ANZIAM J. 50(2008), 129–136
doi:10.1017/S1446181108000308

ANOTHER SHARP L_2 INEQUALITY OF OSTROWSKI TYPE

ZHENG LIU[1]

(Received 11 August, 2008)

Abstract

A new sharp L_2 inequality of Ostrowski type is established, which provides some other interesting results as special cases. Applications in numerical integration are also given.

2000 *Mathematics subject classification*: primary 26D15.

Keywords and phrases: sharp bound, Ostrowski type inequality, corrected Simpson rule, corrected averaged midpoint-trapezoid rule, numerical integration.

1. Introduction

In [1] and [2], the author has proved the following two interesting sharp L_2 inequalities of Ostrowski type.

THEOREM 1.1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function whose derivative $f' \in L_2[a, b]$. Then for any $\theta \in [0, 1]$ and $x \in [a, b]$*

$$\left|\int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] + (1-\theta)\left(x - \frac{a+b}{2}\right)[f(b)-f(a)]\right| \le \left[\theta(1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)^2 + \frac{3\theta^2-3\theta+1}{12}(b-a)^3\right]^{1/2}\sqrt{\sigma(f')}, \tag{1.1}$$

where $\sigma(\cdot)$ is defined by

$$\sigma(f) = \|f\|_2^2 - \frac{1}{b-a}\left(\int_a^b f(t)\,dt\right)^2 \tag{1.2}$$

[1]Institute of Applied Mathematics, School of Science, University of Science and Technology Liaoning, Anshan 114051, Liaoning, People's Republic of China; e-mail: lewzheng@163.net.

and $\|f\|_2 := [\int_a^b f^2(t)\,dt]^{1/2}$. *Inequality* (1.1) *is sharp in the sense that the right-hand side cannot be replaced by a multiple* C *of itself with* $C < 1$.

THEOREM 1.2. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* f' *is absolutely continuous on* $[a, b]$ *and* $f'' \in L_2[a, b]$. *Then for any* $\theta \in [0, 1]$ *and* $x \in [a, b]$

$$\begin{aligned}
&\bigg|\int_a^b f(t)\,dt - (b-a)\bigg[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\bigg] \\
&+ (1-\theta)(b-a)\bigg(x - \frac{a+b}{2}\bigg)f'(x) \\
&- \bigg[\frac{1-\theta}{2}\bigg(x - \frac{a+b}{2}\bigg)^2 + \frac{1-3\theta}{24}(b-a)^2\bigg][f'(b) - f'(a)]\bigg| \\
&\quad \le \bigg[\frac{\theta(1-\theta)}{4}(b-a)\bigg(x - \frac{a+b}{2}\bigg)^4 + \frac{3\theta^2 - 5\theta + 2}{24}(b-a)^3\bigg(x - \frac{a+b}{2}\bigg)^2 \\
&\qquad + \frac{15\theta^2 - 15\theta + 4}{2880}(b-a)^5\bigg]^{1/2}\sqrt{\sigma(f'')}.
\end{aligned} \tag{1.3}$$

Inequality (1.3) *is sharp in the same sense as* (1.1).

In this work, we will derive a new sharp inequality of Ostrowski type for functions whose second derivatives are absolutely continuous and whose third derivatives belong to $L_2[a, b]$. We also give some other interesting sharp inequalities as special cases. Moreover, we show that the corrected Simpson rule (see [3–5]) gives a better result than the Simpson rule and, in particular, the corrected averaged midpoint-trapezoid quadrature rule is optimal. Applications in numerical integration are also given.

2. The results

THEOREM 2.1. *Let* $f : [a, b] \to \mathbb{R}$ *be such that* f'' *is absolutely continuous on* $[a, b]$ *and* $f''' \in L_2[a, b]$. *Then for any* $\theta \in [0, 1]$ *and* $x \in [a, b]$

$$\begin{aligned}
&\bigg|\int_a^b f(t)\,dt - (b-a)\bigg[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\bigg] \\
&+ (1-\theta)(b-a)\bigg(x - \frac{a+b}{2}\bigg)f'(x) - \frac{1-\theta}{2}(b-a)\bigg(x - \frac{a+b}{2}\bigg)^2 f''(x) \\
&- \frac{1-3\theta}{24}(b-a)^2[f'(b) - f'(a)] + \frac{1-\theta}{6}\bigg(x - \frac{a+b}{2}\bigg)^3[f''(b) - f''(a)]\bigg| \\
&\quad \le \bigg[\frac{\theta(1-\theta)}{36}(b-a)\bigg(x - \frac{a+b}{2}\bigg)^6 + \frac{3\theta^2 - 5\theta + 2}{96}(b-a)^3\bigg(x - \frac{a+b}{2}\bigg)^4 \\
&\qquad + \frac{63\theta^2 - 63\theta + 16}{483\,840}(b-a)^7\bigg]^{1/2}\sqrt{\sigma(f''')}.
\end{aligned} \tag{2.1}$$

Inequality (2.1) *is sharp in the same sense as* (1.1).

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press

PROOF. Let us define the function

$$K(x,t) := \begin{cases} \dfrac{(t-a)^3}{6} - \dfrac{\theta(b-a)}{4}(t-a)^2 - \dfrac{(1-3\theta)(b-a)^2}{24}(t-a), & t \in [a,x], \\ \dfrac{(t-b)^3}{6} + \dfrac{\theta(b-a)}{4}(t-b)^2 - \dfrac{(1-3\theta)(b-a)^2}{24}(t-b), & t \in (x,b]. \end{cases}$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_a^b K(x,t) f'''(t)\,dt \\ &\quad = \frac{1-\theta}{2}(b-a)\left(x-\frac{a+b}{2}\right)^2 f''(x) - (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right) f'(x) \\ &\quad\quad + \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)] + (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \\ &\quad\quad - \int_a^b f(t)\,dt. \end{aligned} \tag{2.2}$$

We also have

$$\int_a^b K(x,t)\,dt = \frac{1-\theta}{6}(b-a)\left(x-\frac{a+b}{2}\right)^3 \tag{2.3}$$

and

$$\int_a^b f'''(t)\,dt = f''(b) - f''(a). \tag{2.4}$$

From (2.2)–(2.4), it follows that

$$\begin{aligned} &\int_a^b \left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right]\left[f'''(t) - \frac{1}{b-a}\int_a^b f'''(s)\,ds\right] dt \\ &\quad = \frac{1-\theta}{2}(b-a)\left(x-\frac{a+b}{2}\right)^2 f''(x) - (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right) f'(x) \\ &\quad\quad + \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)] + (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right] \\ &\quad\quad - \int_a^b f(t)\,dt - \frac{1-\theta}{6}\left(x-\frac{a+b}{2}\right)^3 [f''(b)-f''(a)]. \end{aligned} \tag{2.5}$$

On the other hand,

$$\begin{aligned} &\left|\int_a^b \left[K(x,t) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right]\left[f'''(t) - \frac{1}{b-a}\int_a^b f'''(s)\,ds\right] dt\right| \\ &\quad \le \left\|K(x,\cdot) - \frac{1}{b-a}\int_a^b K(x,s)\,ds\right\|_2 \left\|f''' - \frac{1}{b-a}\int_a^b f'''(s)\,ds\right\|_2. \end{aligned} \tag{2.6}$$

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press

We also have

$$\begin{aligned}
&\left\| K(x,\cdot) - \frac{1}{b-a}\int_a^b K(x,s)\,ds \right\|_2^2 \\
&\quad = \frac{\theta(1-\theta)}{36}(b-a)\left(x-\frac{a+b}{2}\right)^6 + \frac{3\theta^2-5\theta+2}{96}(b-a)^3\left(x-\frac{a+b}{2}\right)^4 \\
&\qquad + \frac{63\theta^2-63\theta+16}{483\,840}(b-a)^7
\end{aligned} \tag{2.7}$$

and

$$\left\| f''' - \frac{1}{b-a}\int_a^b f'''(s)\,ds \right\|_2^2 = \|f'''\|_2^2 - \frac{(f''(b)-f''(a))^2}{b-a}. \tag{2.8}$$

From (2.5)–(2.8), we can easily get (2.1), since by (1.2)

$$\sqrt{\sigma(f''')} = \left[\|f'''\|_2^2 - \frac{(f''(b)-f''(a))^2}{b-a}\right]^{1/2}.$$

In order to prove that the inequality (2.1) is sharp, we now suppose that (2.1) holds with a constant $C > 0$ as

$$\begin{aligned}
&\left| \int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f(x) + \theta\frac{f(a)+f(b)}{2}\right]\right. \\
&\quad + (1-\theta)(b-a)\left(x-\frac{a+b}{2}\right)f'(x) - \frac{1-\theta}{2}(b-a)\left(x-\frac{a+b}{2}\right)^2 f''(x) \\
&\quad \left. - \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)] + \frac{1-\theta}{6}\left(x-\frac{a+b}{2}\right)^3[f''(b)-f''(a)]\right| \\
&\quad \le C\left[\frac{\theta(1-\theta)}{36}(b-a)\left(x-\frac{a+b}{2}\right)^6 + \frac{3\theta^2-5\theta+2}{96}(b-a)^3\right. \\
&\qquad \left.\times\left(x-\frac{a+b}{2}\right)^4 + \frac{63\theta^2-63\theta+16}{483\,840}(b-a)^7\right]^{1/2}\sqrt{\sigma(f''')}.
\end{aligned} \tag{2.9}$$

We may find a function $f:[a,b]\to\mathbb{R}$ such that f'' is absolutely continuous on $[a,b]$ as

$$f''(x) = \begin{cases}
\frac{1}{24}[(t-a)^4-(x-a)^4] - \frac{\theta(b-a)}{12}[(t-a)^3-(x-a)^3] \\
\quad -\frac{(1-3\theta)(b-a)^2}{48}[(t-a)^2-(x-a)^2], & t\in[a,x], \\
\frac{1}{24}[(t-b)^4-(x-b)^4] - \frac{\theta(b-a)}{12}[(t-b)^3-(x-b)^3] \\
\quad -\frac{(1-3\theta)(b-a)^2}{48}[(t-b)^2-(x-b)^2], & t\in(x,b].
\end{cases}$$

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press

It follows that

$$f'''(x) = \begin{cases} \dfrac{(t-a)^3}{6} - \dfrac{\theta(b-a)}{4}(t-a)^2 - \dfrac{(1-3\theta)(b-a)^2}{24}(t-a), & t \in [a, x], \\ \dfrac{(t-b)^3}{6} + \dfrac{\theta(b-a)}{4}(t-b)^2 - \dfrac{(1-3\theta)(b-a)^2}{24}(t-b), & t \in (x, b]. \end{cases} \tag{2.10}$$

By (2.5), (2.7), (2.8) and (2.10), it is not difficult to find that the left-hand side of the inequality (2.9) becomes

$$\frac{\theta(1-\theta)}{36}(b-a)\left(x-\frac{a+b}{2}\right)^6 + \frac{3\theta^2-5\theta+2}{96}(b-a)^3 \times \left(x-\frac{a+b}{2}\right)^4 + \frac{63\theta^2-63\theta+16}{483\,840}(b-a)^7 \tag{2.11}$$

and the right-hand side of the inequality (2.9) is

$$C\left[\frac{\theta(1-\theta)}{36}(b-a)\left(x-\frac{a+b}{2}\right)^6 + \frac{3\theta^2-5\theta+2}{96}(b-a)^3 \times \left(x-\frac{a+b}{2}\right)^4 + \frac{63\theta^2-63\theta+16}{483\,840}(b-a)^7\right]. \tag{2.12}$$

From (2.9), (2.11) and (2.12), we find that $C \geq 1$, proving that the coefficient constant 1 is the best possible in (2.1). □

COROLLARY 2.2. *Let the assumptions of Theorem 2.1 hold. Then for any $\theta \in [0, 1]$,*

$$\left|\int_a^b f(t)\,dt - (b-a)\left[(1-\theta)f\left(\frac{a+b}{2}\right) + \theta\frac{f(a)+f(b)}{2}\right] - \frac{1-3\theta}{24}(b-a)^2[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{7/2}}{48\sqrt{210}}(63\theta^2-63\theta+16)^{1/2}\sqrt{\sigma(f''')}. \tag{2.13}$$

PROOF. We set $x = (a+b)/2$ in (2.1) to get (2.13). □

REMARK 1. If we take $\theta = 1$ and $\theta = 0$ in (2.13), then we get a sharp corrected trapezoid inequality as

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{2}[f(a)+f(b)] + \frac{(b-a)^2}{12}[f'(b)-f'(a)]\right| \leq \frac{(b-a)^{7/2}}{12\sqrt{210}}\sqrt{\sigma(f''')},$$

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press

and a sharp corrected midpoint inequality as

$$\left|\int_a^b f(t)\,dt-(b-a)f\left(\frac{a+b}{2}\right)-\frac{(b-a)^2}{24}[f'(b)-f'(a)]\right|$$
$$\leq\frac{(b-a)^{7/2}}{12\sqrt{210}}\sqrt{\sigma(f''')}.$$

REMARK 2. If we take $\theta=1/3$ in (2.13), we get a sharp Simpson-type inequality as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]\right|\leq\frac{(b-a)^{7/2}}{48\sqrt{105}}\sqrt{\sigma(f''')}. \tag{2.14}$$

If we take $\theta=7/15$ in (2.13), we get a sharp corrected Simpson-type inequality as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{30}\left[7f(a)+16f\left(\frac{a+b}{2}\right)+7f(b)\right]+\frac{(b-a)^2}{60}[f'(b)-f'(a)]\right|$$
$$\leq\frac{(b-a)^{7/2}}{120\sqrt{105}}\sqrt{\sigma(f''')}. \tag{2.15}$$

From (2.14) and (2.15), we see that the corrected Simpson rule gives better results than the Simpson rule.

REMARK 3. If we take $\theta=1/2$ in (2.13), we get a sharp corrected averaged midpoint-trapezoid type inequality as

$$\left|\int_a^b f(t)\,dt-\frac{b-a}{4}\left[f(a)+2f\left(\frac{a+b}{2}\right)+f(b)\right]+\frac{(b-a)^2}{48}[f'(b)-f'(a)]\right|$$
$$\leq\frac{(b-a)^{7/2}}{96\sqrt{210}}\sqrt{\sigma(f''')}. \tag{2.16}$$

It is interesting to notice that the smallest bound for (2.1) is obtained at $x=(a+b)/2$ and $\theta=1/2$. Thus the corrected averaged midpoint-trapezoid rule is optimal in the current situation.

3. Application in numerical integration

We restrict further considerations to the corrected averaged midpoint-trapezoid quadrature rule. We also emphasize that similar considerations can be given for all quadrature rules considered in the previous section.

THEOREM 3.1. *Let $\pi=\{x_0=a<x_1<\cdots<x_n=b\}$ be a given subdivision of the interval $[a,b]$ such that $h_i=x_{i+1}-x_i=h=(b-a)/n$ and let the assumptions of*

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press

Theorem 2.1 hold. Then

$$\left|\int_a^b f(t)\,dt - \frac{b-a}{4n}\sum_{i=0}^{n-1}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right] + \frac{(b-a)^2}{48n^2}[f'(b)-f'(a)]\right| \le \frac{(b-a)^{7/2}}{96\sqrt{210}n^3}\sqrt{\sigma(f''')}. \tag{3.1}$$

PROOF. From (2.16) we obtain

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - \frac{h}{4}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right] + \frac{h^2}{48}[f'(x_{i+1})-f'(x_i)]\right| \le \frac{h^{7/2}}{96\sqrt{210}}\left[\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}(f'(x_{i+1})-f'(x_i))^2\right]^{1/2}. \tag{3.2}$$

By summing (3.2) over i from 0 to $n-1$ and using the generalized triangle inequality, we get

$$\left|\int_a^b f(t)\,dt - \frac{h}{4}\sum_{i=0}^{n-1}\left[f(x_i)+2f\left(\frac{x_i+x_{i+1}}{2}\right)+f(x_{i+1})\right] + \frac{h^2}{48}[f'(b)-f'(a)]\right| \le \frac{h^{7/2}}{96\sqrt{210}}\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+i}} (f''(t))^2\,dt - \frac{1}{h}(f'(x_{i+1})-f'(x_i))^2\right]^{1/2}. \tag{3.3}$$

By using the Cauchy inequality twice, it is not difficult to obtain

$$\sum_{i=0}^{n-1}\left[\int_{x_i}^{x_{i+1}} (f'''(t))^2\,dt - \frac{1}{h}(f''(x_{i+1})-f''(x_i))^2\right]^{1/2} \le \sqrt{n}\left[\|f'''\|_2^2 - \frac{n}{b-a}\sum_{i=0}^{n-1}(f''(x_{i+1})-f''(x_i))^2\right]^{1/2} \le \sqrt{n}\left[\|f'''\|_2^2 - \frac{(f''(b)-f''(a))^2}{b-a}\right]^{1/2} = \sqrt{n\sigma(f''')}. \tag{3.4}$$

Consequently, the inequality (3.1) follows from (3.3) and (3.4). □

References

[1] Z. Liu, "Note on a paper by N. Ujević", *Appl. Math. Lett.* **20** (2007) 659–663.
[2] Z. Liu, "A sharp L_2 inequality of Ostrowski type", *ANZIAM J.* **49** (2008) 423–429.
[3] J. Pečarić and I. Franjić, "Generalization of corrected Simpson's formula", *ANZIAM J.* **47** (2006) 367–385.
[4] N. Ujević and A. J. Roberts, "A corrected quadrature formula and applications", *ANZIAM J.* **45(E)** (2004) E41–E56.
[5] N. Ujević, "A generalization of the modified Simpson's rule and error bounds", *ANZIAM J.* **47(E)** (2005) E1–E13.

https://doi.org/10.1017/S1446181108000308 Published online by Cambridge University Press