_{ij})$, and let $E_{\mathrm{tr}}(Z_{ij}) := E(Z_{ij}) - E_{\mathrm{reg}}(Z_{ij})$. Then we have

$$\begin{aligned} E(T_{\mathrm{Dol}}(C,\mathrm{SL}_2)) &= \sum_{i,j} \epsilon_{ij} E_{\mathrm{tr}}(Z_{ij}) + \sum_{ij} \epsilon_{ij} E_{\mathrm{reg}}(Z_{ij}) \\ &= \sum_{ij} \epsilon_{ij} E_{\mathrm{tr}}(Z_{ij}) + 2^{2g} \sum_{ij} \epsilon_{ij} E_{\mathrm{reg}}(Z_{ij})^{\Gamma} \\ E(T_{\mathrm{Dol}}(C,\mathrm{SL}_2))^{\Gamma} &= \sum_{ij} \epsilon_{ij} E_{\mathrm{tr}}(Z_{ij}) + \sum_{ij} \epsilon_{ij} E_{\mathrm{reg}}(Z_{ij})^{\Gamma}. \end{aligned} \tag{27}$$

Via the decomposition theorem, the same holds for $IE(M_{\mathrm{Dol}}(C,\mathrm{SL}_2))$, and so for $IP_t(M(C,\mathrm{SL}_2))$, by the purity of $IH^*(M(C,\mathrm{SL}_2))$, as explained in the proof of Theorem 1.9.

Remark 4.3. Since in our case the varieties Z_{ij} are completely explicit, we can check that all of the coefficients 2^{2g} in Theorem 1.8 come from the E-polynomial of the regular part. So we obtain $IE(M_{\mathrm{Dol}}(C,\mathrm{SL}_2))^{\Gamma}$ by replacing 2^{2g} by 1. By (5) and (6) this gives $IE(M_{\mathrm{Dol}}(C,G))$ with $G=\mathrm{PGL}_2,\mathrm{GL}_2$. Analogously, knowing $IP_t(M(C,\mathrm{SL}_2))$, $E(T_{\mathrm{Dol}}(C,\mathrm{SL}_2))$ and $P_t(T(C,\mathrm{SL}_2))$, we can write their invariant counterparts, as well as their variants for $G=\mathrm{PGL}_2,\mathrm{GL}_2$.

Remark 4.4. By the vanishing of the odd part of $IH^*_{\mathrm{var}}(M(C,\mathrm{SL}_2))$ (cf. Corollary 1.11), every nontrivial $(\mathrm{Gr}^W_{r+s} IH^d_{\mathrm{var}}(M(C,\mathrm{SL}_2))^{r,s}$ will contribute with nonnegative coefficient to $IE_{\mathrm{var}}(M(C,\mathrm{SL}_2))$. Therefore, there is no cancellation, and the Γ-modules $IH^*(M(C,\mathrm{SL}_2))$ and $H^*(T(C,\mathrm{SL}_2))$ are direct sums of copies of the trivial and of the regular representation of Γ by (27). Comparing with [12], one can check that the same holds for $H^*(M(C,\mathrm{SL}_2))$.

5. P=W conjectures

5.1. P=W conjecture for twisted character varieties

The computation of E-polynomials of character varieties was initiated in [33] for twisted character varieties $M_{\mathrm{B}}^{\mathrm{tw}}=M_{\mathrm{B}}^{\mathrm{tw}}(C,G,d)$,

$$M_{\mathrm{B}}^{\mathrm{tw}}(C,G,d) := \left\{(A_1,B_1,\ldots,A_g,B_g)\in G^{2g} \,\middle|\, \prod_{j=1}^{g}[A_j,B_j]=e^{2\pi i/d}\cdot 1_G\right\} /\!\!/ G$$

with $G=\mathrm{GL}_n,\mathrm{SL}_n$ or PGL_n and $\gcd(n,d)=1$; see also [52].

As in the untwisted case, a nonabelian Hodge correspondence holds for $M_{\mathrm{B}}^{\mathrm{tw}}$: there exists a diffeomorphism $\Psi\colon M_{\mathrm{Dol}}^{\mathrm{tw}}\to M_{\mathrm{B}}^{\mathrm{tw}}$, from the Dolbeault moduli space $M_{\mathrm{Dol}}^{\mathrm{tw}}=M_{\mathrm{Dol}}^{\mathrm{tw}}(C,G,d)$ of semistable G-Higgs bundles over C of degree d; see [36]. However, contrary to the general untwisted character variety, $M_{\mathrm{B}}^{\mathrm{tw}}$ is smooth (a significant advantage!).

Surprisingly, Hausel and Rodriguez-Villegas [33] in rank 2, and Mellit [51] for GL_n, observed that the cohomology of $M_{\mathrm{B}}^{\mathrm{tw}}$ enjoys symmetries *typical* of smooth projective varieties, despite the fact that $M_{\mathrm{B}}^{\mathrm{tw}}$ is not projective. They called these symmetries curious hard Lefschetz theorem: there exists a class $\alpha\in H^2(M_{\mathrm{B}}^{\mathrm{tw}})$ which induces the isomorphism

$$\cup\alpha^k\colon\ \mathrm{Gr}^W_{\dim M_{\mathrm{B}}-2k} H^*(M_{\mathrm{B}}^{\mathrm{tw}}) \xrightarrow{\simeq} \mathrm{Gr}^W_{\dim M_{\mathrm{B}}+2k} H^{*+2k}(M_{\mathrm{B}}^{\mathrm{tw}}).$$

Note that, as an immediate consequence of the curious hard Lefschetz theorem, the E-polynomial of $M_{\mathrm{B}}^{\mathrm{tw}}$ is palindromic.

In the attempt to explain the curious hard Lefschetz theorem, de Cataldo, Hausel and Migliorini conjectured the P=W conjecture, and they verified it for rank 2; see [13]. This conjecture posits that the nonabelian Hodge correspondence exchanges two filtrations on the cohomology of $M_{\mathrm{Dol}}^{\mathrm{tw}}$ and $M_{\mathrm{B}}^{\mathrm{tw}}$ of very different origin, respectively the perverse Leray filtration (10) associated to the Hitchin fibration χ on $M^{\mathrm{tw}}_{\mathrm{Dol}}$ (the analogue of the map defined in (11)) and the weight filtration on $M^{\mathrm{tw}}_{\mathrm{B}}$.

Conjecture 5.1 (P=W conjecture for twisted moduli spaces).

$$P_k H^*(M^{\mathrm{tw}}_{\mathrm{Dol}}(C,G,d)) = \Psi^* W_{2k} H^*(M^{\mathrm{tw}}_{\mathrm{B}}(C,G,d)).$$

This suggests that the symmetries of the mixed Hodge structure of the cohomology of twisted character varieties, noted by Hausel and Rodriguez-Villegas, should be understood as a manifestation of the standard relative hard Lefschetz symmetries for the proper map χ on the Dolbeault side. The latter is an isomorphism between graded pieces of the perverse Leray filtration induced by cup product with a relative χ-ample class $\alpha \in H^2(M^{\mathrm{tw}}_{\mathrm{Dol}}(C,G,d))$:

$$\cup\alpha^k\colon\ \mathrm{Gr}^P_{\dim M_{\mathrm{Dol}}/2-k} H^*(M^{\mathrm{tw}}_{\mathrm{Dol}}) \xrightarrow{\simeq} \mathrm{Gr}^W_{\dim M_{\mathrm{Dol}}/2+2} H^{*+2k}(M^{\mathrm{tw}}_{\mathrm{Dol}});$$

see, for instance, [18, Theorem 2.1.1.(a)].

5.2. PI=WI and the intersection curious hard Lefschetz

In the untwisted (singular) case, curious hard Lefschetz fails in general; for example, [26, Remark 7.6], and the E-polynomial of $M_{\mathrm{B}}(C,G)$ is not palindromic; see, for instance, [46, Theorem 1.2], [48, Theorem 2] or [4, Theorem 1.3]. In order to restore the symmetries, de Cataldo and Maulik suggested considering the intersection cohomology of $M_{\mathrm{B}}(C,G)$, and in [15, Question 4.1.7] they conjectured the following.

Conjecture 5.2 (PI=WI conjecture).

$$P_k IH^*(M_{\mathrm{Dol}}(C,G)) = \Psi^* W_{2k} IH^*(M_{\mathrm{B}}(C,G)).$$

As in the twisted case, the PI=WI conjecture and the relative hard Lefschetz theorem for χ would imply the intersection curious hard Lefschetz theorem.

Conjecture 5.3 (intersection curious hard Lefschetz). There exists a class $\alpha \in H^2(M_{\mathrm{B}}(C,G))$ which induces the isomorphisms

$$\cup\alpha^k\colon\ \mathrm{Gr}^W_{\dim M_{\mathrm{B}}-2k} IH^*(M_{\mathrm{B}}(C,G)) \xrightarrow{\simeq} \mathrm{Gr}^W_{\dim M_{\mathrm{B}}+2k} IH^{*+2k}(M_{\mathrm{B}}(C,G)).$$

In particular, the intersection E-polynomial of $M_{\mathrm{B}}(C,G)$ is palindromic.

In this article we provide some numerical evidence for Conjecture 5.3.

Theorem 5.4 (Corollary 1.5). *The intersection E-polynomial $IE(M_{\mathrm{B}}(C,G))$ is palindromic for $G = \mathrm{GL}_2, \mathrm{SL}_2, \mathrm{PGL}_2$.*

5.3. PI=WI for SL_2 is equivalent to PI=WI for GL_2

The P=W conjecture for SL_n implies the P=W conjectures for PGL_n and GL_n; see [26, §3.3]. The converse holds true in the twisted case for n prime by [17]. By (5) and (6), this reduction boils down to prove the P=W conjecture for the variant cohomology. The

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proof in Theorem 5.5 does not rely on the smoothness of twisted character varieties, and it extends to the singular case verbatim.

Theorem 5.5 ([17]). *Suppose that*

1. $q^{(1-n^2)(2g-2)} IE_{\mathrm{var}}(M_{\mathrm{Dol}}(C,\mathrm{SL}_n);q,q) =: q^{(1-n^2)(2g-2)} E(q)$ *is palindromic;*
2. $IE_{\mathrm{var}}(M_{\mathrm{B}}(C,\mathrm{SL}_n);\sqrt{q},\sqrt{q}) = q^{(2-n-n^2)(g-1)} E(q)$.

Set $c_n := n(n-1)(g-1)$. *Then we have*

$$IH^d_{\mathrm{var}}(M(C,\mathrm{SL}_n)) \simeq \mathrm{Gr}^P_{d-c_n} IH^d_{\mathrm{var}}(M_{\mathrm{Dol}}(C,\mathrm{SL}_n)) \simeq \mathrm{Gr}^W_{2(d-c_n)} IH^d_{\mathrm{var}}(M_{\mathrm{B}}(C,\mathrm{SL}_n)).$$

Unfortunately, in the untwisted case the variant intersection E-polynomials are available only in rank 2; see Corollary 1.11.

Corollary 5.6. *The PI=WI conjecture for* $M(C,\mathrm{SL}_2)$ *is equivalent to the PI=WI conjecture for* $M(C,\mathrm{GL}_2)$.

5.4. Tautological classes

In [35] Hausel and Thaddeus proved that $H^*(M^{\mathrm{tw}}(C,\mathrm{SL}_2))^\Gamma$ is generated by tautological classes.[8] This is an essential ingredient of the proof of the P=W conjecture in the twisted case [13] and [16] and a missing desirable piece of information in the untwisted case. Here we provide a partial result: we show that tautological classes do generate the low-degree intersection cohomology of M.

Let $B(C,\mathrm{SL}_2)$ be the (infinite-dimensional and contractible) space of SL_2-Higgs bundles on C of degree zero and $B^{ss}(C,\mathrm{SL}_2)$ be the corresponding locus of semistable Higgs bundles. Let $\mathcal{G}$ be the group of real gauge transformations with fixed determinant acting on this spaces by precomposition and $\mathcal{G}^{\mathbb{C}}$ its complexification.

We can identify the classifying space $B\mathcal{G} \simeq B(C,\mathrm{SL}_2)$ with the space of continuous maps $\mathrm{Map}(C,\mathrm{SU}_2)$. The second Chern class of the tautological (flat) SU_2-bundle $\mathcal{T}$ on $C \times \mathrm{Map}(C,\mathrm{SU}_2)$ admits the Künneth decomposition

$$c_2(\mathcal{T}) = \sigma \otimes \alpha + \sum_{j=1}^{2g} e_j \otimes \psi_j + 1 \otimes \beta,$$

where $\sigma \in H^2(C)$ is the fundamental cohomology class, and $e_1,\ldots,e_{2g}$ is a standard symplectic basis of $H^1(C)$. Atiyah and Bott showed in [3] that the rational cohomology of $B\mathcal{G}$ is freely generated by the tautological classes α, ψ_j and β. That is, $H^*(B\mathcal{G})$ is the tensor product of the polynomial algebra on the classes α and β of degree 2 and 4 with an exterior algebra on the classes ψ_j of degree 3,

$$H^*(B\mathcal{G}) \simeq \mathbb{Q}[\alpha,\beta] \otimes \Lambda(\psi_j). \tag{28}$$

In particular, the Poincaré polynomial of the classifying space $B\mathcal{G}$ is

$$P_t(B\mathcal{G}) = \frac{(t^3+1)^{2g}}{(t^2-1)(t^4-1)}. \tag{29}$$

[8] The result have been generalised to arbitrary rank in [47].

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Now the nonabelian Hodge correspondence induces the following isomorphism in equivariant cohomology:

$$H^*_{\mathcal{G}}(B^{ss}(C, \mathrm{SL}_2)) \simeq H^*_{\mathrm{SL}_2}(\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)); \tag{30}$$

see [12, Theorem 1.2]. Together with Kirwan surjectivity, [10, Theorem 1.4]

$$H^*(B\mathcal{G}) \simeq H^*_{\mathcal{G}}(B(C, \mathrm{SL}_2)) \twoheadrightarrow H^*_{\mathcal{G}}(B^{ss}(C, \mathrm{SL}_2))^{\Gamma}, \tag{31}$$

this implies that the Γ-invariant SL_2-equivariant cohomology of $\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)$ is generated by tautological classes.

Theorem 5.7. *$IH^{<4g-6}(M(C, \mathrm{SL}_2))$ has a canonical structure of graded ring freely generated by the tautological classes α, ψ_j, β of degree 2, 3, 4, respectively, and weight 4. Among the tautological classes, only α is a cohomology class; that is, it is in the image of the natural map $H^*(M(C, \mathrm{SL}_2)) \to IH^*(M(C, \mathrm{SL}_2))$.*

Proof. Since Σ has codimension $4g-6$ in M, we have

$$IH^{<4g-6}(M(C, \mathrm{SL}_2)) \simeq H^{<4g-6}(M^{\mathrm{sm}}(C, \mathrm{SL}_2)); \tag{32}$$

see, for instance, [23, Lemma 1]. In particular, $IH^{<4g-6}(M(C, \mathrm{SL}_2))$ has a natural structure of graded ring. The open subset of simple representations $\mathrm{Hom}^s(\pi_1(C), \mathrm{SL}_2)$ in $\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)$ is a PGL_2-principal bundle over the smooth locus $M^{\mathrm{sm}}_B(C, \mathrm{SL}_2)$, and so

$$H^*(M^{\mathrm{sm}}(C, \mathrm{SL}_2)) \simeq H^*_{\mathrm{SL}_2}(\mathrm{Hom}^s(\pi_1(C), \mathrm{SL}_2)). \tag{33}$$

We claim that the composition of (31), (30), (34), the inverse of (33) and (32),

$$\begin{aligned} H^{<4g-6}(B\mathcal{G}) &\xrightarrow{a} H^{<4g-6}_{\mathrm{SL}_2}(\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)) \\ &\xrightarrow{b} H^{<4g-6}_{\mathrm{SL}_2}(\mathrm{Hom}^s(\pi_1(C), \mathrm{SL}_2)) \simeq IH^{<4g-6}(M(C, \mathrm{SL}_2)), \end{aligned}$$

is an isomorphism. Indeed, a is surjective by (31) (and Corollary 1.11) and actually bijective since by [11, Corollary 1.3] we have

$$P^{\mathrm{SL}_2}_t(\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)) := \sum_d \dim H^d_{\mathrm{SL}_2}(\mathrm{Hom}(\pi_1(C), \mathrm{SL}_2)) t^d = P_t(B\mathcal{G}) + O(t^{4g-4}).$$

Further, b is injective by Lemma 5.9 and actually bijective due to (3) and (29). The free generation of $IH^{<4g-6}(M(C, \mathrm{SL}_2))$ now follows from (28). See [64] for the weight of the tautological classes. Finally, ψ_i and β are not cohomology classes by Corollary 1.10 and preceding lines. □

Remark 5.8. Theorem 5.7 holds for PGL_2, as

$$IH^{<4g-6}(M(C, \mathrm{SL}_2)) = IH^{<4g-6}(M(C, \mathrm{SL}_2))^{\Gamma} \simeq IH^{<4g-6}(M(C, \mathrm{PGL}_2))$$

by Corollary 1.11, and so for GL_2, too. In the latter case, however, mind that there are additional generators ϵ_j which are pullback via the map (4) of a standard basis of $H^1(M(C, \mathrm{GL}_1)) \simeq H^1(C)$.

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As a final remark, note that the proof of Theorem 5.7 shows the more general statement that Kirwan surjectivity implies the tautological generation of the low-degree intersection cohomology for $M(C,\mathrm{SL}_n)$. However, this surjectivity is an open problem for $n>2$; cf. [9].

We prove the lemma used in the proof of Theorem 5.7.

Lemma 5.9. *The natural restriction map*

$$H^d_{\mathrm{SL}_2}(\mathrm{Hom}(\pi_1(C),\mathrm{SL}_2)) \to H^d_{\mathrm{SL}_2}(\mathrm{Hom}^s(\pi_1(C),\mathrm{SL}_2)) \tag{34}$$

is bijective for $d<4g-7$ and injective for $d=4g-7$.

Proof. Set $c:=\operatorname{codim}\operatorname{Sing}\mathrm{Hom}(\pi_1(C),\mathrm{SL}_2)=\dim M-\dim\Sigma+\dim\mathrm{Stab}_\Sigma=4g-5$, where $\mathrm{Stab}_\Sigma\simeq\mathbb{G}_m$ is the stabiliser of a closed orbit over Σ. Since $\mathrm{Hom}(\pi_1(C),\mathrm{SL}_2)$ is a complete intersection (adapt [24, Theorem 1.2] or [66, Proposition 11.3]), there exists an isomorphism

$$H^d_{\mathrm{SL}_2}(\mathrm{Hom}(\pi_1(C),\mathrm{SL}_2)) \simeq H^d_{\mathrm{SL}_2}(\mathrm{Hom}^{\mathrm{sm}}(\pi_1(C),\mathrm{SL}_2)) \text{ for } d<c-1. \tag{35}$$

Indeed, take an approximation E_k of the universal SL_2-bundle $E\mathrm{SL}_2$; that is, a smooth variety E_k with a free SL_2-action and such that $H^{<k}(X\times_{\mathrm{SL}_2}E_k)\simeq H^{<k}_{\mathrm{SL}_2}(X\times_{\mathrm{SL}_2}E\mathrm{SL}_2)=: H^{<k}_{\mathrm{SL}_2}(X)$; see [1, Lemma 1.3]. By Luna slice theorem $X\times_{\mathrm{SL}_2}E_k$ is a local complete intersection, and the singular locus has again codimension c. Then (35) follows from [31, p.199].

Further, the complement of $\mathrm{Hom}^s(\pi_1(C),\mathrm{SL}_2)$ in the smooth locus has codimension $2g-3$; see, for instance, [29, §7.2] where the complement is denoted $\mathfrak{X}^\rho_g$. Therefore, by the equivariant Thom isomorphism, the restriction map

$$H^d_{\mathrm{SL}_2}(\mathrm{Hom}^{\mathrm{sm}}(\pi_1(C),\mathrm{SL}_2)) \to H^d_{\mathrm{SL}_2}(\mathrm{Hom}^s(\pi_1(C),\mathrm{SL}_2))$$

is bijective for $d<4g-7$ and injective for $d=4g-7$. □

5.5. P=W vs PI=WI: nonpurity of $H^*(M_{\mathrm{B}})$

Despite the failure of curious hard Lefschetz, it still makes sense to conjecture $P=W$ phenomena for the ordinary cohomology of M_{B}.

Conjecture 5.10 (P=W conjecture for untwisted character varieties).

$$P_kH^*(M_{\mathrm{Dol}}(C,G))=\Psi^*W_{2k}H^*(M_{\mathrm{B}}(C,G)).$$

It was proved in [26, Theorem 6.1] that the PI=WI conjecture for genus 2 and rank 2 implies the P=W conjecture simply by restriction, since $H^*(M_{\mathrm{B}}(C,\mathrm{SL}_2))$ injects into $IH^*(M_{\mathrm{B}}(C,\mathrm{SL}_2))$ or, equivalently, by the purity of $H^*(M_{\mathrm{B}}(C,\mathrm{SL}_2))$; see Proposition 2.4. In higher genus the situation is more subtle, as the following theorem shows.

Theorem 5.11. *Let C be a curve of genus $g>3$. Then the natural map $H^*(M)\to IH^*(M)$ is not injective. Equivalently, M_{Dol} has no pure cohomology.*

Proof. Otherwise, the polynomial $IP_t(M)-P_t(M)$ would have only positive coefficients, but this is not the case by Corollary 1.10 and preceding lines. □

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Remark 5.12 (Torelli group). We propose an alternative proof of Theorem 5.11. The Torelli group is the subgroup of the mapping class group acting trivially on the cohomology of the curve C. The Torelli group acts nontrivially on $H^*(M)$ by [12, Proposition 4.7] (already in degree 6 by [7, Theorem 1.1]), but $IH^{<4g-6}(M)$ is generated by tautological classes due to Theorem 5.7, and so the Torelli group acts trivially on $IH^{<4g-6}(M)$ as in [7, Theorem 2.1.(c)]. Since the natural map $H^*(M) \to IH^*(M)$ is equivariant with respect to the Torelli group, we conclude that it has nontrivial kernel for $g > 3$.

5.6. P=W for resolution fails when no symplectic resolution exists

In [26] Camilla Felisetti and the author proposed a strong version of PI=WI conjecture, called P=W for resolution, and proved it for character varieties which admit a symplectic resolution.

Conjecture 5.13 (P=W conjecture for resolution). There exist resolutions of singularities $f_{\mathrm{Dol}}\colon \widetilde{M}_{\mathrm{Dol}}(C,G) \to M_{\mathrm{Dol}}(C,G)$ and $f_{\mathrm{B}}\colon \widetilde{M}_{\mathrm{B}}(C,G) \to M_{\mathrm{B}}(C,G)$ and a diffeomorphism $\widetilde{\Psi}\colon \widetilde{M}_{\mathrm{Dol}}(C,G) \to \widetilde{M}_{\mathrm{B}}(C,G)$, such that the following square commutes:

$$\begin{tikzcd} H^*(\widetilde{M}_{\mathrm{Dol}}(C,G),\mathbb{Q}) & H^*(\widetilde{M}_{\mathrm{B}}(C,G),\mathbb{Q}) \arrow[l, "\widetilde{\Psi}^*"'] \\ H^*(M_{\mathrm{Dol}}(C,G),\mathbb{Q}) \arrow[u, "f^*_{\mathrm{Dol}}"] & H^*(M_{\mathrm{B}}(C,G),\mathbb{Q}), \arrow[l, "\Psi^*"'] \arrow[u, "f^*_{\mathrm{B}}"'] \end{tikzcd} \tag{36}$$

and the lift $\widetilde{\Psi}^*$ of the nonabelian Hodge correspondence Ψ^* satisfies the property

$$P_k H^*(\widetilde{M}_{\mathrm{Dol}}(C,G),\mathbb{Q}) = \widetilde{\Psi}^* W_{2k} H^*(\widetilde{M}_{\mathrm{B}}(C,G),\mathbb{Q}). \tag{37}$$

In [26, Theorem 3.4] Camilla Felisetti and the author proved that resolutions of singularities satisfying (36) do exist; for instance, the Kirwan–O'Grady desingularisations are such.

In Theorem 5.14 we show, however, that if $M(C,\mathrm{GL}_n)$ does not admit a symplectic resolution, no resolution of $M(C,\mathrm{GL}_n)$ satisfies (37), despite the palindromicity of the E-polynomial of T_B; see Theorem 1.12. A fortiori, the same negative result holds for $G = \mathrm{SL}_n$.

This means that the hypotheses of [25, Main Theorem, 3] were optimal for $G = \mathrm{GL}_n, \mathrm{SL}_n$: the proof of Theorem 5.14 suggests that the semismallness of the desingularisation may be a necessary requirement for the P=W conjecture for resolutions to hold for a G-character variety with G arbitrary reductive group. This is compatible with the expectation of [14, §4.4].

Theorem 5.14. *Let $M(C,\mathrm{GL}_n)$ be an untwisted GL_n-character variety with no symplectic resolution; that is, for $g,n > 1$ and $(g,n) \neq (2,2)$. Then the P=W conjecture for any resolution of $M(C,\mathrm{GL}_n)$ does not hold.*

Proof. Let $f\colon \widetilde{M} \to M(C,\mathrm{GL}_n)$ be a resolution of singularities of $M(C,\mathrm{GL}_n)$ as in (36) and E be an f-exceptional divisor whose image is contained in the singular locus $\Sigma :=$

$\operatorname{Sing} M(C,\mathrm{GL}_n)$. Recall that $\chi\colon M_{\mathrm{Dol}}(C,\mathrm{GL}_n) \to \Lambda := \bigoplus_{i=1}^n H^0(C,K_C^{\otimes i})$ is the Hitchin fibration (11).

The locus $\chi(\Sigma)$ consists of reducible characteristic polynomials, and it has codimension

$$\frac{1}{2}(\dim M(C,\mathrm{GL}_n) - \max\{\dim M(C,\mathrm{GL}_{n_1}) + \dim M(C,\mathrm{GL}_{n_2}) : n = n_1 + n_2\}) \geq 2.$$

The last inequality follows, for instance, from [6, Lemma 2.2.(2)]. In particular, the general affine line in Λ avoids $\chi \circ f(E) \subseteq \chi(\Sigma)$. Then by (10) the Poincaré dual of E belongs to $P_0H^2(\widetilde{M})$. However, since $\widetilde{M}_{\mathrm{B}}$ is smooth, $H^2(\widetilde{M}_B)$ has weight not smaller than 2. This contradicts (37). $\square$

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References

[1] D. Anderson, Introduction to equivariant cohomology in algebraic geometry, in *Contributions to Algebraic Geometry*, EMS Ser. Congr. Rep., (European Mathematical Society, Zürich, Switzerland, 2012), 71–92.

[2] E. Arbarello and G. Saccà, Singularities of moduli spaces of sheaves on K3 surfaces and Nakajima quiver varieties, *Adv. Math.* **329** (2018), 649–703,.

[3] M. F. Atiyah and R. Bott, The Yang-Mills equations over Riemann surfaces, *Philos. Trans. Roy. Soc. London Ser. A* **308**(1505) (1983), 523–615.

[4] D. Baraglia and P. Hekmati, Arithmetic of singular character varieties and their E-polynomials, *Proc. Lond. Math. Soc. (3)* **114**(2) (2017), 293–332,.

[5] A. A. Beilinson, J. N. Bernstein and P. Deligne, Faisceaux pervers, in *Analysis and Topology on Singular Spaces, I (Luminy,1981)*, Vol. **100** of *Astérisque* (Société Mathématique de France, Paris, 1982), 5–171.

[6] G. Bellamy and T. Schedler, Symplectic resolutions of character varieties, Preprint, 2019, arXiv:1909.12545.

[7] S. E. Cappell, R. Lee and E. Y. Miller, The action of the Torelli group on the homology of representation spaces is nontrivial, *Topology* **39**(4) (2000), 851–871.

[8] J. Choy and Y. H. Kiem, Nonexistence of a crepant resolution of some moduli spaces of sheaves on a $K3$surface, *J. Korean Math. Soc.* **44**(1) (2007), 35–54.

[9] E. Cliff, T. Nevins and S. Shen, On the Kirwan map for moduli of Higgs bundles, Preprint, 2018, arXiv:1808.10311.

[10] G. Daskalopoulos, J. Weitsman, R. A. Wentworth and G. Wilkin, Morse theory and hyperkähler Kirwan surjectivity for Higgs bundles, *J. Differ. Geom.* **87**(1) (2011), 81–115.

[11] G. D. Daskalopoulos and K. K. Uhlenbeck, An application of transversality to the topology of the moduli space of stable bundles, *Topology* **34**(1) (1995), 203–215.

[12] G. D. Daskalopoulos, R. A. Wentworth and G. Wilkin, Cohomology of SL$(2,\mathbb{C})$ character varieties of surface groups and the action of the Torelli group. *Asian J. Math.* **14**(3) (2010), 359–383.
[13] M. A. de Cataldo, T. Hausel and L. Migliorini, Topology of Hitchin systems and Hodge theory of character varieties: the case A_1, *Ann. Math. (2)* **175**(3) (2012), 1329–1407.
[14] M. A. de Cataldo, T. Hausel and L. Migliorini, Exchange between perverse and weight filtration for the Hilbert schemes of points of two surfaces, *J. Singul.* **7** (2013), 23–38.
[15] M. A. de Cataldo and D. Maulik, The perverse filtration for the Hitchin fibration is locally constant, *Pure Appl. Math. Q.* **16**(5) (2020), 1441–1464.
[16] M. A. de Cataldo, D. Maulik and J. Shen, Hitchin fibrations, abelian surfaces, and the P=W conjecture, Preprint, 2019, arXiv:1909.11885.
[17] M. A. de Cataldo, D. Maulik andJ. Shen, On the P=W conjecture for SL$_n$, Preprint, 2020, arXiv:2002.03336.
[18] M. A. de Cataldo and L. Migliorini, The Hodge theory of algebraic maps, *Ann. Sci. École Norm. Sup. (4)* **38**(5) (2005), 693–750.
[19] M. A. de Cataldo and L. Migliorini, The decomposition theorem, perverse sheaves and the topology of algebraic maps, *Bull. Amer. Math. Soc. (N.S.)* **46**(4) (2009), 535–633.
[20] M. A. de Cataldo and L. Migliorini, The perverse filtration and the Lefschetz hyperplane theorem, *Ann. Math. (2)* **171**(3) (2010), 2089–2113.
[21] M. Demazure, Anneaux gradués normaux, in *Introduction à la théorie des singularités, II*, Vol. **37** of *Travaux en Cours* (Hermann, Paris, 1988), 35–68.
[22] R. Donagi, L. Ein and R. Lazarsfeld, Nilpotent cones and sheaves on $K3$surfaces, in *Birational Algebraic Geometry (Baltimore, MD, 1996)*, Vol. **207** of *Contemp. Math.* (American Mathematical Society, Providence, RI, 1997), 51–61.
[23] A. H. Durfee, Intersection homology Betti numbers, *Proc. Amer. Math. Soc.* **123**(4) (1995), 989–993.
[24] P. Etingof, Casimirs of the Goldman Lie algebra of a closed surface, *Int. Math. Res. Not.* **2006** (2006), Art. ID 24894.
[25] C. Felisetti, Intersection cohomology of the moduli space of Higgs bundles on a genus 2 curve, *J. Inst. Math. Jussieu* (2021), 1–50.
[26] C. Felisetti and M. Mauri, P=W conjectures for character varieties with symplectic resolution, Preprint, 2020, arXiv:2006.08752.
[27] V. Ginzburg, Lectures on Nakajima's quiver varieties, in *Geometric Methods in Representation Theory. I*, Vol. **24** of *Sémin. Congr.* (Société Mathématique de France, Paris, 2012), 145–219.
[28] W. M. Goldman and J. J. Millson, The deformation theory of representations of fundamental groups of compact Kähler manifolds, *Inst. Hautes Études Sci. Publ. Math.* (**67**) (1988), 43–96.
[29] A. González-Prieto, Topological Geometric Invariant Theory and character varieties, Preprint, 2018, arXiv:1807.08540.
[30] M. Goresky and R. MacPherson, Lefschetz fixed point theorem for intersection homology, *Comment. Math. Helv.* **60**(3) (1985), 366–391.
[31] M. Goresky and R. MacPherson, *Stratified Morse theory*, Vol. 14 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)* (Springer, Berlin, 1988).
[32] T. Hausel. Global topology of the Hitchin system, in *Handbook of Moduli. Vol. II*, Vol. **25** of *Adv. Lect. Math. (ALM)* (International Press, Somerville, MA, 2013), 29–69.
[33] T. Hausel and F. Rodriguez-Villegas, Mixed Hodge polynomials of character varieties, *Invent. Math.* **174**(3) (2008), 555–624.

https://doi.org/10.1017/S1474748021000487 Published online by Cambridge University Press

[34] T. Hausel and F. Rodriguez-Villegas, Cohomology of large semiprojective hyperkähler varieties, *Astérisque* **370** (2015), 113–156.

[35] T. Hausel and M. Thaddeus, Mirror symmetry, Langlands duality, and the Hitchin system, *Invent. Math.* **153**(1) (2003), 197–229.

[36] T. Hausel and M. Thaddeus, Generators for the cohomology ring of the moduli space of rank 2 Higgs bundles, *Proc. London Math. Soc. (3)* **88**(3) (2004), 632–658.

[37] N. J. Hitchin, The self-duality equations on a Riemann surface, *Proc. London Math. Soc. (3)* **55**(1) (1987), 59–126.

[38] D. Huybrechts and M. Lehn, *The Geometry of Moduli Spaces of Sheaves*, second edition (Cambridge University Press, Cambridge, 2010).

[39] D. Kaledin, M. Lehn and Ch. Sorger, Singular symplectic moduli spaces, *Invent. Math.* **164**(3) (2006), 591–614.

[40] Y. H. Kiem, Intersection cohomology of representation spaces of surface groups, *Internat. J. Math.* **17**(2) (2006), 169–182.

[41] Y. H. Kiem and S. B. Yoo, The stringy E-function of the moduli space of Higgs bundles with trivial determinant, *Math. Nachr.* **281**(6) (2008), 817–838.

[42] F. Kirwan, Partial desingularisations of quotients of nonsingular varieties and their Betti numbers, *Ann. of Math. (2)* **122**(1) (1985), 41–85.

[43] F. Kirwan, On the homology of compactifications of moduli spaces of vector bundles over a Riemann surface, *Proc. London Math. Soc. (3)* **53**(2) (1986), 237–266.

[44] F. Kirwan, Rational intersection cohomology of quotient varieties, *II. Invent. Math.*, **90**(1) (1987), 153–167.

[45] F. Kirwan and J. Woolf, *An Introduction to Intersection Homology Theory*, second edition (Chapman & Hall/CRC, Boca Raton, FL, 2006).

[46] M. Logares, V. Muñoz and P. Newstead, Hodge polynomials of $\mathrm{SL}(2,\mathbb{C})$-character varieties for curves of small genus, *Rev. Mat. Complut.* **26**(2) (2013), 635–703.

[47] E. Markman, Generators of the cohomology ring of moduli spaces of sheaves on symplectic surfaces, *J. Reine Angew. Math.* **544** (2002), 61–82.

[48] J. Martínez and V. Muñoz, E-polynomials of the $\mathrm{SL}(2,\mathbb{C})$-character varieties of surface groups, *Int. Math. Res. Not. IMRN* **3** (2016), 926–961.

[49] M. Mauri, Topological mirror symmetry for rank two character varieties of surface groups, *Abh. Math. Semin. Univ. Hambg.* (2021).

[50] M. Mayrand, Local structure of singular hyperkahler quotients, Preprint, 2018, arXiv:1807.05992.

[51] A. Mellit, Cell decompositions of character varieties, Preprint, 2019, arXiv:1905.10685.

[52] M. Mereb, On the E-polynomials of a family of SL_n-character varieties, *Math. Ann.* **363**(3-4) (2015), 857–892.

[53] S. Mozgovoy and M. Reineke, Intersection cohomology of moduli spaces of vector bundles over curves, Preprint, 2015, arXiv:1512.04076.

[54] H. Nakajima, t-Analogs of q-characters of quantum affine algebras of type E_6,E_7,E_8, in *Representation Theory of Algebraic Groups and Quantum Groups*, Vol. **284** of *Progr. Math.* (Birkhäuser/Springer, New York, 2010), 257–272.

[55] Y. Namikawa, Flops and Poisson deformations of symplectic varieties, *Publ. Res. Inst. Math. Sci.* **44**(2) (2008), 259–314.

[56] K. G. O'Grady, Desingularized moduli spaces of sheaves on a $K3$, *J. Reine Angew. Math.* **512** (1999), 49–117.

[57] P. Orlik and P. Wagreich, Isolated singularities of algebraic surfaces with $\mathbb{C}^*$action, *Ann. Math.* **93**(2) (1971), 205–228.

[58] A. Perego and A. Rapagnetta, Deformation of the O'Grady moduli spaces, *J. Reine Angew. Math.* **678** (2013), 1–34.

[59] A. Perego and A. Rapagnetta, The moduli spaces of sheaves on K3 surfaces are irreducible symplectic varieties, Preprint, 2018, arXiv:1802.01182.

[60] C. Peters and J. Steenbrink, *Mixed Hodge Structures*, Vol. **52** of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* (Springer, Berlin, 2008).

[61] M. Saito, Introduction to mixed Hodge modules. Actes du Colloque de Théorie de Hodge (Luminy, 1987), *Astérisque* **179–180**, (1989), 145–162.

[62] M. Saito. Mixed Hodge modules, *Publ. Res. Inst. Math. Sci.* **26**(2) (1990), 221–333.

[63] C. Schnell, An overview of Morihiko Saito's theory of mixed Hodge modules, Preprint, 2014, arXiv:1405.3096.

[64] V. Shende, The weights of the tautological classes of character varieties, *Int. Math. Res. Not. IMRN* **22** (2017), 6832–6840.

[65] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety, *I. Inst. Hautes Études Sci. Publ. Math.* **79** (1994), 47–129.

[66] C. T. Simpson, Moduli of representations of the fundamental group of a smooth projective variety. *II, Inst. Hautes Études Sci. Publ. Math.* **80** (1994), 5–79.

[67] C. Voisin, *Hodge theory and complex algebraic geometry. I*, Vol. **76** of *Cambridge Studies in Advanced Mathematics* (Cambridge University Press, Cambridge, 2007).

[68] H. Whitney, Local properties of analytic varieties, in *Differential and Combinatorial Topology (A Symposium in Honor of Marston Morse)* (Princeton University Press, Princeton, NJ, 1965), 205–244.

[69] Z. Zhang, A note on formality and singularities of moduli spaces, *Mosc. Math. J.* **12**(4) (2012), 863–879, 885.