The first derivative test and the classification of stationary points

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1. *Introduction*

Given a real differentiable function f we say that a point x_0 is a stationary point of f if $f'(x_0) = 0$.

In any standard single-variable calculus class, students learn how to determine the nature of a stationary point by checking the sign of $f(x)$ in intervals to the left and to the right of the stationary point. In doing so, they are performing the *first derivative test*.

Suppose that x_0 is an isolated zero for f' (i.e. x_0 is an isolated stationary point for f). With respect to the sign of f, two things can happen near x_0:

(1) either the derivative changes sign at x_0 (from positive to negative or vice versa), or

(2) the derivative keeps the same sign on both sides of x_0.

By the first derivative test, in the first case we have a local extremum (a local maximum or a local minimum). In the second case we can only conclude that near x_0 the function is strictly increasing (or decreasing) on both sides of x_0 and so x_0 is not a local extremum. Unfortunately, however, it is a common belief among students and also practitioners that this second case can always be classified as 'inflection point', a point at which there is a change in concavity. The goal of this Article is to point out that this is not always the case, unless we add more restrictive hypotheses (e.g. f analytic).

If we google 'classification of stationary points' we stumble upon many interesting websites that discuss the first derivative test at length. Focusing just on the second case (derivative with the same sign on both sides of the isolated stationary point) we read, for example:

- In [1]: Isolated stationary points of a C^1 real-valued function $f : \mathbb{R} \to \mathbb{R}$ are classified into four kinds, by the first derivative test: [...] a rising point of inflection (or inflexion) is one where the derivative of the function is positive on both sides of the stationary point; such a point marks a change in concavity; a falling point of inflection (or inflexion) is one where the derivative of the function is negative on both sides of the stationary point; such a point marks a change in concavity.
- In [2]: from *Mathworld–Wolfram* Suppose $f(x)$ is continuous at a stationary point x_0. [...] If $f'(x)$ has the same sign on an open interval extending left from x_0 and on an open interval extending right from x_0, then $f(x)$ has an inflection point at x_0.

In both these websites (and many others in the Google list) it is implied that, no matter what, the second case can always be classified as 'inflection point'. But the truth is that this is too optimistic: stationary points, even

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isolated stationary points, can happen to be neither a local extremum nor an inflection point.

In Section 2 we state some well-known useful facts concerning real differentiable functions and we introduce the first derivative test, explaining when it can be applied; that section can be skipped by those who are already familiar with its main concepts. In Section 3, we list some original examples of functions with isolated stationary points that are neither local extrema nor inflection points. We start from a simple and drawable C^1 example and then move on to more regular C^k and also C^∞ functions.

2. *Preliminaries*

2.1. *Shapes of graphs*. In this Article we will be dealing with geometric aspects of real functions, and we will use the terms 'convex', 'concave', 'inflection point'. Probably the reader is already familiar with this terminology. Anyway, for the sake of completeness, let us recall these classical definitions:

Definition 1 (Convex, concave function): Let f be a real-valued function of a real variable and let I be an interval contained in the domain of f. Then f is said to be *convex* on I, or *concave up*, if for every $a, b \in I$, $a < b$ and for every x we have $f(x) \leqslant f(a) + \dfrac{f(b) - f(a)}{b - a}(x - a)$ and f is said to be *concave* on I, or *concave down*, if $-f$ is convex. (See [3, def. 16.11.1]).

Geometrically, f is convex if the secant that connects the points $(a, f(a))$ and $(b, f(b))$ of the graph of f is above the graph of f, f is concave if the secant is below the graph. For example x^2 is convex, while $-x^2$ is concave. If f is also differentiable on I then f is convex (concave) on I if, and only if, f is increasing (decreasing) on I (see [3, prop. 16.11.2]).

Definition 2 (Inflection point): Let f be a real-valued function of a real variable, and let x_0 be a point inside its domain. If there exists a neighbourhood $[x_0 - \delta, x_0 + \delta]$ contained in the domain of f such that f is convex on $[x - \delta, x_0]$ and concave on $[x_0, x_0 + \delta]$ (or vice versa) then we say that x_0 is an *inflection point* for f, or *point of inflection*, or *flex*. (See [3, p. 402], [4, p. 248]).

In other words, an inflection point is a point at which the concavity of the function changes direction (from up to down or from down to up).

For simplicity, all the examples that we will provide in this Article will show some symmetry. It is worth recalling that a real-valued function f of a real variable is called *odd* if $f(-x) = -f(x)$, while it is called *even* if $f(-x) = f(x)$. Examples of the former are the functions x^{2k+1}, examples of the latter are the functions $f(x) = x^{2k}$ (with $k \in \mathbb{N}$). Odd functions are symmetric with respect to the origin of the axes, meaning that their graphs remain unchanged after a rotation of 180° about the origin. Even functions are symmetric with respect to the y-axis, meaning that their graphs remain

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unchanged after a reflection over the y-axis. Clearly, in order to plot an odd or even function we just need to study the function in the half-plane $x \geqslant 0$. By the very definition of the derivative, we see that if f is differentiable and odd, then f' is even, while if f is differentiable and even, then f' is odd. Differentiating changes the parity of the function. For example the function f_4 in Section 3 is an odd function (see Figure 2), while its derivative $f_4{}'$ is even (see Figure 3).

2.2. *Stationary points*: Students learn in any calculus course that in order to study a real function and plot it, the zeros of the first derivative are of primary importance. That is why they deserve a name on their own:

Definition 3 (Stationary point): Let f be a real-valued function of a real variable, differentiable on an open interval I. A point $x_0 \in I$ is called a stationary point of f if the first derivative of f vanishes at x_0, i.e. $f'(x_0) = 0$.

In more geometric terms, a stationary point of f is a point at which the tangent to the graph of f at $(x_0, f(x_0))$ is horizontal.

One of the cornerstones of calculus, Rolle's celebrated theorem ([5, thm. 2.3.4]), is actually a theorem on the existence of stationary points. Most of the readers are no stranger to it, but it might be useful to recall it. Let f be a continuous function on a proper, closed and bounded interval $[a, b]$. Assume that f is also differentiable on (a, b). Rolle's theorem states that if $f(a) = f(b)$, i.e. if the line joining the two points $(a, f(a))$ and $(b, f(b))$ on the graph of f is a horizontal line, then there exists a stationary point inside the interval, i.e. there is point on the graph of f with a horizontal tangent. With a simple argument, just applying Rolle's theorem to $f(x) - \frac{f(b) - f(a)}{b - a}$ instead of $f(x)$, Rolle's theorem can be extended to the case $f(a) \neq f(b)$, thus obtaining Lagrange's Mean Value Theorem ([5, thm. 2.3.4]). The Mean Value Theorem says that there exists a point $c \in (a, b)$ such that $f'(c)$, which measures the slope of the tangent to the graph of f at $(c, f(c))$, is equal to the slope $\frac{f(b) - f(a)}{b - a}$ of the chord joining the two points $(a, f(a))$ and $(b, f(b))$ on the graph of f. Rolle's theorem rests on two principles: the first is the Extreme Value Theorem ([5, thm. 1.7.6]), that asserts that a continuous function in a bounded closed interval attains both its maximum and its minimum in the interval; the second is the reason why stationary points are so important: when a function is differentiable, a local (or relative) extremum point (i.e. a local maximum or a local minimum) is a stationary point ([5, def. 2.3.1, thm. 2.3.2]). The converse of this last fact is not true in general, but stationary points often indicate a local extremum or a horizontal-tangent inflection point. An example of the former is the point $x = 0$ for the function x^2, an example of the latter is the point $x = 0$ for the function x^3, where the function goes from being concave to being convex .

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2.3. *Regularity of derivatives and differentiability classes*: Many readers are certainly aware that if a function is differentiable then it is continuous ([5, thm. 2.2.4]), but its derivative may not be continuous (e.g. the function ϕ_2 of Sect. 2.4).

Anyway, the derivative f' of a differentiable function f cannot be discontinuous at will. We refer to [6] for a thorough discussion on this topic. For our current purposes it suffices to recall that even if we do not assume that f' is continuous, f' satisfies a typical property pertaining to continuous functions: the Intermediate Value Property ([5, thm. 1.7.7]). This result is called Darboux's theorem ([5, thm. 2.4.6], [6, thm. 2]). Probably it is not well-known, but its proof just depends on Rolle's theorem, with an argument that is not much more complicated than the proof of the Mean Value Theorem from Rolle's. For these reasons it is worth stating and proving it.

Theorem (Darboux): Let f be a real-valued function of a real variable, differentiable on an open interval I. Let $[a, b] \subseteq I$ and let y be a real number between $f'(a)$ and $f'(b)$. Then there exists $\lambda \in [a, b]$ such that $f'(\lambda) = y$.

Proof: Consider first the special case $y = 0$. If $f(a) = f(b)$ the result follows at once by Rolle's theorem. Suppose next that $f(a) < f(b)$. The hypothesis that $f'(a) < 0$ implies that there exists $c > a$ such that $f(c) < f(a)$. Moreover, by the Intermediate Value Property for continuous functions, it follows that there exists $\xi \in (a, b)$ such that $f(a) = f(\xi)$, and hence, by Rolle's theorem, there exists $\lambda \in (a, \xi)$ such that $f'(\lambda) = 0$. The case $f(a) > f(b)$ is similar: we use instead that $f'(b) > 0$. The general case follows by setting $g(x) = f(x) - yx$, so that $g'(x) = f'(x) - y$.

Remark: The shape of the graph of f between a and b, in the special case, resembles a ‘tick’ from a teacher marking a correctly presented solution. Indeed some teachers call it the ‘tick argument’ so that students can recall the theorem and its proof.

In particular Darboux's theorem states that if $f'(a)f'(b) < 0$ then there necessarily exists a point $\lambda \in (a, b)$ such that $f'(\lambda) = 0$. More simply: f' can change sign only if it passes through 0. We will use this simple observation later on (see Sect. 2.6).

Of course, in many examples f is a little bit more regular that simply differentiable, so f' is continuous, and Darboux's theorem is equivalent to the classical Intermediate Value Property.

Actually, functions can be classified according to their regularity properties, i.e. according to the highest order of derivatives that exist and are continuous:

Definition 4 (Functions of class C^k). Let f be a real-valued function of a real variable defined on an open interval I. If f is continuous on I, we say that f

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is of *class* C^0 on I. The set of all continuous functions on I is denoted by $C^0(I)$. Here f is said to be *continuously differentiable* on I, or of (differentiability) class C^1, if f is differentiable on I and its first derivative is continuous on I. The set of all such functions is denoted by $C^1(I)$ and f is said to be *k-times continuously differentiable* on I, or of class C^k, if f is k-times differentiable on I, and the k th order derivative is continuous. The set of all such functions is denoted by $C^k(I)$ and f is said to be *infinitely differentiable* on I, or of *class* C^∞, or *smooth*, if we can compute the derivatives of f of any order. The set of all such functions is denoted by $C^\infty(I)$. (See [5, def. 2.4.3]).

We have that $C^k(I) \subset C^{k-1}(I)$. The inclusion is proper; see Sect. 2.4 or Sect. 3 for examples of functions that belong to $C^{k-1}(\mathbb{R})$ but not to $C^k(\mathbb{R})$.

2.4. *Non-isolated stationary points*: Sometimes stationary points are not isolated. This is trivially true for constant functions, but we can also produce non-constant functions with stationary points that are surrounded by infinitely many other stationary points. There are famous examples that can be brought (see [7, chap. 3]). The idea is to consider a bounded function, infinitely oscillating in a finite interval that contains 0 (for example $\sin\frac{1}{x}$), multiplied by monomials of sufficiently high degree (like x^k) that can make the function defined and continuous at 0 as well, and also differentiable or even more regular.

Let us consider the odd function

$$\phi_2(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Since $\sin\frac{1}{x}$ is bounded and $\lim_{x \to 0} x^2 = 0$ then ϕ_2 is continuous at 0. It is also clearly differentiable everywhere, except at most at 0. If we check the definition of the derivative as the limit of the difference quotient (see [5, def. 2.2.1] for the definition and [6, ex. 1] or [5, ex. 2.4.4] for the computations) we see that ϕ_2 is differentiable also at 0 and $\phi_2'(0) = 0$. So $x = 0$ is a stationary point, ϕ_2 is continuous and differentiable on all $\mathbb{R}$ and its first derivative is:

$$\phi_2'(x) = \begin{cases} 2x \sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function ϕ_2 is plotted in Figure 1. Each local extremum in Figure 1 is a stationary point, hence 0 is not an isolated stationary point.

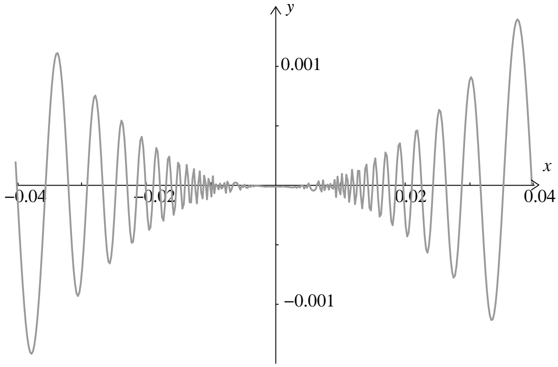

FIGURE 1: Plot of the function $\phi_2(x)$

Although ϕ_2 is differentiable, $\phi_2{}'$ is not continuous at 0, in fact $\lim_{x \to 0} \cos \frac{1}{x}$ does not exist. We can produce more regular functions by increasing the exponent of x.

Consider the following generalisation of the previous example ($k \in \mathbb{N}$);

$$\phi_k(x) = \begin{cases} x^k \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

The function ϕ_0 is not continuous at 0, ϕ_1 is continuous but not differentiable at 0. As we have just discussed, ϕ_2 is differentiable at 0, but the first derivative is not continuous at 0 (therefore – a fortiori – the first derivative is not differentiable at 0). By the same token, we can see that ϕ_3 is differentiable at 0 and the first derivative is continuous at 0, but the first derivative is still not differentiable at 0. ϕ_4 is twice differentiable at 0, but the second derivative is not continuous at 0. ϕ_5 is twice differentiable at 0 and the second derivative is continuous at 0, but it is not differentiable, and so on ... . In all the cases with $k \geqslant 2$, 0 is a non-isolated stationary point (which is actually neither a local extremum nor an inflection point).

This discussion on ϕ_k can be summed up by saying that ϕ_{2k-1} and ϕ_{2k} belong to $C^{k-1}(\mathbb{R})$ but not to $C^k(\mathbb{R})$, the former because it is not k-differentiable at 0, the latter because, even if it is k-differentiable at 0, the k th order derivative is not continuous at 0 (see [5, p. 65, ex. (4)]).

We have seen that non-isolated stationary points can occur in C^k functions (for any k). That is true also for C^∞ functions, but in order to produce an example we need an infinitesimal of higher order with respect to any monomial x^k. Let us consider

$$\varepsilon(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

We can repeatedly compute the derivatives of ε using the usual calculus rules, except at $x = 0$. For $x = 0$ we can compute the derivatives either by

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computing limits of difference quotients, or by computing limits for $x \to 0$ of the derivatives outside 0 (see [6, thm. 1]). In both cases, since $\lim_{x \to 0} \frac{e^{-1/x^2}}{x^k} = 0$ for every k, we see that ε is infinitely differentiable on all $\mathbb{R}$ and the derivatives of any order vanish at 0. We could have used $e^{-1/|x|}$ as well, but the exponent 2 is more convenient. The function

$$\gamma(x) = \begin{cases} e^{-1/x^2} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is an odd function that infinitely oscillates around 0, and so 0 is a non-isolated stationary point (that is actually neither a local extremum nor an inflection point).

2.5. *Real analytic functions*: In our quest for non-isolated stationary points, clearly we could not have used polynomial functions, since polynomials have just a finite number of zeros. Actually we could not have used any of the elementary functions (sums, products, roots and compositions of finitely many polynomial, rational, trigonometric, hyperbolic, and exponential functions, including their inverse functions). Let us see why.

Definition 5 (Real analytic function): Let f be a real-valued function of a real variable, defined on an open interval I. Then f is said to *be real analytic* on I, or of *class* C^ω, if for every $x_0 \in I$ there is an open interval $(x_0 - \delta, x_0 + \delta)$ in I such that there exists a power series $\sum_{k=0}^{+\infty} c_k (x - x_0)^k$ centred at x_0 which converges to f on $(x_0 - \delta, x_0 + \delta)$. The set of all such functions is denoted by $C^\omega(I)$. (See [8, def. 4.2.1]).

In other words, real analytic functions are the functions that can be locally expressed as a convergent real power series. By the theory of power series (see [5, sect. 4.2] and [8, sect. 4.2]), if f is a real analytic function on I then

- f is infinitely differentiable;
- all the derivatives of f are real analytic;
- unless $f = 0$, the zeros of f are isolated;
- f is equal to its Taylor series centred at x_0 (for any $x_0 \in I$), i.e. $f = \sum_{k=0}^{+\infty} \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k$ in a neighbourhood of x_0.

This last property entails an important consequence: unless f = constant, at least one derivative $f^{(k)}$ does not vanish at x_0. Examples of real analytic functions include all the elementary functions. On the contrary, the function ε (and also γ) of Sect. 2.4 is not real analytic. In fact, all the derivatives of ε vanish at 0, hence its Taylor series at 0 is the null function, that does not coincide with ε on any interval. This means that the inclusion $C^\omega(I) \subset C^\infty(I)$ is strict.

Let f be a real analytic function on I. f' is real analytic as well, so, unless f is constant, the zeros of f' (i.e. the stationary points) are isolated.

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2.6. *Isolated stationary points and the first derivative test*: From now on we will consider only isolated stationary points. Let f be a real-valued function of a real variable, differentiable on an open interval I, and let $x_0 \in I$ be an isolated stationary point for f, that is: in a neighbourhood of x_0, f vanishes only at x_0.

By Darboux's theorem , f' can change sign only if it passes through 0. This means that only two things can happen near the isolated stationary point x_0:

(1) either the first derivative changes sign at x_0 (from positive to negative or vice-versa), or

(2) the first derivative keeps the same sign on both sides of x_0.

The nature of an isolated stationary point can then be established by checking the sign of $f'(x)$ in a small interval to the left and to the right of the stationary point. This is the so-called *first derivative test* (see [4], [5, thm. 2.4.2], [9, thm. 3.1]).

The first derivative test (even if not in its uttermost generality, for which we refer to [9]) can be summed up in this way: if there exists a $\delta > 0$ such that $f'(x) < 0$ on $(x_0 - \delta, x_0)$ and $f'(x) > 0$ on $(x_0, x_0 + \delta)$ then f is strictly decreasing on $(x_0 - \delta, x_0)$ and strictly increasing on $(x_0, x_0 + \delta)$ (see [5, thm. 2.3.7]), therefore x_0 is a local extremum, in particular a local minimum; if $f'(x) > 0$ on $(x_0 - \delta, x_0)$ and $f'(x) < 0$ on $(x_0, x_0 + \delta)$ then f is strictly increasing on $(x_0 - \delta, x_0)$ and strictly decreasing on $(x_0, x_0 + \delta)$, therefore x_0 is a local extremum, in particular a local maximum; if $f'(x) > 0$ on $(x_0 - \delta, x_0)$ and $f'(x) > 0$ on $(x_0, x_0 + \delta)$ then f is strictly increasing both on the left and on the right of x_0, so x_0 is not a local extremum; if $f'(x) < 0$ on $(x_0 - \delta, x_0)$ and $f'(x) < 0$ on $(x_0, x_0 + \delta)$ then f is strictly decreasing both on the left and on the right of x_0 and so, as in the previous case, x_0 is not a local extremum. In other words, if the derivative changes sign at x_0 then we have a local extremum, otherwise we do not have a local extremum.

Since we need this $\delta > 0$, the first derivative test can be applied only to isolated stationary points. As we have seen in the introduction, it is commonly believed that if the first derivative does not change sign at the stationary point then the stationary point is a flex. As we will see in the next section, this is not always the case.

3. *Isolated stationary points that are neither local extrema nor inflection points*

We want to find a differentiable function f with an isolated stationary point at 0 that is neither a local extremum nor an inflection point. That is, we look for a monotone function such that near 0 the function changes its concavity infinitely often, i.e. near 0 the function has infinitely many inflection points (and so the δ in the Definition 2.2 cannot be found).

The trick is to start with x^3 (the easiest function with an isolated horizontal-tangent inflection point at 0) and then perturb it slightly by an

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oscillating function (e.g. ϕ_4 of Section 2.4) so that we can produce infinitely many non-stationary inflection points. Let us consider the odd function

$$f_4(x) = \begin{cases} x^3 + x^4 \sin\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Like ϕ_4, f_4 is continuous and differentiable on all $\mathbb{R}$. Its first derivative is

$$f_4'(x) = \begin{cases} 3x^2 + 4x^3 \sin\frac{1}{x} - x^2 \cos\frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

Like ϕ_4, f_4 is $C^1(\mathbb{R})$ and twice differentiable everywhere, but the second derivative is not continuous at 0.

We see that 0 is the only stationary point of f_4, f_4' is an even function that is positive everywhere (except at 0) so f_4 is strictly increasing. The function f_4 is plotted in Figure 2, and its first derivative in Figure 3. We see

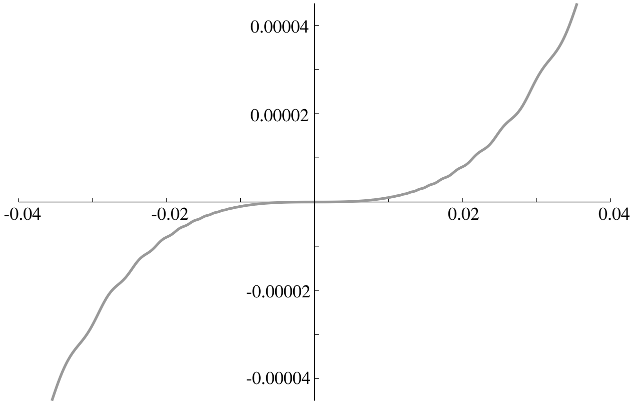

FIGURE 2: Plot of the function $f_4(x)$

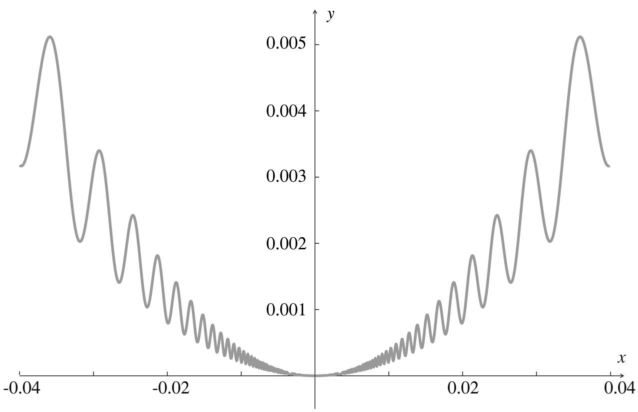

FIGURE 3: Plot of the function $f_4{}'(x)$

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that even if f_4 is positive on both sides of 0, f_4 repeatedly changes its trend: the upward trends of Figure 3 correspond to intervals in which the function is convex, the downward trends to intervals in which the function is concave. So, while approaching 0, the curve changes its concavity infinitely often. Each local extremum in Figure 3 (except $x = 0$) corresponds to a point of inflection in Fig. 2, hence the points of inflection accumulate around 0. For these reasons 0 is not an inflection point.

The previous example can be generalised using ϕ_{2k} of Section 2.4. For every $k \in \mathbb{N}, k \geqslant 2$, consider the odd function

$$f_{2k}(x) = \begin{cases} x^{2k-1} + x^{2k} \sin \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

its first derivative

$$f'_{2k}(x) = \begin{cases} (2k - 1)x^{2k-2} + 2kx^{2k-1} \sin \frac{1}{x} - x^{2k-2} \cos \frac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

and its second derivative

$$f''_{2k}(x) = \begin{cases} x^{2k-4}\mu(x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

where

$$\mu(x) = (2k - 1)(2k - 2)x + 2k(2k - 1)x^2 \sin \frac{1}{x} - (4k - 2)x \cos \frac{1}{x} - \sin \frac{1}{x}.$$

Like ϕ_{2k} of Section 2.4, $f_{2k} \in C^{2k-1}(\mathbb{R})$ and it is k-times differentiable on $\mathbb{R}$, but the k th derivative is not continuous at 0.

We see that f_{2k} has an isolated stationary point at 0, f'_{2k} is positive everywhere (except at 0), so f_{2k} is strictly increasing. Anyhow, 0 is not an inflection point, since, when approaching 0, the function changes its concavity infinitely often. This last statement is not obvious, and we need to justify it. That is why also the second derivative has been reported. Consider that the second derivative is continuous everywhere (except maybe at 0): this means that for each point at which the second derivative is positive there is a neighborhood for which the first derivative is increasing and hence the function is convex, and for each point at which the second derivative is negative there is a neighborhood for which the first derivative is decreasing and hence the function is concave. When x approaches 0 the second derivative changes sign infinitely often, because the first factor x^{2k-4} is positive, while the second factor has an addend $\sin \frac{1}{x}$ that repeatedly moves from -1 to 1, while the other summands tend to 0 when x tends to 0.

For each k we have found examples of C^k functions with isolated stationary points that are neither local extrema nor inflection points. It is possible to find C^∞ functions with the same property.

In this regard, consider the odd function

$$g(x) = \begin{cases} e^{-1/x} + \frac{1}{2}e^{-1/x}\sin\frac{1}{x} & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -e^{1/x} + \frac{1}{2}e^{1/x}\sin\frac{1}{x} & \text{if } x < 0, \end{cases}$$

its first derivative

$$g'(x) = \begin{cases} \frac{1}{x^2}e^{-1/x}\left(1 + \frac{1}{2}\sin\frac{1}{x} - \frac{1}{2}\cos\frac{1}{x}\right) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ \frac{1}{x^2}e^{1/x}\left(1 - \frac{1}{2}\sin\frac{1}{x} - \frac{1}{2}\cos\frac{1}{x}\right) & \text{if } x < 0, \end{cases}$$

and its second derivative

$$g''(x) = \begin{cases} \frac{1}{x^4}e^{-1/x}\left(1 - \cos\frac{1}{x} + x\left(-2 - \sin\frac{1}{x} + \cos\frac{1}{x}\right)\right) & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -\frac{1}{x^4}e^{-1/x}\left(1 - \cos\frac{1}{x} + x\left(2 - \sin\frac{1}{x} - \cos\frac{1}{x}\right)\right) & \text{if } x < 0. \end{cases}$$

Notice that $g \in C^\infty(\mathbb{R})$ (see [8, ex. 4.5.4] and Sect. 2.4). Again, g has an isolated stationary point at 0, g' is positive everywhere (except at 0), so g is strictly increasing. Anyway, 0 is not an inflection point, since, when approaching 0, the function changes its concavity infinitely often. In fact, consider the points $p_n = \frac{1}{n\pi}$ with $n \in \mathbb{N}\setminus\{0\}$. When n is even $g''(p_n) < 0$, when n is odd $g''(p_n) > 0$. We conclude as before.

Remark: Let f be a real analytic function on an open interval I. Let $x_0 \in I$ be a stationary point for f. Then in this case x_0 is actually either a local extremum or an inflection point, *tertium non datur*.

In fact the zeros of analytic functions are isolated, and in particular zeros are isolated for f' and f''. This means that not only x_0 is an isolated stationary point, but also that in a neighbourhood of x_0, f'' vanishes at most at x_0. If x_0 is not a local extremum then f'' must change sign at x_0 (and vanish there), otherwise f' would be strictly increasing or decreasing and would change sign at x_0 as well. Therefore x_0 is an inflection point.

In other words, since the second derivative has isolated zeros, then f cannot change its concavity infinitely often.

4. *Conclusion*

In summary, this article is a remainder to students and practitioners that some care is required when making more general statements about the first derivative test. The first derivative test is a number line test that can discern if an isolated stationary point is a local extremum or not, and also the nature of the local extremum (minimum or maximum), but without other pieces of information it cannot be used to conclude that an isolated stationary point is an inflection point.

In this Article we purposely focused on the first derivative test, but it is worth mentioning both the famous second derivative test ([3, thm. 16.10.1] or also [4] for its extended versions) and a higher-order derivative test (see [4, Conclusion] or [3, thm. 16.10.3]). In particular, the latter classifies any stationary point as a local minimum or a local maximum or a point of inflection, *provided that a non-zero derivative shows up eventually*. It is not coincidence that for all the examples $\phi_k, \gamma, f_{2k}, g$ in Sections 2 and 3, the n th order derivative either does not exist or it vanishes at x_0.

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