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Typical properties of periodic Teichmüller geodesics: Lyapunov exponents

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Abstract. Consider a component $\mathcal{Q}$ of a stratum in the moduli space of area-one abelian differentials on a surface of genus g. Call a property $\mathcal{P}$ for periodic orbits of the Teichmüller flow on $\mathcal{Q}$ *typical* if the growth rate of orbits with property $\mathcal{P}$ is maximal. We show that the following property is typical. Given a continuous integrable cocycle over the Teichmüller flow with values in a vector bundle $V \to \mathcal{Q}$, the logarithms of the eigenvalues of the matrix defined by the cocycle and the orbit are arbitrarily close to the Lyapunov exponents of the cocycle for the Masur–Veech measure.

Key words: abelian differentials, Teichmüller flow, periodic orbits, Lyapunov exponents, equidistribution
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1. *Introduction*

The *mapping class group* $\mathrm{Mod}(S)$ of a closed surface S of genus $g \geq 0$ with $m \geq 0$ punctures and $3g - 3 + m \geq 2$ acts by precomposition of marking on the *Teichmüller space* $\mathcal{T}(S)$ of marked complex structures on S. The action is properly discontinuous, with quotient the *moduli space* $\mathcal{M}$ of complex structures on S.

If $m = 0$, then the *Hodge bundle* $\mathcal{H} \to \mathcal{M}$ over moduli space is the bundle whose fibre over a Riemann surface x equals the vector space of holomorphic one-forms on x. This is a holomorphic vector bundle (in the orbifold sense) of complex dimension g which decomposes into *strata* of differentials with zeros of given number and multiplicities. Strata need not be connected, but their components are classified [**KZ03**]. The sphere subbundle for the natural norm obtained by integration of a holomorphic one-form over the base surface is the moduli space of area-one abelian differentials on S. There is a natural $SL(2, \mathbb{R})$-action on this sphere bundle preserving every component $\mathcal{Q}$ of a stratum. The action of the diagonal subgroup is called the *Teichmüller flow* Φ^t.

Similarly, for arbitrary (g, m), the sphere bundle in the cotangent bundle of $\mathcal{M}$ can be identified with the (orbifold) bundle of area-one quadratic differentials over $\mathcal{M}$. It decomposes into strata of differentials with fixed number and multiplicities of zeros and simple poles. Strata need not be connected, but connected components are classified [**L08**]. Each connected component of such a stratum is invariant under a natural $SL(2, \mathbb{R})$-action whose diagonal subgroup acts as the Teichmüller flow Φ^t.

Let now $\mathcal{Q}$ be such a component of area-one abelian or quadratic differentials. Let $k \geq 1$ be the total number of zeros and poles of the differentials in $\mathcal{Q}$ and let $h = 2g - 1 + k$ in case $\mathcal{Q}$ is a component of abelian differentials, and $h = 2g - 2 + k$ otherwise. Let Γ be the set of all periodic orbits for Φ^t in $\mathcal{Q}$. The length of a periodic orbit $\gamma \in \Gamma$ is denoted by $\ell(\gamma)$.

As an application of [**EMR19**] (see also [**EM11**]), we showed in [**H13**] that

$$\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Call a subset $\mathcal{A}$ of Γ *typical* if

$$\sharp\{\gamma \in \mathcal{A} \mid \ell(\gamma) \leq R\}\frac{hR}{e^{hR}} \to 1 \quad (R \to \infty).$$

Thus, a subset of Γ is typical if its growth rate is maximal. The intersection of two typical subsets of Γ is typical.

Now let us assume that we are given a finite-dimensional real or complex vector space V and a representation $\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(V)$. Using this representation, we can form the flat vector bundle $\mathcal{N}_0 = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} V \to \mathcal{M}$, where $\mathrm{Mod}(S)$ acts on V via the representation Ψ. This bundle is equipped with a natural flat connection. For each component $\mathcal{Q}$ of a stratum of area-one abelian or quadratic differentials, we can consider the pullback $P^*\mathcal{N}_0 = \mathcal{N}$ of $\mathcal{N}_0$ to $\mathcal{Q}$, where $P : \mathcal{Q} \to \mathcal{M}$ is the natural projection. The pullback bundle is equipped with the flat pullback connection.

An important example for $m = 0$ arises from the representation

$$\Psi : \mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$$

defined by the action of $\mathrm{Mod}(S)$ on the first real cohomology group $H^1(S, \mathbb{R})$ of S which preserves the intersection form ι on $H^1(S, \mathbb{Z})$. The corresponding flat bundle over moduli space is just the Hodge bundle, and the flat connection obtained in this way is called the *Gauss–Manin* connection.

Given a flat vector bundle $\mathcal{N} \to \mathcal{Q}$, we can parallel transport the fibres of $\mathcal{N}$ along the flow lines of the Teichmüller flow Φ^t. This defines a continuous cocycle over Φ^t.

A periodic orbit $\gamma \in \Gamma$ for Φ^t determines a holonomy map for $\mathcal{N}$, which is the first return map of the parallel transport along γ for a fixed choice of a base point in γ. Different base points give rise to conjugate maps. Thus, a periodic orbit $\gamma \in \Gamma$ determines the conjugacy class $[A(\gamma)]$ of a matrix $A(\gamma) \in \mathrm{GL}(V)$, which is just the image under the representation Ψ of the conjugacy class of a pseudo-Anosov mapping class ϕ which defines the periodic orbit γ.

The $SL(2, \mathbb{R})$-action on $\mathcal{Q}$ preserves a Borel probability measure λ in the Lebesgue measure class, the so-called *Masur–Veech measure* [**M82**, **V86**]. If the cocycle defined by the flat bundle $\mathcal{N}$ is integrable for the action of the Teichmüller flow with respect to this measure, then we can apply the Oseledets multiplicative ergodic theorem [**O68**] to obtain *Lyapunov exponents* of the cocycle with respect to λ. These Lyapunov exponents

$$\kappa_1 \geq \cdots \geq \kappa_n$$

listed in descending order describe the asymptotic growth rate of vectors in subcones determined by the Lyapunov filtration of $\mathcal{N}$ as described by Oseledec's theorem. As an example, the flat bundle defined by the standard representation $\mathrm{Mod}(S) \to \mathrm{Sp}(2g, \mathbb{Z})$ is integrable in this sense.

For $\gamma \in \Gamma$, let $\hat{\alpha}_i(\gamma)$ be the logarithm of the absolute value of the ith eigenvalue of the matrix $A(\gamma)$, ordered in decreasing order, and write $\alpha_i(\gamma) = \hat{\alpha}_i(\gamma)/\ell(\gamma)$. Since eigenvalues of matrices are invariant under conjugation, this does not depend on the choice of a representative in the class $[A(\gamma)]$ and, for $i \leq n$, we obtain in this way a function $\alpha_i : \Gamma \to \mathbb{R}$.

We show the following result.

THEOREM 1. *For each $\epsilon > 0$, the set $\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \kappa_i| < \epsilon\}(1 \leq i \leq n)$ is typical.*

The main technical tool towards this result is a hyperbolicity property for the action of the Teichmüller flow on a component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials. For its formulation, recall [**M82**, **V86**] that for every $q \in \mathcal{Q}$, the set of differentials in $\mathcal{Q}$ with the same vertical (or horizontal) measured foliation as q is a local suborbifold $W^{ss}_{\mathcal{Q}}(q)$ (or $W^{su}_{\mathcal{Q}}(q)$) of $\mathcal{Q}$. These local suborbifolds can be equipped with a natural distance function d_H obtained from the *modified Hodge norm* [**ABEM12**]. For $r > 0$, we denote by $B^i_{\mathcal{Q}}(q, r) \subset W^i_{\mathcal{Q}}(q)(i = ss, su)$ the ball of radius r about q for the distance function d_H.

Denote the set of manifold points in $\mathcal{Q}$ by $\mathcal{Q}_{\mathrm{good}}$. Then $\mathcal{Q}_{\mathrm{good}}$ is an open dense Φ^t-invariant subset of $\mathcal{Q}$ (see the discussion in §3 for more details). Call a point $q \in \mathcal{Q}$ *recurrent* if it is contained in its own α- and ω-limit set for the action of Φ^t. This means that there are sequences $s_i, t_i \to \infty$ so that $\Phi^{t_i} q \to q$ and $\Phi^{-s_i} q \to q (i \to \infty)$. We show in §4 the following strengthening of contraction results, which can be found in [**ABEM12**, **H13**].

THEOREM 2. *Let $q \in \mathcal{Q}_{\mathrm{good}}$ be a recurrent point. Then there is a number $r_0 = r_0(q) > 0$ and there is a neighborhood U of q in $\mathcal{Q}_{\mathrm{good}}$ with the following property. Let $z \in U$ be recurrent; then, for every $a > 0$, there is a number $T(z, a) > 0$ so that for all $T > T(z, a)$, we have $\Phi^T B^{ss}_{\mathcal{Q}}(z, r_0) \subset B^{ss}_{\mathcal{Q}}(\Phi^T(z), a)$ and $\Phi^T B^{su}_{\mathcal{Q}}(z, a) \supset B^{su}_{\mathcal{Q}}(\Phi^T(z), r_0)$.*

Theorem 2 states in a precise and quantitative way that the Teichmüller flow is non-uniformly hyperbolic, that is, it contracts and expands balls of locally uniform size on the strong stable and strong unstable manifolds along any recurrent orbit. Thus, it applies to any orbit which is typical for any finite invariant Borel probability measure, and it does not rely on assumptions like the percentage of time the orbit spends in the thick part of moduli space used in [**ABEM12**]. This is particularly valuable as there are periodic orbits and hence invariant measures for the Teichmüller flow which are entirely contained in any prescribed neighborhood of the cusp of moduli space [**H16**].

The proof of Theorem 2 rests on two facts. First, although Teichmüller space equipped with the Teichmüller metric is far from being hyperbolic or even resembling in the large a CAT(0) metric space, convex subsets of its thick part are hyperbolic in the sense of Gromov in a way which can be made quantitative [**H10**]. Second, Teichmüller space is *acylindrically hyperbolic* in the sense that it admits a coarsely well defined coarsely Lipschitz Mod(S)-equivariant map onto the *curve graph* of S, a Gromov hyperbolic Mod(S)-graph. This curve graph records hyperbolic behavior in Teichmüller space and hence of its geodesic flow, the Teichmüller flow. In spite of being a locally infinite graph, it can effectively be used as a witness of hyperbolicity. As far as we know, quantitative results on non-uniform hyperbolicity of the Teichmüller flow as strong as Theorem 2 are out of reach using more standard methods. What is more, such ideas are most likely applicable to other non-uniformly hyperbolic geometric flows, like geodesic flows on closed rank-one manifolds of non-positive curvature with acylindrically hyperbolic fundamental group.

The structure of this article is as follows. In §2 we recall some properties of flat bundles needed later on. Section 3 introduces the curve graph, geodesic laminations and some additional basic tools used in the proofs and explains some technical results from [**H13**]. This section was included to make the article essentially self-contained. Section 4 is devoted to a study of some specific properties of the Teichmüller flow which resemble properties of a hyperbolic flow, extending the results of [**H13**]. We also prove Theorem 2, which is the main technical tool for the proof of Theorem 1. In §5 we establish a local criterion for a property to be typical, which is applied in §6 to show Theorem 1.

2. *Flat bundles*

Let S be an oriented surface of finite type, that is, S is a closed surface of genus $g \geq 0$ from which $m \geq 0$ points, so-called *punctures*, have been deleted. We assume that $3g - 3 + m \geq 2$, which means that S is not a sphere with at most four punctures or a torus with at most one puncture.

The *Teichmüller space* $\mathcal{T}(S)$ of S is the quotient of the space of all complete finite-volume hyperbolic metrics on S under the action of the group of diffeomorphisms of S which are isotopic to the identity. The fibre bundle $\tilde{\mathcal{Q}}(S)$ over $\mathcal{T}(S)$ of all *marked holomorphic quadratic differentials* of area one can be viewed as the unit cotangent bundle of $\mathcal{T}(S)$ for the *Teichmüller metric* $d_{\mathcal{T}}$. Each such differential is holomorphic on the complement of the punctures and has at most a simple pole at each puncture. The *Teichmüller flow* Φ^t on $\tilde{\mathcal{Q}}(S)$ commutes with the action of the *mapping class group* Mod(S) of all isotopy classes of orientation-preserving self-homeomorphisms of S. Therefore, this flow descends to a flow on the quotient orbifold $\mathcal{Q}(S) = \tilde{\mathcal{Q}}(S)/\mathrm{Mod}(S)$, again denoted by Φ^t. This orbifold is a fibre bundle in the orbifold sense over the *moduli space of curves* $\mathcal{M} = \mathcal{T}(S)/\mathrm{Mod}(S)$.

Let V be a finite-dimensional real or complex vector space and let $\Psi : \mathrm{Mod}(S) \to \mathrm{GL}(V)$ be an irreducible representation. Then we obtain a flat bundle

$$\mathcal{N}_0 = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} V \to \mathcal{M},$$

where Mod(S) acts on $\mathcal{T}(S) \times V$ from the right by $(x, Z)\phi = (x\phi, \Psi(\phi^{-1})(Z))$. The pullback of $\mathcal{N}_0$ to a component $\mathcal{Q}$ of a stratum of abelian or quadratic differentials is again a flat vector bundle $\mathcal{N} \to \mathcal{Q}$, equipped with the pullback connection.

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The Teichmüller flow Φ^t acts on $\mathcal{N}$ as a one-parameter group of bundle automorphisms. It associates to (t, v, Z) the image of Z under parallel transport along the subsegment of the flow line of the Teichmüller flow through v of oriented length t.

The holonomy group of the bundle $\mathcal{N}_0 = \mathcal{T}(S) \times_{\mathrm{Mod}(S)} V \to \mathcal{M}$ is the image of Mod(S) under the homomorphism Ψ. As the action of the mapping class group on $\mathcal{T}(S)$ is not free, we have to be slightly careful when computing the holonomy about a closed loop in $\mathcal{M}$. The following discussion is geared at circumventing this difficulty.

Let $\mathrm{Sing} \subset \mathcal{T}(S)$ be the Mod(S)-invariant subvariety of surfaces with non-trivial automorphisms. The complex codimension of Sing is at least two. We will not need this fact in the following; all we need is that this set is closed and nowhere dense. Let $\tilde{\alpha} : [0, 1] \to \mathcal{T}(S)$ be any smooth path with $\tilde{\alpha}(0), \tilde{\alpha}(1) \in \mathcal{T}(S) - \mathrm{Sing}$. Assume that there is an element $\phi \in \mathrm{Mod}(S)$ so that $\phi(\tilde{\alpha}(0)) = \tilde{\alpha}(1)$. Then ϕ is unique. Furthermore, $\tilde{\alpha}$ projects to a closed path α in $\mathcal{M}$. Up to conjugation, the holonomy along α for the flat connection on $\mathcal{N}_0$ equals the map $\Psi \circ \phi^{-1}$ which maps the fibre of $\mathcal{N}_0$ over $\tilde{\alpha}(1)$ to the fibre of $\mathcal{N}_0$ over $\tilde{\alpha}(0)$. It depends only on the end points of $\tilde{\alpha}$. In particular, it is well defined even if the path $\tilde{\alpha}$ is not entirely contained in $\mathcal{T}(S) - \mathrm{Sing}$.

As $\mathrm{Sing} \subset \mathcal{T}(S)$ is closed and nowhere dense, there exists a contractible neighborhood U of $\tilde{\alpha}(0)$ which is entirely contained in $\mathcal{T}(S) - \mathrm{Sing}$ and such that $\eta(U) \cap U = \emptyset$ for all $\mathrm{Id} \neq \eta \in \mathrm{Mod}(S)$. Let $\gamma : [0, 1] \to \mathcal{T}(S) - \mathrm{Sing}$ be any smooth path which connects a point $\gamma(0) \in U$ to a point $\gamma(1) = \phi(\gamma(0))$ in $\phi(U)$. The discussion in the previous paragraph shows that the holonomy of the parallel transport of $\mathcal{N}$ along the projection of γ to $\mathcal{M}$ is conjugate to the holonomy of the parallel transport along α. In particular, the absolute values of the eigenvalues of these holonomy maps coincide. The same consideration is also valid for the holonomy of the pullback bundle $\mathcal{N} \to \mathcal{Q}$ with respect to the flat pullback connection.

We use this fact as follows. Define the *good subset* $\mathcal{Q}_{\mathrm{good}}$ of $\mathcal{Q}$ to be the set of all points $q \in \mathcal{Q}$ with the following property. Let $\tilde{\mathcal{Q}}$ be a component of the preimage of $\mathcal{Q}$ in the Teichmüller space of *marked* abelian or quadratic differentials and let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a lift of q; then an element of Mod(S) which fixes $\tilde{q}$ acts as the identity on $\tilde{\mathcal{Q}}$ (compare [**H13**] for more information on this technical condition) and hence it centralizes the stabilizer of $\tilde{\mathcal{Q}}$ in Mod(S) and fixes every point in the projection of $\tilde{\mathcal{Q}}$ to $\mathcal{T}(S)$. Due to the existence of hyperelliptic components of strata, this group may be non-trivial.

To avoid technical issues related to this fact, we always assume in the following that the restriction of the representation to the stabilizer of$\tilde{\mathcal{Q}}$ in Mod(S) factors through the quotient by the finite subgroup which acts trivially on $\tilde{\mathcal{Q}}$. We then always pass to this quotient whenever the component $\mathcal{Q}$ has this property. This is consistent with the fact that the orbifold fundamental group of $\mathcal{Q}$ equals this quotient. Note that $\mathcal{Q}_{\mathrm{good}}$ is precisely the subset of $\mathcal{Q}$ of manifold points. Lemma 4.4 of [**H13**] shows that the good subset $\mathcal{Q}_{\mathrm{good}}$ of $\mathcal{Q}$ is open, dense and Φ^t-invariant.

Definition 2.1. A closed curve $\eta : [0, a] \to \mathcal{Q}_{\mathrm{good}}$ *defines the conjugacy class of a pseudo-Anosov mapping class* $\phi \in \mathrm{Mod}(S)$ if the following holds true. Let $\tilde{\eta}$ be a lift of η to an arc in the Teichmüller space of abelian differentials. Then $\psi\tilde{\eta}(0) = \tilde{\eta}(a)$ for some $\psi \in \mathrm{Mod}(S)$ and we require that ψ is conjugate to ϕ.

In the above definition, we associate a conjugacy class in $\mathrm{Mod}(S)$ to a *parameterized* curve η. That is, we think of the curve as defining an element in the (orbifold) fundamental group of $\mathcal{M}$ by projecting it to a loop in $\mathcal{M}$. As the loop η is contained in $\mathcal{Q}_{\text{good}}$, its lifts to the Teichmüller space of abelian differentials consist of a countable collection of arcs whose end points are not fixed by any element in $\mathrm{Mod}(S)$ (unless the element fixes a component $\tilde{\mathcal{Q}}$ of the preimage of $\tilde{\mathcal{Q}}$ pointwise and hence is contained in the center of the stabilizer of $\tilde{\mathcal{Q}}$ and, in this case, by the above convention, the definition still makes sense).

Since η is a closed curve, its projection to $\mathcal{M}$ is closed as well and hence the end points of such a lift (which are points in the cotangent space of $\mathcal{T}(S)$) are paired by an element in $\mathrm{Mod}(S)$, which moreover is unique. A different choice of lift defines an element of $\mathrm{Mod}(S)$ in the same conjugacy class. Moreover, changing the parameterization preserves the conjugacy class as well. Note also that a closed curve in $\mathcal{Q}_{\text{good}}$ may define the conjugacy class of an element of $\mathrm{Mod}(S)$ which is not pseudo-Anosov. As we will see later, those curves play no role for us.

Consider a smooth closed parameterized curve $\alpha : [0, 1] \to \mathcal{Q}_{\text{good}}$. As before, the parallel transport along α of the bundle $\mathcal{N} = P^*\mathcal{N}_0 \to \mathcal{Q}$ with respect to the flat pullback connection is defined.

Using Definition 2.1, the following is now immediate from the above discussion.

LEMMA 2.2. *Let $\eta \subset \mathcal{Q}_{\text{good}}$ be a closed curve which defines the conjugacy class of a pseudo-Anosov mapping class $\phi \in \mathrm{Mod}(S)$. Then the eigenvalues of the holonomy map obtained by parallel transport of the bundle $\mathcal{N}$ along η coincide with the eigenvalues of the map $\Psi \circ \phi^{-1}$.*

Proof. As discussed above, if $\eta : [0, a] \to \mathcal{Q}_{\text{good}}$ is a closed curve and if $\tilde{\eta}$ is a lift of η to the Teichmüller space of marked abelian differentials, then there is a unique element $\psi \in \mathrm{Mod}(S)$ with $\psi(\tilde{\eta}(0)) = \tilde{\eta}(a)$. This element pairs the end points of the projection of $\tilde{\eta}$ to $\mathcal{T}(S)$. The conjugacy class of the element $\Psi \circ \psi^{-1} \in \mathrm{GL}(V)$ does not depend on choices. Since $\mathcal{N}$ is the pullback of a bundle on $\mathcal{M}$, the absolute values of the eigenvalues of $\Psi \circ \psi^{-1}$ are precisely the absolute values of the eigenvalues of the holonomy map for parallel transport of $\mathcal{N}$ along η. The lemma now follows from the definition of a curve which defines the conjugacy class of ϕ. □

3. *Laminations and the curve graph*

The goal of this section is to summarize some results from [**H13**] used later on. It was included here to make the article self-contained. An introduction to the complex of curves can be found in [**S06**]. Geodesic laminations are introduced and discussed in detail in [**CEG06**, **Mar16**]. The surveys [**Wr16**, **Z99**] give an accessible introduction to flat surfaces and dynamics on the moduli space of abelian and quadratic differentials.

3.1. *Geodesic laminations.*

A *geodesic lamination* for a complete hyperbolic structure on S of finite volume is a *compact* subset of S which is foliated into simple geodesics. A geodesic lamination ν is called *minimal* if each of its half-leaves is dense in ν. Thus, a simple closed geodesic is a minimal geodesic lamination. A minimal geodesic lamination with more than one leaf has uncountably many leaves and is called *minimal arational*.

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Every geodesic lamination ν consists of a disjoint union of finitely many minimal components and a finite number of isolated leaves. Each of the isolated leaves of ν either is an isolated closed geodesic and hence a minimal component, or it *spirals* about one or two minimal components. A geodesic lamination ν *tightly fills* S if its complementary components are topological discs or once-punctured monogons, that is, once-punctured discs bounded by a single leaf of ν. Note that this definition deviates from the standard definition of filling, which only requires that a geodesic lamination decomposes S into discs and once-punctured discs. A geodesic lamination which tightly fills a surface with punctures is not orientable.

The set $\mathcal{L}$ of all geodesic laminations on S can be equipped with the restriction of the *Hausdorff topology* for compact subsets of S. With respect to this topology, the space $\mathcal{L}$ is compact.

A *measured geodesic lamination* is a geodesic lamination ν equipped with a translation-invariant transverse measure ξ such that the ξ-weight of every compact arc in S with end points in $S - \nu$ which intersects ν non-trivially and transversely is positive. We say that ν is the *support* of the measured geodesic lamination. The geodesic lamination ν is *uniquely ergodic* if, up to scale, ξ is the only transverse measure with support ν.

The space $\mathcal{ML}$ of measured geodesic laminations equipped with the weak*-topology admits a natural continuous action of the multiplicative group $(0, \infty)$. The quotient under this action is the space $\mathcal{PML}$ of *projective measured geodesic laminations* which is homeomorphic to the sphere $S^{6g-7+2m}$.

Every simple closed geodesic c on S defines a measured geodesic lamination. The geometric intersection number between simple closed curves on S extends to a continuous function ι on $\mathcal{ML} \times \mathcal{ML}$, the *intersection form*. We say that a pair $(\xi, \mu) \in \mathcal{ML} \times \mathcal{ML}$ of measured geodesic laminations *binds* S if, for every measured geodesic lamination $\eta \in \mathcal{ML}$, we have $\iota(\eta, \xi) + \iota(\eta, \mu) > 0$. This is equivalent to stating that every complete simple (possibly infinite) geodesic on S intersects either the support of ξ or the support of μ transversely.

3.2. *The curve graph.* The *curve graph* $\mathcal{C}(S)$ of S is the locally infinite metric graph whose vertices are the free homotopy classes of essential simple closed curves on S, that is, curves which are neither contractible nor freely homotopic into a puncture. Two such curves are connected by an edge of length one if and only if they can be realized disjointly. The mapping class group $\mathrm{Mod}(S)$ of S acts on $\mathcal{C}(S)$ as a group of simplicial isometries.

The curve graph $\mathcal{C}(S)$ is a hyperbolic geodesic metric space [**MM99**] and hence it admits a *Gromov boundary* $\partial\mathcal{C}(S)$. For $c \in \mathcal{C}(S)$, there is a complete distance function δ_c on $\partial\mathcal{C}(S)$ of uniformly bounded diameter and there is a number $\rho > 0$ such that

$$\delta_c \leq e^{\rho d(c,a)} \delta_a \quad \text{for all } c, a \in \mathcal{C}(S).$$

The group $\mathrm{Mod}(S)$ acts on $\partial\mathcal{C}(S)$ as a group of homeomorphisms.

Let $\kappa_0 > 0$ be a *Bers constant* for S, that is, κ_0 is such that for every complete hyperbolic metric on S of finite volume, there is a pants decomposition of S consisting of pants curves of length at most κ_0. Define a map

$$\Upsilon_{\mathcal{T}} : \mathcal{T}(S) \to \mathcal{C}(S) \tag{1}$$

by associating to $x \in \mathcal{T}(S)$ a simple closed curve of x-length at most κ_0. Then there is a number $c > 0$ such that

$$d_{\mathcal{T}}(x, y) \geq d(\Upsilon_{\mathcal{T}}(x), \Upsilon_{\mathcal{T}}(y))/c - c \tag{2}$$

for all $x, y \in \mathcal{T}(S)$ [**MM99**] (see also the discussion in [**H10**]).

For a number $L > 1$, a map $\gamma : [0, s) \to \mathcal{C}(S) (s \in (0, \infty])$ is an *L-quasi-geodesic* if, for all $t_1, t_2 \in [0, s)$, we have

$$|t_1 - t_2|/L - L \leq d(\gamma(t_1), \gamma(t_2)) \leq L|t_1 - t_2| + L.$$

A map $\gamma : [0, \infty) \to \mathcal{C}(S)$ is called an *unparameterized L-quasi-geodesic* if there is an increasing homeomorphism $\phi : [0, s) \to [0, \infty) (s \in (0, \infty])$ such that $\gamma \circ \phi$ is an L-quasi-geodesic. We say that an unparameterized quasi-geodesic is *infinite* if its image set has infinite diameter. The following important result was established in [**MM99**].

THEOREM 3.1. *There is a number $p > 1$ such that the image under $\Upsilon_{\mathcal{T}}$ of every Teichmüller geodesic is an unparameterized p-quasi-geodesic.*

Choose a smooth function $\sigma : [0, \infty) \to [0, 1]$ with $\sigma[0, \kappa_0] \equiv 1$ and $\sigma[2\kappa_0, \infty) \equiv 0$. For each $x \in \mathcal{T}(S)$, the number of essential simple closed curves c on S whose x-length $\ell_x(c)$ (that is, the length of a geodesic representative in its free homotopy class) does not exceed $2\kappa_0$ is bounded from above by a constant not depending on x, and the diameter of the subset of $\mathcal{C}(S)$ containing these curves is uniformly bounded as well. Thus, we obtain for every $x \in \mathcal{T}(S)$ a finite Borel measure μ_x on $\mathcal{C}(S)$ by defining

$$\mu_x = \sum_{c \in \mathcal{C}(S)} \sigma(\ell_x(c)) \Delta_c,$$

where Δ_c denotes the Dirac mass at c. The total mass of μ_x is bounded from above and below by a universal positive constant, and the diameter of the support of μ_x in $\mathcal{C}(S)$ is uniformly bounded as well. Moreover, the measures μ_x depend continuously on $x \in \mathcal{T}(S)$ in the weak*-topology. This means that for every bounded function $f : \mathcal{C}(S) \to \mathbb{R}$, the function $x \to \int f \, d\mu_x$ is continuous.

For $x \in \mathcal{T}(S)$, define a distance δ_x on $\partial\mathcal{C}(S)$ by

$$\delta_x(\xi, \zeta) = \int \delta_c(\xi, \zeta) \, d\mu_x(c)/\mu_x(\mathcal{C}(S)). \tag{3}$$

The distances δ_x are equivariant with respect to the action of $\mathrm{Mod}(S)$ on $\mathcal{T}(S)$ and $\partial\mathcal{C}(S)$. Moreover, there is a constant $\kappa > 0$ such that

$$\delta_x \leq e^{\kappa d_{\mathcal{T}}(x,y)} \delta_y \quad \text{and} \quad \kappa^{-1}\delta_y \leq \delta_{\Upsilon_{\mathcal{T}}(y)} \leq \kappa \delta_y \tag{4}$$

for all $x, y \in \mathcal{T}(S)$ (see [**H09**, pp. 230 and 231]).

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3.3. *Quadratic differentials.* An area-one quadratic differential $z \in \tilde{Q}(S)$ is determined by a pair (μ, ν) of measured geodesic laminations which bind S and such that $\iota(\mu, \nu) = 1$. The laminations μ, ν are called *vertical* and *horizontal*, respectively. Namely, Levitt [**L83**] constructed from a measured foliation on S a measured geodesic lamination, and the measured geodesic lamination determines the measured foliation up to Whitehead moves. On the other hand, a pair $(\hat{\mu}, \hat{\nu})$ of measured foliations is the pair consisting of the horizontal and the vertical measured foliations for a quadratic differential z on S if and only if the corresponding measured geodesic laminations bind S.

For $z \in \tilde{Q}(S)$, let $W^u(z) \subset \tilde{Q}(S)$ be the set of all quadratic differentials whose horizontal projective measured geodesic laminations coincide with the horizontal projective measured geodesic lamination of z. The space $W^u(z)$ is called the *unstable* manifold of z and these unstable manifolds define the *unstable foliation* W^u of $\tilde{Q}(S)$. The *strong unstable manifold* $W^{su}(z) \subset W^u(z)$ is the set of all quadratic differentials whose horizontal measured geodesic laminations coincide with the horizontal measured geodesic lamination of z. These sets define the *strong unstable* foliation W^{su} of $\tilde{Q}(S)$. The *flip* $\mathcal{F} : z \to \mathcal{F}(z) = -z$ exchanges the vertical and the horizontal measured geodesic laminations of a quadratic differential z. The image of the unstable (or the strong unstable) foliation of $\tilde{Q}(S)$ under the flip $\mathcal{F}$ is the *stable foliation* W^s (or the *strong stable foliation* W^{ss}).

By the Hubbard–Masur theorem [**HM79**], for each $z \in \tilde{Q}(S)$, the restriction to $W^u(z)$ of the canonical projection

$$P : \tilde{Q}(S) \to \mathcal{T}(S)$$

is a homeomorphism. Thus, the Teichmüller metric lifts to a complete path metric d^u on $W^u(z)$ (that is, a complete distance function so that any two points can be connected by a minimal geodesic). Denote by d^{su} the restriction of this distance function to $W^{su}(z)$. Then $d^s = d^u \circ \mathcal{F}$, $d^{ss} = d^{su} \circ \mathcal{F}$ are distance functions on the leaves of the stable and strong stable foliations, respectively. For $z \in \tilde{Q}(S)$ and $r > 0$, let $B^i(z, r) \subset W^i(z)$ be the closed ball of radius r about z with respect to d^i $(i = u, su, s, ss)$.

Let

$$\tilde{\mathcal{A}} \subset \tilde{Q}(S) \tag{5}$$

be the set of all marked quadratic differentials z such that the unparameterized quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t z)(t \in [0, \infty))$ is infinite. Then $\tilde{\mathcal{A}}$ is the set of all quadratic differentials whose vertical measured geodesic lamination fills up S (that is, its support decomposes S into ideal polygons and once-punctured polygons; see [**H06**] for a comprehensive discussion of this result of Klarreich [**Kl99**]). There is a natural surjective map

$$F : \tilde{\mathcal{A}} \to \partial C(S)$$

which associates to a point $z \in \tilde{\mathcal{A}}$ the end point of the infinite unparameterized quasi-geodesic $t \to \Upsilon_{\mathcal{T}}(P\Phi^t z)(t \in [0, \infty))$. This map is equivariant with respect to the action of the mapping class group on $\tilde{\mathcal{A}} \subset \tilde{Q}(S)$ and on $\partial C(S)$.

Call a marked quadratic differential $z \in \tilde{Q}(S)$ *uniquely ergodic* if the support of its vertical measured geodesic lamination is uniquely ergodic and fills up S. A uniquely ergodic quadratic differential is contained in the set $\tilde{\mathcal{A}}$. The following is Lemma 2.3 of [**H13**].

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LEMMA 3.2.

(1) *The map $F : \tilde{\mathcal{A}} \to \partial C(S)$ is continuous and closed.*
(2) *If $z \in \tilde{Q}(S)$ is uniquely ergodic, then the sets $F(B^{su}(z, r) \cap \tilde{\mathcal{A}})(r > 0)$ form a neighborhood basis for $F(z)$ in $\partial C(S)$.*

For $z \in \tilde{\mathcal{A}}$ and $r > 0$, let

$$D(z, r)$$

be the closed ball of radius r about $F(z)$ with respect to the distance function δ_{Pz}. As a consequence of Lemma 3.2, if $z \in \tilde{Q}(S)$ is uniquely ergodic, then, for every $r > 0$, there are numbers $r_0 < r$ and $\beta > 0$ such that

$$F(B^{su}(z, r_0) \cap \tilde{\mathcal{A}}) \subset D(z, \beta) \subset F(B^{su}(z, r) \cap \tilde{\mathcal{A}}). \tag{6}$$

3.4. *Strata.* A tuple $(m_1, \ldots, m_\ell; -m)$ of positive integers $1 \le m_1 \le \cdots \le m_\ell$ with $\sum_i m_i = 4g - 4 + m$ defines a *stratum* $\tilde{Q}(m_1, \ldots, m_\ell; -m)$ in $\tilde{Q}(S)$. This stratum consists of all marked area-one quadratic differentials with m simple poles and ℓ zeros of order $m_1, \ldots, m_\ell$ which are not squares of holomorphic one-forms. The stratum is a real hypersurface in a complex manifold of dimension $2g - 2 + m + \ell$.

The closure in $\tilde{Q}(S)$ of a stratum is a union of components of strata. Strata are invariant under the action of the mapping class group $\mathrm{Mod}(S)$ of S and hence they project to strata in the moduli space $Q(S) = \tilde{Q}(S)/\mathrm{Mod}(S)$ of quadratic differentials on S with at most simple poles at the punctures. We denote the projection of the stratum $\tilde{Q}(m_1, \ldots, m_\ell; -m)$ by $Q(m_1, \ldots, m_\ell; -m)$.

The strata in moduli space need not be connected, but their connected components have been identified by Lanneau [**L08**]. A stratum in $Q(S)$ has at most three connected components. The entropy h of the invariant Lebesgue measure on a component Q of a stratum $Q(m_1, \ldots, m_\ell; -m)$ just equals the dimension $2g - 2 + m + \ell$ [**M82**, **V86**], that is, we have

$$h = 2g - 2 + m + \ell. \tag{7}$$

Similarly, if $m = 0$, then we let $\tilde{\mathcal{H}}(S)$ be the bundle of marked area-one holomorphic one-forms over Teichmüller space $\mathcal{T}(S)$ of S. For a tuple $k_1 \le \cdots \le k_\ell$ of positive integers with $\sum_i k_i = 2g - 2$, the stratum $\tilde{\mathcal{H}}(k_1, \ldots, k_\ell)$ of marked area-one holomorphic one-forms on S with ℓ zeros of order $k_i (i = 1, \ldots, \ell)$ is a real hypersurface in a complex manifold of dimension $2g - 1 + \ell$. It projects to a stratum $\mathcal{H}(k_1, \ldots, k_\ell)$ in the moduli space $\mathcal{H}(S)$ of area-one holomorphic one-forms on S. Strata of holomorphic one-forms in moduli space need not be connected, but the number of connected components of a stratum is at most three [**KZ03**]. Moreover, as before, the entropy of the invariant Lebesgue measure on a component of a stratum $\mathcal{H}(k_1, \ldots, k_\ell)$ coincides with the dimension $2g - 1 + \ell$, that is, we have

$$h = 2g - 1 + \ell. \tag{8}$$

Let $\tilde{Q}$ be a component of a stratum $\tilde{Q}(m_1, \ldots, m_\ell; -m)$ of marked quadratic differentials or of a stratum $\tilde{\mathcal{H}}(k_1, \ldots, k_\ell)$ of marked abelian differentials. Using period coordinates, one sees that every $q \in \tilde{\mathcal{Q}}$ has a connected neighborhood U in $\tilde{\mathcal{Q}}$ with the following properties [H13]. For $u \in U$, let $[u^v]$ (or $[u^h]$) be the vertical (or the horizontal) projective measured geodesic lamination of u. Then $\{[u^v] \mid u \in U\}$ is homeomorphic to an open ball in $\mathbb{R}^{h-1}$ (where $h > 0$ is as in equations (7) and (8)). Moreover, for $q \in U$, the set

$$\{u \in U \mid [u^v] = [q^v]\} = W^s_{\tilde{\mathcal{Q}},\mathrm{loc}}(q) \subset W^s(q)$$

is a smooth connected local submanifold of U of (real) dimension h, which is called the *local stable manifold* of q in $\tilde{\mathcal{Q}}$. Similarly, we define the *local unstable manifold* $W^u_{\tilde{\mathcal{Q}},\mathrm{loc}}(q)$ of q in $\tilde{\mathcal{Q}}$. If two such local stable (or unstable) manifolds intersect, then their union is again a local stable (or unstable) manifold. The maximal connected set containing q which is a union of intersecting local stable (or unstable) manifolds is the *stable manifold* $W^s_{\tilde{\mathcal{Q}}}(q)$ (or the *unstable manifold* $W^u_{\tilde{\mathcal{Q}}}(q)$) of q in $\tilde{\mathcal{Q}}$. Note that $W^i_{\tilde{\mathcal{Q}}}(q) \subset W^i(q)(i = s, u)$. A stable (or unstable) manifold is invariant under the action of the Teichmüller flow Φ^t.

The stable and unstable manifolds define smooth foliations $W^s_{\tilde{\mathcal{Q}}}$, $W^u_{\tilde{\mathcal{Q}}}$ of $\tilde{\mathcal{Q}}$, which are called the *stable* and *unstable* foliations of $\tilde{\mathcal{Q}}$, respectively. Define the *strong stable* foliation $W^{ss}_{\tilde{\mathcal{Q}}}$ (or the *strong unstable* foliation $W^{su}_{\tilde{\mathcal{Q}}}$) of $\tilde{\mathcal{Q}}$ by requiring that the leaf $W^{ss}_{\tilde{\mathcal{Q}}}(q)$ (or $W^{su}_{\tilde{\mathcal{Q}}}(q)$) through q is the subset of $W^s_{\tilde{\mathcal{Q}}}(q)$ (or of $W^u_{\tilde{\mathcal{Q}}}(q)$) of all marked quadratic differentials whose vertical (or horizontal) measured geodesic lamination equals the vertical (or horizontal) measured geodesic lamination of q. The strong stable foliation of $\tilde{\mathcal{Q}}$ is transverse to the unstable foliation of $\tilde{\mathcal{Q}}$.

The foliations $W^i_{\tilde{\mathcal{Q}}}(i = ss, s, su, u)$ are invariant under the action of the stabilizer $\mathrm{Stab}(\tilde{\mathcal{Q}})$ of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S)$ and they project to Φ^t-invariant singular foliations $W^i_{\mathcal{Q}}$ of $\mathcal{Q} = \tilde{\mathcal{Q}}/\mathrm{Stab}(\tilde{\mathcal{Q}})$.

As in §2, let $\tilde{\mathcal{Q}}_{\mathrm{good}}$ be the subset of $\tilde{\mathcal{Q}}$ of all points $\tilde{q}$ so that the stabilizer of $\tilde{q}$ in $\mathrm{Mod}(S)$ is contained in the subgroup of all elements which stabilize the component $\tilde{\mathcal{Q}}$ pointwise. For $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathrm{good}}$ and $z \in W^s_{\tilde{\mathcal{Q}}}(\tilde{q})$, there is a neighborhood V of $\tilde{q}$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde{q})$, and there is a homeomorphism

$$\Xi_z : V \to \Xi_z(V) \subset W^{su}_{\tilde{\mathcal{Q}}}(z) \tag{9}$$

with $\Xi_z(\tilde{q}) = z$ which is determined by the requirement that $\Xi_z(u) \in W^s_{\tilde{\mathcal{Q}}}(u)$ for all $u \in V$. We call Ξ_z a *holonomy map* for the strong unstable foliation along the stable foliation.

To be more precise, since $z \in W^s_{\tilde{\mathcal{Q}}}(\tilde{q})$, the vertical measured geodesic lamination $\tilde{q}^v$ of $\tilde{q}$ and the horizontal measured lamination z^h of z bind S. Since binding S is an open condition for pairs of measured laminations, there is a neighborhood Z of $[\tilde{q}^v]$ in $\mathcal{PML}$ such that for every $[\nu] \in Z$ and every representative ν of the projective class $[\nu]$, the laminations ν and z^h bind S. But this just means that there is a point in $W^{su}_{\tilde{\mathcal{Q}}}(z)$ whose projective vertical measured lamination equals $[\nu]$.

Similarly, for $\tilde{q} \in \tilde{\mathcal{Q}}_{\mathrm{good}}$ and $z \in W^u_{\tilde{\mathcal{Q}}}(\tilde{q})$, there is a neighborhood Y of $\tilde{q}$ in $W^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q})$, and there is a homeomorphism

$$\Theta_z : Y \to \Theta_z(Y) \subset W^{ss}_{\tilde{\mathcal{Q}}}(z) \tag{10}$$

with $\Theta_z(\tilde{q}) = z$ which is determined by the requirement that $\Theta_z(u) \in W^u_{\tilde{Q}}(u)$ for all $u \in Y$. We call Θ_z a *holonomy map* for the strong stable foliation along the unstable foliation. The holonomy maps are equivariant under the action of the mapping class group and hence they project to locally defined holonomy maps in the projection $\mathcal{Q}$ of $\tilde{\mathcal{Q}}$ to the moduli space of area-one quadratic or abelian differentials, which are denoted by the same symbols.

The tangent bundle of the strong stable and strong unstable foliations of the component $\mathcal{Q}$ can be equipped with the so-called *modified Hodge norm*, which induces a *Hodge distance* d_H on the leaves of the foliation. This distance is locally uniformly equivalent to any other distance defined by a continuous norm on that tangent bundle.

Let

$$\Pi : \tilde{\mathcal{Q}}(S) \to \mathcal{Q}(S)$$

be the canonical projection. For $q \in \mathcal{Q}_{\mathrm{good}}$ and $r > 0$, let

$$B^i_{\mathcal{Q}}(q, r)$$

be the closed ball of radius r about q in $W^i_{\mathcal{Q}}(q) (i = ss, su)$ with respect to the metric d_H. Call such a ball $B^i_{\mathcal{Q}}(q, r)$ *admissible* if the following holds true. Let $\tilde{q} \in \tilde{\mathcal{Q}}$ be a preimage of q; then the restriction of Π to the component $B^i_{\tilde{\mathcal{Q}}}(\tilde{q}, r)$ containing $\tilde{q}$ of the preimage of $B^i_{\mathcal{Q}}(q, r)$ is a diffeomorphism.

For every point $q \in \mathcal{Q}_{\mathrm{good}}$, there is a number

$$a_{\mathcal{Q}}(q) > 0$$

such that the balls $B^i_{\mathcal{Q}}(q, a_{\mathcal{Q}}(q))$ are admissible $(i = ss, su)$ and that for any preimage $\tilde{q}$ of q in $\tilde{\mathcal{Q}}$ and any $z \in B^{ss}_{\mathcal{Q}}(\tilde{q}, a_{\mathcal{Q}}(q))$ (or $z \in B^{su}_{\mathcal{Q}}(\tilde{q}, a_{\mathcal{Q}}(q))$), the holonomy map Ξ_z (or Θ_z) is defined on $B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$ (or on $B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$).

Now let

$$W_1 \subset B^{ss}_{\mathcal{Q}}(q, a_{\mathcal{Q}}(q)), \quad W_2 \subset B^{su}_{\mathcal{Q}}(q, a_{\mathcal{Q}}(q))$$

be Borel sets and let $\tilde{W}_1 \subset B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$, $\tilde{W}_2 \subset B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$ be the preimages of W_1, W_2 in $B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$, $B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))$. Define

$$V(\tilde{W}_1, \tilde{W}_2) = \bigcup_{z \in \tilde{W}_1} \Xi_z \tilde{W}_2 \quad \text{and} \quad V(W_1, W_2) = \Pi V(\tilde{W}_1, \tilde{W}_2).$$

Note that the map $\Omega : \tilde{W}_1 \times \tilde{W}_2 \to V(\tilde{W}_1, \tilde{W}_2)$ defined by $\Omega(z, u) = \Xi_z(u)$ is a homeomorphism. If W_1, W_2 are path connected and contain the point q, then the set $V(\tilde{W}_1, \tilde{W}_2)$ is path connected and $V(W_1, W_2)$ is path connected as well.

Similarly, define

$$Y(\tilde{W}_1, \tilde{W}_2) = \bigcup_{u \in \tilde{W}_2} \Theta_u \tilde{W}_1 \quad \text{and} \quad Y(W_1, W_2) = \Pi Y(\tilde{W}_1, \tilde{W}_2).$$

Then there is a continuous function

$$\sigma : V(B^{ss}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q)), B^{su}_{\tilde{\mathcal{Q}}}(\tilde{q}, a_{\mathcal{Q}}(q))) \to \mathbb{R} \tag{11}$$

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which vanishes on $B^{ss}_{\tilde{Q}}(\tilde{q}, a_Q(q)) \cup B^{su}_{\tilde{Q}}(\tilde{q}, a_Q(q))$ and such that

$$Y(\tilde{W}_1, \tilde{W}_2) = \{\Phi^{\sigma(z)} z \mid z \in V(\tilde{W}_1, \tilde{W}_2)\}.$$

In particular, for every number $\kappa > 0$, there is a number $r(q, \kappa) > 0$ such that the restriction of the function σ to $V(B^{ss}_{\tilde{Q}}(\tilde{q}, r(q, \kappa)), B^{su}_{\tilde{Q}}(\tilde{q}, r(q, \kappa)))$ assumes values in $[-\kappa, \kappa]$.

For $t_0 > 0$, define

$$V(\tilde{W}_1, \tilde{W}_2, t_0) = \bigcup_{-t_0 \le s \le t_0} \Phi^s V(\tilde{W}_1, \tilde{W}_2) \tag{12}$$

$$\text{and} \quad V(W_1, W_2, t_0) = \Pi V(\tilde{W}_1, \tilde{W}_2, t_0).$$

Then, for sufficiently small t_0, say for all $t_0 \le t_Q(q)$, the following properties are satisfied.

(a) $V(W_1, W_2, t_0)$ is homeomorphic to $V(\tilde{W}_1, \tilde{W}_2) \times [-t_0, t_0]$.

(b) Every connected component of the intersection of an orbit of Φ^t with $V(W_1, W_2, t_0)$ is an arc of length $2t_0$.

We call a set $V(W_1, W_2, t_0)$ as in (12) which satisfies the assumptions (a) and (b) a set with a *local product structure*. Note that every point $q \in Q_{\text{good}}$ has a neighborhood in Q with a local product structure, e.g. the set $V(B^{ss}_Q(q, r), B^{su}_Q(q, r), t)$ for $r \in (0, a_Q(q))$ and $t \in (0, t_Q(q))$. Moreover, the neighborhoods of q with a local product structure form a basis of neighborhoods of q in Q.

Period coordinates can be used to pull the standard Lebesgue measure on $\mathbb{C}^k$ back to a Φ^t-invariant Borel probability measure λ on Q in the Lebesgue measure class (see [**M82**, **V86**]—the point here is that this measure is finite). The measure λ admits a natural family of conditional measures $\lambda^{ss}, \lambda^{su}$ on strong stable and strong unstable manifolds. The conditional measures λ^i are well defined up to a universal constant, and they transform under the Teichmüller geodesic flow Φ^t via

$$d\lambda^{ss} \circ \Phi^t = e^{-ht} d\lambda^{ss} \quad \text{and} \quad d\lambda^{su} \circ \Phi^t = e^{ht} d\lambda^{su}.$$

Let $\mathcal{F} : Q(S) \to Q(S)$ be the flip $q \to \mathcal{F}(q) = -q$ and let dt be the Lebesgue measure on the flow lines of the Teichmüller flow. Any given choice of conditional measures λ^{su} on the strong unstable manifolds determines a choice of conditional measures λ^{ss} on the strong stable manifolds by the requirement that $\mathcal{F}_* \lambda^{su} = \lambda^{ss}$. The measure which can be written with respect to a local product structure in the form

$$d\lambda^{ss} \times d\lambda^{su} \times dt$$

is invariant under the Teichmüller flow and contained in the Lebesgue measure class. This implies that there is a unique choice of conditionals λ^{su} such that

$$d\lambda = d\lambda^{ss} \times d\lambda^{su} \times dt,$$

that is, so that the measure on the right-hand side of the equation is a probability measure. The measures λ^u on unstable manifolds defined by $d\lambda^u = d\lambda^{su} \times dt$ are invariant under holonomy along strong stable manifolds.

The following is Lemma 3.2 of [**H13**].

LEMMA 3.3. *Let $q \in \mathcal{Q}_{\rm good}$ be a smooth point. Then, for every $\epsilon > 0$, there is a number $a(q,\epsilon) \in (0, a_{\mathcal{Q}}(q))$ with the following property. For every $a \le a(q,\epsilon)$, the holonomy maps define a homeomorphism*

$$\Psi : B^{ss}_{\mathcal{Q}}(q,a) \times B^{su}_{\mathcal{Q}}(q,a) \times [-t_{\mathcal{Q}}(q), t_{\mathcal{Q}}(q)] \to V(B^{ss}_{\mathcal{Q}}(q,a), B^{su}_{\mathcal{Q}}(q,a), t_{\mathcal{Q}}(q))$$

whose Jacobian with respect to the measure $\lambda^{ss} \times \lambda^{su} \times dt$ and the measure λ is contained in the interval $[(1+\epsilon)^{-1}, 1+\epsilon]$.

4. *Contraction and recurrence control*

In this section we establish a contraction property for the modified Hodge distances d_H and use this to obtain some quantitative recurrence properties for the Teichmüller flow using the tools from the previous sections.

The following is the first part of Theorem 3.15 of [**ABEM12**].

THEOREM 4.1. *There exists a number $c_H > 0$ not depending on choices such that for any $q \in \mathcal{Q}$, any $q' \in W^{ss}_{\mathcal{Q},\rm loc}(q)$ and all $t > 0$, we have*

$$d_H(\Phi^t q, \Phi^t q') \le c_H d_H(q, q').$$

Theorem 4.1 is in general not sufficient to obtain contraction of distances on strong stable manifolds. We aim at a contraction result which is more general than what can be extracted from [**ABEM12**].

A point $q \in \mathcal{Q}$ is called *forward recurrent* (or *backward recurrent*) if it is contained in its own ω-limit set (or in its own α-limit set) under the action of Φ^t. A point $q \in \mathcal{Q}$ is *recurrent* if it is forward and backward recurrent. The set $\mathcal{R} \subset \mathcal{Q}$ of recurrent points is a Φ^t-invariant Borel subset of $\mathcal{Q}$. It follows from the work of Masur [**M82**] that a recurrent point $q \in \mathcal{R}$ has uniquely ergodic vertical and horizontal measured geodesic laminations whose supports fill up S. As a consequence, the preimage $\tilde{\mathcal{R}}$ of $\mathcal{R}$ in $\tilde{\mathcal{Q}}(S)$ is contained in the set $\tilde{\mathcal{A}}$ defined in (5) of §3.

Using the notation from §3, by Theorem 3.1, there is a number $p > 1$ such that for every $q \in \tilde{\mathcal{Q}}(S)$, the map $t \to \Upsilon_{\mathcal{T}}(P\Phi^t q)$ is an unparameterized p-quasi-geodesic in the curve graph $\mathcal{C}(S)$. If q is a lift of a recurrent point in $\mathcal{Q}(S)$, then this unparameterized quasi-geodesic is of infinite diameter (see [**H13**] for more and references).

Recall from (3) of §3 the definitions of the distances $\delta_x (x \in \mathcal{T}(S))$ on $\partial\mathcal{C}(S)$ and of the sets $D(q,r) \subset \partial\mathcal{C}(S)$ $(q \in \tilde{\mathcal{A}}, r > 0)$. The following lemma is Lemma 4.2 of [**H13**], which is going to be used as a substitute for hyperbolicity.

LEMMA 4.2. *There are numbers $c_0 > 0, \ell > 0, b > 0$ with the following property. Let $q \in \tilde{\mathcal{R}}$ and, for $s > 0$, write $\sigma(s) = d(\Upsilon_{\mathcal{T}}(Pq), \Upsilon_{\mathcal{T}}(P\Phi^s q))$; then*

$$\ell e^{-b\sigma(s)}\delta_{P\Phi^s q} \le \delta_{Pq} \le \ell^{-1} e^{-b\sigma(s)}\delta_{P\Phi^s q} \quad \text{on } D(\Phi^s q, c_0).$$

Let $\tilde{Q} \subset \tilde{\mathcal{Q}}(S)$ be a component of the preimage of $\mathcal{Q}$ and let $\mathrm{Stab}(\tilde{\mathcal{Q}}) < \mathrm{Mod}(S)$ be the stabilizer of $\tilde{\mathcal{Q}}$ in $\mathrm{Mod}(S)$. The Φ^t-invariant Borel probability measure λ on $\mathcal{Q}$ in the Lebesgue measure class (that is, the normalized Masur–Veech measure) lifts to a

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$\mathrm{Stab}(\tilde{Q})$-invariant locally finite measure on $\tilde{Q}$, which we again denote by λ. The conditional measures λ^{ss}, λ^{su} of λ on the leaves of the strong stable and strong unstable foliations of Q lift to a family of locally finite Borel measures on the leaves of the strong stable and strong unstable foliations $W^{ss}_{\tilde{Q}}$, $W^{su}_{\tilde{Q}}$ of $\tilde{Q}$, respectively, which we again denote by λ^{ss}, λ^{su} (see the discussion in §3.3).

The following observation is Lemma 4.3 of [H13]. In its formulation, the number $c_0 > 0$ is the constant from Lemma 4.2.

LEMMA 4.3. *There exists $c_0 > 0$ and, for every $\epsilon > 0$, for every $\tilde{q} \in \tilde{Q}_{\mathrm{good}} \cap \tilde{\mathcal{R}}$ and for all compact neighborhoods $W_1 \subset W_2$ of $\tilde{q}$ in $W^{su}_{\tilde{Q}}(\tilde{q})$, there are compact neighborhoods $K \subset C \subset W_1$ of $\tilde{q}$ in $W^{su}_{\tilde{Q}}(\tilde{q})$ with the following properties.*

(1) *There are numbers $0 < r_1 < r_2 < c_0/2$ such that*

$$K = \overline{W_1 \cap F^{-1}D(\tilde{q}, r_1)},\ C = \overline{W_1 \cap F^{-1}D(\tilde{q}, r_2)}.$$

(2) $\lambda^{su}(K)(1 + \epsilon) \geq \lambda^{su}(C)$.

(3) *If $z \in K \cap \tilde{\mathcal{A}}$, then $\overline{F^{-1}D(z, (r_2 - r_1)/2) \cap W_2} \subset C$.*

We next show Theorem 2 from the introduction. It translates some structural results on non-uniform hyperbolicity of the Teichmüller flow which are described in [H13] using the curve graph into a property for the modified Hodge distance d_H. To this end, denote as before by $B^i_Q(q, r)$ the d_H-ball of radius r about q in $W^i_Q(q)$.

The main point of the theorem is that the local strong stable and strong unstable manifolds which appear in the statement do not depend on the constant $a > 0$.

THEOREM 4.4. *Let $q \in Q_{\mathrm{good}}$ be a recurrent point. Then there is a number $r_0 = r_0(q) > 0$, and there is a neighborhood U of q in Q_{good} with the following property. Let $z \in U$ be recurrent; then, for every $a > 0$, there is a number $T(z, a) > 0$ so that for all $T > T(z, a)$, we have $\Phi^T B^{ss}_Q(z, r_0) \subset B^{ss}_Q(\Phi^T(z), a)$ and $\Phi^T B^{su}_Q(z, a) \supset B^{su}_Q(\Phi^T(z), r_0)$.*

Proof. We only show the statement for strong unstable manifolds, that is, the containment $\Phi^T B^{su}_Q(z, a) \supset B^{su}_Q(\Phi^T(z), r_0)$. The statement for strong stable manifolds is completely analogous and will be left to the reader.

Let $q \in Q$ be recurrent and let $\tilde{q} \in \tilde{Q}$ be a lift of q. By the second part of Lemma 3.2, there is a number $r_0 > 0$ such that

$$B^{su}_{\tilde{Q}}(\tilde{q}, 2c_H r_0) \subset \overline{F^{-1}D(\tilde{q}, c_0/2)},$$

where $c_0 > 0$ is as in Lemma 4.2. Namely, $\tilde{q}$ is uniquely ergodic and the balls $D(\tilde{q}, \epsilon)(\epsilon > 0)$ form a neighborhood basis of $F(\tilde{q})$ in the Gromov boundary of the curve graph of S. We may also assume that the balls $B^{su}_{\tilde{Q}}(\tilde{w}, r)$ are contractible for all $r \leq 2c_H r_0$ and all points $\tilde{w}$ in a neighborhood $\tilde{W}$ of $\tilde{z}$ in $\tilde{Q}$.

By continuity of the Hodge distance on the leaves of the strong stable foliation and by continuity of the dependence of the distances $\delta_{P\tilde{z}}$ on $\tilde{z} \in \tilde{Q}$ as made precise in inequality (4), there is a neighborhood $\tilde{U} \subset \tilde{W}$ of $\tilde{q}$ in $\tilde{Q}$ such that the following holds true. Let $\tilde{z} \in \tilde{U}$

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be a lift of a recurrent point $z \in \mathcal{Q}$; then

$$B^{su}_{\tilde{\mathcal{Q}}}(\tilde{z}, c_H r_0) \subset \overline{F^{-1} D(\tilde{z}, c_0)}. \tag{13}$$

Denote by U the projection of $\tilde{U}$ to $\mathcal{Q}$. Let $z \in U$ be recurrent and let $a \in (0, c_H r_0)$. Consider the preimage $\tilde{z}$ of z in $\tilde{U}$. Since $B^{su}_{\tilde{\mathcal{Q}}}(\tilde{z}, a/2)$ is a neighborhood of $\tilde{z}$ in $W^{su}_{\tilde{\mathcal{Q}}}(\tilde{z})$, by Lemma 4.3 there exists a number $c_1 = c_1(z, a) > 0$ such that

$$B^{su}_{\tilde{\mathcal{Q}}}(\tilde{z}, a/2) \supset \overline{F^{-1} D(\tilde{z}, 2c_1)} \cap B^{su}_{\tilde{\mathcal{Q}}}(\tilde{z}, 2c_H r_0).$$

Using once more continuity as in the beginning of this proof, we then can find a neighborhood $\tilde{V} \subset \tilde{U}$ of $\tilde{z}$ so that

$$B^{su}_{\tilde{\mathcal{Q}}}(\tilde{u}, a) \supset \overline{F^{-1} D(\tilde{u}, c_1)} \cap B^{su}_{\tilde{\mathcal{Q}}}(\tilde{u}, c_H r_0) \tag{14}$$

for all $\tilde{u} \in \tilde{V}$ which project to recurrent points in U.

Let $V \subset U$ be the projection of $\tilde{V}$. Lemma 4.2 shows that there exists a number $R(z, a) > 0$ so that for all $R > R(z, a)$, we have

$$D(\Phi^R \tilde{z}, c_0) \subset D(\tilde{z}, c_1), \tag{15}$$

where $c_1 > 0$ is as in (14). As z is recurrent, there exists furthermore a number $T > R(z, a)$ so that $\Phi^T z \in V$.

By equivariance under the action of the mapping class group, the assumption that $\Phi^T z \in V$ and the estimate (13), we have $B^{su}_{\tilde{\mathcal{Q}}}(\Phi^T \tilde{z}, c_H r_0) \subset \overline{F^{-1} D(\Phi^T \tilde{z}, c_0)}$ and together with (15) we conclude that

$$\Phi^{-T} B^{su}_{\tilde{\mathcal{Q}}}(\Phi^T \tilde{z}, c_H r_0) \subset \overline{F^{-1} D(\tilde{z}, c_1)}. \tag{16}$$

On the other hand, as $\tilde{z} \in \tilde{V}$, the estimate (14) holds true. But $a < c_H r_0$ and consequently (14) and (16) together with the assumption that for $u \in V$ the ball $B^{su}_{\tilde{\mathcal{Q}}}(u, c_H r_0)$ is contractible yield

$$\Phi^{-T} B^{su}_{\mathcal{Q}}(\Phi^T z, c_H r_0) \subset B^{su}_{\mathcal{Q}}(z, a)$$

or, equivalently,

$$B^{su}_{\mathcal{Q}}(\Phi^T z, c_H r_0) \subset \Phi^T B^{su}_{\mathcal{Q}}(z, a).$$

By Theorem 4.1, this implies that $\Phi^R B^{su}_{\mathcal{Q}}(z, a) \supset B^{su}_{\mathcal{Q}}(\Phi^R z, r_0)$ for all $R > T = T(z, a)$. As $z \in U$ and $a \in (0, c_H r_0)$ were arbitrary, this is what we wanted to show. □

Definition 4.5. A flow Φ^t on an orbifold X admits a *uniformly hyperbolic localization* near a point $p \in X$ if there exists a neighborhood U of p in X with the properties stated in Theorem 4.4.

To summarize, Theorem 4.4 shows that the Teichmüller flow on a component of a stratum of abelian or quadratic differentials admits a uniformly hyperbolic localization near every recurrent point.

Let as before λ be the normalized Masur–Veech measure on $\mathcal{Q}$. A version of the following technical result was established in [**H13**]. As it is not explicitly formulated in

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[H13] in the form we need, we provide a proof. Its proof is a variation of the arguments of Margulis [Ma04], adapted to our situation.

Before we can proceed, we introduce the notion of a *characteristic curve*. To this end, let $U \subset \mathcal{Q}_{\rm good}$ be an open contractible set, let $u \in U$ and let $T >> 0$ be such that $\Phi^T u \in U$. Connect $\Phi^T u$ to u by a path in U and let ζ be the resulting closed curve. We call ζ a *characteristic curve* of the *pseudo-orbit* $(u, \Phi^T u)$ *with end points in* U.

As $U \subset \mathcal{Q}_{\rm good}$ is contractible, there exists an open subset $\tilde{U}$ of $\tilde{\mathcal{Q}}_{\rm good}$ so that the canonical projection $\Pi : \tilde{\mathcal{Q}} \to \mathcal{Q}$ maps $\tilde{U}$ diffeomorphically onto U. Furthermore, there exists a unique mapping class $\phi \in \mathrm{Mod}(S)$ so that for the lift $\tilde{u} \in \tilde{U}$ of u, we have $\Phi^T \tilde{u} \in \phi(\tilde{U})$. The curve ζ then lifts to a curve in $\tilde{\mathcal{Q}}$ which connects $\tilde{u}$ to $\phi(\tilde{u})$; in particular, the conjugacy class of the element ϕ depends only on u and T and not on the particular choice of the curve ζ. In other words, a characteristic curve of a pseudo-orbit with end points in U determines uniquely the conjugacy class of an element in $\mathrm{Mod}(S)$.

Lemma 5.1 of [H13] shows that even more is true. By perhaps decreasing U and increasing T, the conjugacy class of the element $\phi \in \mathrm{Mod}(S)$ is pseudo-Anosov, and it defines a periodic orbit γ contained in $\mathcal{Q}$ which passes through a neighborhood of U of controlled size. We use these facts for the following strengthening of the main technical result in [H13].

In its statement, there appear two *a priori* chosen constants $\delta > 0$, $\eta > 0$. The number δ is used to control volume ratios. As we are interested in properties which are true almost surely, we can think of δ as the total measure of points which fail to have a sufficiently controlled recurrence behavior. The number $\eta > 0$ controls the lengths of connected orbit segments which are contained in a suitably chosen set with a product structure. It will only be used in §6. What matters later on is that η can be chosen as small as we wish, independent of δ. The result translates uniformly hyperbolic localization into measure control.

PROPOSITION 4.6. *Let $\mathcal{Q}$ be a component of a stratum and let $q \in \mathcal{Q}_{\rm good}$ be a good recurrent point. Then, for every neighborhood U of q and for all $\delta > 0$, $\eta > 0$, there are contractible closed neighborhoods $Z_1 \subset Z_2 \subset U$ of q with local product structures, there are a Borel set $Z_0 \subset Z_1$ and a number $R_0 > 0$ with the following properties.*

(1) $\lambda(Z_2) \leq (1-\delta)^{-1}\lambda(Z_0)$.

(2) *For some integer $m > 1/\delta$, a Φ^t-orbit intersects Z_1, Z_2 in arcs of length $2t_0 < \eta/m$.*

(3) *The set*

$$Z_4 = \bigcup_{-t_0 m \leq t \leq t_0 m} \Phi^t Z_2 \subset U$$

has a product structure and the same holds true for

$$Z_3 = \bigcup_{-t_0(m-2) \leq t \leq t_0(m-2)} \Phi^t Z_1 \subset Z_4.$$

Furthermore, Z_3 is contained in the interior of Z_4.

(4) *Let $z \in Z_0$ and let $T > R_0$ be such that $\Phi^T z \in Z_3$. Then there exists a path-connected set $B(z) \subset Z_2$ containing z, with*

$$\Phi^T B(z) \subset Z_4 \quad \text{and} \quad \lambda(B(z)) \in [(1-\delta)e^{-hT}\lambda(Z_1), (1-\delta)^{-1}e^{-hT}\lambda(Z_1)].$$

There is a periodic orbit γ for Φ^t of length contained in $[T - mt_0, T + mt_0]$ such that for each $u \in B(z)$, the characteristic curve of the pseudo-orbit $(u, \Phi^T u)$ with end points in Z_4 determines the same component $\gamma(z, T)$ of $\gamma \cap Z_4$. If $u \in Z_0 - B(z)$ and if $\Phi^T u \in Z_3$, then the arc $\gamma(u, T)$ is disjoint from $\gamma(z, T)$.

Proof. The basic idea goes back to Margulis, who proved a similar but somewhat stronger result for Anosov flows on closed manifolds, using uniform expansion (or contraction) of strong unstable (or strong stable) manifolds. The book [**Ma04**] contains a very detailed account on these by now classical arguments.

The Teichmüller flow is not hyperbolic, but we shall show that uniformly hyperbolic localization as established in Theorem 4.4 and non-uniform recurrence suffice to establish the proposition. The price we pay is that the set Z_0 which appears in its statement and which can be chosen to be open in the setting of Anosov flows is now only a Borel set. As counting results rely on mixing properties, this is not a problem for our main goal.

We start with obtaining some volume control using absolute continuity of the strong stable and strong unstable foliations. Namely, recall [**M82**, **V86**] that there is a family of conditional measures $\lambda^{ss}, \lambda^{su}$ for the Masur–Veech measure λ on strong stable and strong unstable manifolds in the Lebesgue measure class, uniquely determined by λ and equivariance under the flip $z \to -z$. We have

$$d\lambda = d\lambda^{su} \times d\lambda^{ss} \times dt \quad \text{and} \quad \frac{d\Phi^t \circ \lambda^{su}}{d\lambda^{su}} = e^{ht}, \ \frac{d\Phi^t \circ \lambda^{ss}}{d\lambda^{ss}} = e^{-ht}.$$

Let $q \in \mathcal{Q}_{\mathrm{good}}$ be a recurrent point and let $U \subset \mathcal{Q}_{\mathrm{good}}$ be an open neighborhood of q. By perhaps making U smaller, we may assume that U satisfies the assumptions in Theorem 4.4. By making U even smaller, we may moreover assume that U has a product structure. More precisely, for the number $r_0 = r_0(q)$ as in Theorem 4.4, we may assume that there are some $r_1 < r_0$ and some $s_0 > 0$ so that

$$U = V(B^{ss}_{\mathcal{Q}}(q, r_1), B^{su}_{\mathcal{Q}}(q, r_1), s_0).$$

Let $\delta > 0$, let $\eta > 0$ and let $\kappa < 1$ be sufficiently small that $(1 + \kappa)^7 < (1 - \delta)^{-1}$. Recall from §3 the definition of the holonomy maps Ξ_u, Θ_z. By the equation (11) and invariance of the construction in §3 under the action of the mapping class group, for $u \in B^{ss}_{\mathcal{Q}}(q, r_1)$ and $z \in B^{su}_{\mathcal{Q}}(q, r_1)$, we have

$$\Theta_z(u) = \Phi^{\sigma(u,z)} \Xi_u(z), \tag{17}$$

for a number $\sigma(u, z)$ which depends continuously on u, z, and $\sigma(q, q) = 0$.

Let $m > 1/\delta$ be an integer. As the Hodge distance is defined by a continuous norm on the tangent bundle of strong stable and strong unstable manifolds of the component $\mathcal{Q}$ and as holonomy maps are smooth, we can find numbers $\epsilon < \min\{\kappa, s_0, 1/4\}$, $r_2 < r_1(1 - 2\epsilon)/4$ and $s_1 < \min\{s_0, \eta\}$ with the following properties.

(a) If $r \le r_2$, then $\lambda^i B^i_{\mathcal{Q}}(q, r) \le (1 + \kappa)\lambda^i B^i_{\mathcal{Q}}(q, r(1 - 2\epsilon))$ $(i = ss, su)$.

(b) If $u \in B^{ss}_{\mathcal{Q}}(q, r_2)$ and $t \in [-s_1, s_1]$, then the restriction of the map $\Phi^t \circ \Xi_u$ to $B^{su}_{\mathcal{Q}}(q, r_2)$ is a $(1 + \epsilon)$-bilipschitz diffeomorphism of $B^{su}_{\mathcal{Q}}(q, r_2)$ into $B^{su}_{\mathcal{Q}}(\Phi^t u, r_1)$. Its Jacobian for the measures λ^{su} is contained in the interval $[(1 + \kappa)^{-1}, 1 + \kappa]$. Furthermore, if $z \in B^{su}_{\mathcal{Q}}(q, r_2)$ and $t \in [-2s_1, 2s_1]$, then the restriction of the map

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$\Phi^t \circ \Theta_z$ to $B^{ss}_Q(q, r_2)$ is a $(1+\epsilon)$-bilipschitz diffeomorphism of $B^{ss}_Q(q, r_2)$ into $B^{ss}_Q(\Phi^t z, r_1)$.

(c) The map $\Lambda : B^{ss}_Q(q, r_2) \times B^{su}_Q(q, r_2) \times (-s_1, s_1) \to U$ defined by $\Lambda(u, z, t) = \Phi^t \Xi_u(z)$ is a diffeomorphism onto its image, and its Jacobian with respect to the product measure $\lambda^{ss} \times \lambda^{su} \times dt$ on $B^{ss}_Q(q, r_2) \times B^{su}_Q(q, r_2) \times (-s_1, s_1)$ and the Masur–Veech measure λ on U is contained in the interval $[(1+\kappa)^{-1}, 1+\kappa]$. Furthermore, the analogous statement also holds true for the map $\Psi : B^{ss}_Q(q, r_2) \times B^{su}_Q(q, r_2) \times (-s_1, s_1) \to U$ defined by $\Psi(u, z, t) = \Phi^t \Theta_z(u)$.

(d) $|\sigma(u, z)| \le s_1/m$ for all $(u, z) \in B^{ss}_Q(q, r_2) \times B^{su}_Q(q, r_2)$.

Constants with these properties can be determined as follows.

The constants s_0, r_1 are invariants of the set U, and the number κ depends only on an *a priori* chosen number $\delta > 0$.

Given s_0, r_1, κ, we next determine a number ϵ as specified in (a) above. Namely, as the measures λ^i can be defined by a smooth volume form on a local strong stable or strong unstable manifold in $\mathcal{Q}$, and as the Hodge distance is defined by a smooth norm, near the point q the volumes of balls of sufficiently small radius are uniformly equivalent to the volumes of euclidean balls. This implies that for the given number $\kappa > 0$, we can find a number $\epsilon \in (0, \min\{\kappa, s_0, 1/4\})$ such that property (a) holds true for this ϵ and for all sufficiently small numbers $r > 0$, say for all $r \le \hat{r}_2$ for some $\hat{r}_2 < r_1$.

Once the numbers $s_0, r_1, \kappa, \epsilon$ are fixed, we have to find r_2, s_1. This is done in two steps. First we determine a number $\tilde{r}_2 < \min\{r_1(1-2\epsilon)/4, \hat{r}_2\}$ such that property (b) holds true for $r_2 = \tilde{r}_2$ and for $t = 0$, and where the number $1+\epsilon$ in the statement is replaced by $\sqrt{1+\epsilon}$ and the number $1+\kappa$ is replaced by $\sqrt{1+\kappa}$. This is possible because the map $(u, z) \in B^{ss}_Q(q, r_1) \times B^{su}_Q(q, r_1) \to \Xi_u(z)$ (or $(u, z) \in B^{ss}_Q(q, r_1) \times B^{su}_Q(q, r_1) \to \Theta_z(u)$) is smooth, its restriction to $\{q\} \times B^{su}_Q(q, r_1)$ (or to $B^{ss}_Q(q, r_1) \times \{q\}$) is just the second (or the first) factor projection and since the Hodge norm is smooth.

By possibly decreasing $\tilde{r}_2$, we may also assume that property (c) holds true for the restriction of the diffeomorphism Λ (or the restriction of the diffeomorphism Ψ) to the set $B^{ss}_Q(q, \tilde{r}_2) \times B^{su}_Q(q, \tilde{r}_2) \times \{0\}$, with $1+\kappa$ replaced by $\sqrt{1+\kappa}$. Namely, Λ (or Ψ) is a diffeomorphism onto its image and its Jacobian equals one at the point $(q, q, 0)$.

Given such a number $\tilde{r}_2$, and for an *a priori* chosen constant $\eta > 0$ which is independent of any other constraint, we determine a number $s_1 < \min\{s_0, \eta\}$ such that property (b) holds true for $\tilde{r}_2$, $t \in [-s_1, s_1]$ and the constants $1+\epsilon$, $1+\kappa$, and that property (c) holds true for $\tilde{r}_2$, the constant $1+\kappa$ and this number s_1. By the choice of $\tilde{r}_2$, such a number s_1 can be found by continuity.

For the numbers $\kappa, \epsilon, \tilde{r}_2, s_1$, properties (a)–(c) hold true as required, but this may not be the case for (d). However, as $\sigma(q, q) = 0$, we can find a number $r_2 < \tilde{r}_2$ such that property (d) holds true if we replace $\tilde{r}_2$ by r_2. As this does not affect the validity of statements (a)–(c) for the fixed number s_1, we have found numbers ϵ, s_1, r_2 for which properties (a)–(d) above hold true.

For this number $r_2 < r_1(1-2\epsilon)/4$ and for any $\rho \in (0, s_1)$, define

$$Z_1(\rho) = V(B^{ss}_Q(q, r_2(1-2\epsilon)), B^{su}_Q(q, r_2(1-2\epsilon)), \rho) \quad \text{and}$$
$$Z_2(\rho) = V(B^{ss}_Q(q, r_2), B^{su}_Q(q, r_2), \rho).$$

It follows from properties (a) and (c) that

$$\lambda(Z_2(\rho)) \le (1+\kappa)^4 \lambda(Z_1(\rho)) \tag{18}$$

for all $\rho \in (0, s_1)$. Namely, by property (c), we know that

$$\lambda(Z_1(\rho)) \ge 2\rho(1+\kappa)^{-1}\lambda^{ss} B_Q^{ss}(q, r_2(1-2\epsilon))\lambda^{su} B_Q^{su}(q, r_2(1-2\epsilon));$$

on the other hand, also

$$\lambda(Z_2(\rho)) \le 2\rho(1+\kappa)\lambda^{ss} B_Q^{ss}(q, r_2)\lambda^{su} B_Q^{su}(q, r_2).$$

Together with property (a), this yields the estimate (18).

Furthermore, by property (b), if $u \in Z_2(s_1)$, then the local strong unstable manifold $W^{su}_{Z_2(s_1),\mathrm{loc}}(u)$ of u in $Z_2(s_1)$ (that is, the intersection of $Z_2(s_1)$ with the local strong unstable manifold of u in Q) is contained in the ball $B_Q^{su}(u, r_1)$ and, if $u \in Z_1(s_1)$, then

$$B_Q^{su}(u, \epsilon r_2) \subset W^{su}_{Z_2(s_1),\mathrm{loc}}(u). \tag{19}$$

Namely, as the distance between a point $u \in B_Q^{su}(q, r_2(1-2\epsilon))$ and the boundary of $B_Q^{su}(q, r_2)$ is not smaller than $2\epsilon r_2$, the distance between the image of u and the image of the boundary of $B_Q^{su}(q, r_2)$ under a $1+\epsilon < 2$ bilipschitz diffeomorphism is at least $2\epsilon r_2/2 = \epsilon r_2$. Thus, the estimate (19) follows from the requirement (b) and the definitions.

By equation (17) and property (d) above, if $u \in Z_2(s_1(1-1/m))$, then the intersection of $Z_2(s_1)$ with the local strong stable manifold through u just equals the image of $B_Q^{ss}(q, r_2)$ under the map $\Phi^t \Theta_z$ for some $z \in B_Q^{su}(q, r_2)$ and some $t \in (-s_1, s_1)$ which is determined by u (this is in general not true for points in $Z_2(s_1) - Z_2(s_1(1-1/m))$). As a consequence, the discussion in the previous paragraph shows that for all $u \in Z_1(s_1(1-1/m))$, we have

$$B_Q^{ss}(u, \epsilon r_2) \subset W^{ss}_{Z_2(s_1),\mathrm{loc}}(u), \tag{20}$$

where $m > 1/\delta$ is as above.

Recall that the number $r_1 > 0$ was chosen so that Theorem 4.4 is valid for the set U and the number r_1. Since the Borel subset of Q of recurrent points has full measure, there are a number $R_0 > 0$ and a Borel subset $E \subset B_Q^{ss}(q, r_2(1-2\epsilon)) \times B_Q^{su}(q, r_2(1-2\epsilon))$ with the following property. For $\rho \in (0, s_1)$, write

$$Z_0(\rho) = \Lambda(E \times (-\rho, \rho)) \subset Z_1(\rho). \tag{21}$$

If $z \in Z_0(\rho)$ and if $T > R_0$, then the d_H-diameter of $\Phi^{-T} B_Q^{su}(\Phi^T z, r_1)$ is at most $\epsilon r_2/2$ and the d_H-diameter of $\Phi^T B_Q^{ss}(u, r_1)$ is at most $\epsilon r_2/2$. Furthermore, we have

$$\lambda(Z_0(\rho)) > (1+\kappa)^{-5}\lambda(Z_2(\rho)) > (1-\delta)\lambda(Z_2(\rho)). \tag{22}$$

Note that this estimate relies on the estimate (18).

By equation (19), we know that if $u \in Z_1(s_1)$, then $B_Q^{su}(u, \epsilon r_2) \subset W^{su}_{Z_2(s_1),\mathrm{loc}}(u)$. Furthermore, as for any $\rho < s_1$ and any $v \in Z_2(\rho)$ the set $W^{su}_{Z_2(\rho),\mathrm{loc}}(v)$ is the image of $B^{su}(q, r_2)$ under a $(1+\epsilon)$-bilipschitz diffeomorphism, the d_H-diameter of $W^{su}_{Z_2(\rho),\mathrm{loc}}(v)$ is at most $2r_2(1+\epsilon) < r_1$.

Let $\rho < s_1$ and let $z \in Z_0(\rho) \subset Z_1(\rho)$ and $T > R_0$ be such that $\Phi^T z \in Z_1(s_1)$. The estimate in the previous paragraph shows that $W^{su}_{Z_2(s_1),\mathrm{loc}}(\Phi^T z) \subset B^{su}_Q(\Phi^T z, r_1)$. Thus, by the choice of the set $Z_0(\rho)$ and the assumption on T, the d_H-diameter of $\Phi^{-T} W^{su}_{Z_2(s_1),\mathrm{loc}}(\Phi^T z)$ is at most $\epsilon r_2/2$. But $z \in \Phi^{-T} W^{su}_{Z_2(s_1),\mathrm{loc}}(\Phi^T z) \cap Z_1(\rho)$ and therefore the estimate (19) shows that

$$\Phi^{-T} W^{su}_{Z_2(s_1),\mathrm{loc}}(\Phi^T z) \subset W^{su}_{Z_2(\rho),\mathrm{loc}}(z). \tag{23}$$

Note that by invariance of strong unstable manifolds under the Teichmüller flow and by the definition of the holonomy maps and the definition of the sets $Z_i(\rho)$, although in the above discussion we may choose $\rho < s_1$ to be different from s_1, it still holds true that $\Phi^{-T} W^{su}_{Z_2(s_1),\mathrm{loc}}(\Phi^T z) \subset W^{su}_{Z_2(\rho),\mathrm{loc}}(z)$ since, by assumption, the point z is contained in $Z_2(\rho)$ and, for $z \in Z_2(\rho) \subset Z_2(s_1)$, we have

$$W^{su}_{Z_2(s_1),\mathrm{loc}}(z) = W^{su}_{Z_2(\rho),\mathrm{loc}}(z).$$

Now let us assume in addition that $\Phi^T z \in Z_1(\rho(1 - 1/m)) \subset Z_1(\rho)$. The above argument, with the roles of Φ^{-T} and Φ^T exchanged and replacing (19) by the estimate (20), which accommodates the time shift in the relation between the holonomy maps Ξ_z and Θ_u, yields that

$$\Phi^T W^{ss}_{Z_2(\rho),\mathrm{loc}}(z) \subset W^{ss}_{Z_2(\rho),\mathrm{loc}}(\Phi^T z). \tag{24}$$

For $t_0 = s_1/2m$, define

$$Z_0 = Z_0(t_0), \quad Z_1 = Z_1(t_0), \quad Z_2 = Z_2(t_0),$$
$$Z_3 = Z_1(s_1(1 - 2/m)), \quad Z_4 = Z_2(s_1).$$

We claim that these sets have the properties (1)–(4) stated in the proposition.

Note that $2t_0 m < \eta$ follows from the assumption $s_1 < \eta$. Property (1) coincides with the estimate (22) for $\rho = t_0$, and Property (2) holds true by the definition of the sets Z_1, Z_2. Property (3) is an immediate consequence of the definitions as well.

To show property (4), assume that $z \in Z_0$ and $T > R_0$ are such that $\Phi^T z \in Z_3$. We first identify the connected component $B(z)$ of z in $Z_2 \cap \Phi^{-T} Z_4$.

To this end, recall that $W^{su}_{Z_3,\mathrm{loc}}(\Phi^T z) = W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z)$ and note that

$$\Phi^{-T} W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z) \subset Z_2$$

by the estimate (23). Since $W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z)$ is diffeomorphic to a ball and holonomy maps are diffeomorphisms, there exist a ball $A \subset B^{su}_Q(q, r_2)$, a point $y \in B^{ss}_Q(q, r_2)$ and a number $s \in [-t_0, t_0]$ such that

$$\Phi^{-T} W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z) = \Phi^s \Xi_y A.$$

Here $y \in B^{ss}_Q(q, r_2)$ and $s \in [-t_0, t_0]$ are such that $z \in \Phi^s \Xi_y B^{su}_Q(q, r_2)$. It then follows from two applications of property (b) and the fact that

$$\lambda^{su}(\Phi^{-T} W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z)) = e^{-hT} \lambda^{su}(W^{su}_{Z_4,\mathrm{loc}}(\Phi^T z))$$

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that

$$\lambda^{su}(A) \in [(1+\kappa)^{-2}e^{-hT}\lambda^{su}(B_Q^{su}(q,r_2)), (1+\kappa)^2 e^{-hT}\lambda^{su}(B_Q^{su}(q,r_2))].$$

In particular, if we define

$$\hat{B}(z) = \bigcup_{-t_0 \le t \le t_0} \bigcup_{u \in B_Q^{ss}(q,r_2)} \Phi^t \Xi_u(A),$$

then property (c) above shows that

$$\lambda(\hat{B}(z)) \in [(1+\kappa)^{-3}e^{-hT}\lambda(Z_2), (1+\kappa)^3 e^{-hT}\lambda(Z_2)].$$

We now claim that in fact $B(z) = \hat{B}(z)$. Since $(1+\kappa)^7 < (1-\delta)^{-1}$ and since $\lambda(Z_2) < (1+\kappa)^4\lambda(Z_1)$ by inequality (18), this then yields the measure control for $B(z)$ stated in part (4) of the proposition.

To show that $\hat{B}(z) \subset B(z)$, we have to show that $\Phi^T \hat{B}(z) \subset Z_4$. For this, let $u \in A \subset B_Q^{su}(q,r_2)$ and $s \in [-t_0, t_0]$ be such that $\Phi^s \Xi_y(u) = z$. By property (b) above, the image of $B_Q^{ss}(q,r_2)$ under the map $\Phi^s\Theta_y$ is contained in the ball $B_Q^{ss}(z,r_1)$. Hence, by the choice of Z_0 and by property (20), we have

$$\Phi^T B_Q^{ss}(z,r_1) \subset B_Q^{ss}(\Phi^T z, \epsilon r_2/2) \subset W^{ss}_{Z_4,\mathrm{loc}}(\Phi^T z).$$

More precisely, the d_H-diameter of $\Phi^T B_Q^{ss}(z,r_1)$ is at most $\epsilon r_2/2$.

Now the Teichmüller flow commutes with holonomy maps, and holonomy maps for Z_4 are $(1+\epsilon)$- bilipschitz by property (b) above. From $(1+\epsilon)^2 < 2$ and by the choice of t_0 and the definition of $Z_3 \subset Z_4$, we conclude that the diameter of $\Phi^T\Phi^s\Theta_v B_Q^{ss}(q,r_2) \subset W^{ss}_{Z_4,\mathrm{loc}}(\Phi^T\Phi^s v)$ is at most ϵr_2 for all $v \in A \subset B_Q^{su}(q,r_2)$. Taking into account the control on the time shift between the holonomy maps Θ_z, Ξ_y and the fact that $\Phi^{T+\tau}(z) \in Z_4$ for all $\tau \in [-4t_0, 4t_0]$, this yields that, indeed, $\Phi^T\hat{B}(z) \subset Z_4$.

Since A is a ball, the set $\hat{B}(z)$ is connected and hence to show that $\hat{B}(z) = B(z)$ it suffices to show the following. Let $y \in B_Q^{ss}(q,r_2)$; then there exists an open neighborhood W of A in $B_Q^{su}(q,r_1)$ such that $\Phi^T(\Xi_y(W-A)) \cap Z_4 = \emptyset$. However, this holds true by equivariance under holonomy, the fact that for every $u \in A$ we have $\Phi^T(\Xi_y(A)) = W^{su}_{Z_4,\mathrm{loc}}(\Phi^T(\Xi_y u))$ and the fact that Φ^T preserves the strong unstable foliation.

Finally, we show that $B(z)$ defines a periodic orbit for the Teichmüller flow. Namely, connect $\Phi^T z$ to z by an arc contained in Z_3 and let ζ be the resulting based loop. Note that $\zeta \subset \mathcal{Q}_{\mathrm{good}}$ by construction. In particular, ζ determines the conjugacy class of an element $\phi \in \mathrm{Mod}(S)$ as explained in §2.

By [**H13**], ϕ is pseudo-Anosov. Furthermore, [**H13**] shows that the periodic orbit for Φ^t determined by the conjugacy class of ϕ passes through Z_4. The period of γ is contained in the interval $[T - mt_0, T + mt_0]$. The choice of the base point $z \in Z_0$ for the loop ζ determines a component $\gamma(z,T)$ of the intersection of γ with Z_4. This means the following. The periodic orbit γ may intersect Z_4 in many different components. As ζ fellow travels γ [**R14**], the choice of the base point z for ζ determines a base point for γ which is contained in Z_4, unique up to a small time shift along a subarc of γ entirely contained in Z_4. Thus, it determines an intersection component of γ with Z_4.

As the orbit segment $\gamma(z, T)$ is contained in $B(z)$ and the set $B(z)$ is a component of $Z_2 \cap \Phi^{-T} Z_4$, we conclude that if $u \in Z_0 - B(z)$ and if $\Phi^T u \in Z_3$, then the orbit segment $\gamma(u, T)$ is disjoint from $\gamma(z, T)$ (see [H13] for more details). This completes the proof of the proposition. □

Remark 4.7. The only information about the Teichmüller flow Φ^t on components $\mathcal{Q}$ of strata of quadratic or abelian differentials used in Proposition 4.6 is local uniform hyperbolicity as established in Theorem 4.4 and the existence of a Φ^t-invariant Borel probability measure which is absolutely continuous with respect to the strong unstable and strong stable foliations, with uniformly expanding (or contracting) conditional measures. Using these hypotheses, the proof of Proposition 4.6 is an adaptation of Margulis' argument as laid out in [Ma04].

As a consequence, the proposition holds true for all locally uniformly hyperbolic flows on a smooth manifold which admits such a measure. We believe that this class includes geodesic flows on compact rank-one manifolds of non-positive curvature, equipped with the unique measure of maximal entropy.

5. *Counting periodic orbits*

This section is devoted to a discussion of counting results for periodic orbits of the Teichmüller flow in a component $\mathcal{Q}$ of a stratum of area-one abelian or quadratic differentials on S. Recall from the introduction that the Teichmüller flow Φ^t acts on $\mathcal{Q}$ preserving a Borel probability measure λ in the Lebesgue measure class, the normalized Masur–Veech measure. Let $k \geq 1$ be the number of zeros of a differential in $\mathcal{Q}$ and let $h = 2g - 2 + k$ be the entropy of Φ^t with respect to the measure λ.

The only properties we use from the Teichmüller flow Φ^t on $\mathcal{Q}$ are as follows. The Masur–Veech measure λ is ergodic and it is obtained from Bowen's construction as formulated in Theorem 5.1 below.

Let

$$\Gamma \subset \mathcal{Q}$$

be the countable collection of all periodic orbits for Φ^t contained in $\mathcal{Q}$. Denote by $\ell(\gamma)$ the period of $\gamma \in \Gamma$ and let δ_γ be the standard Φ^t-invariant Lebesgue measure on γ of total mass $\ell(\gamma)$.

Using an approach by Margulis [Ma04], the asymptotic growth rate of the number of these periodic orbits can explicitly be determined. In fact, these orbits determine the Lebesgue measure λ on $\mathcal{Q}$ via a construction due to Bowen [Bw73]. The following is the main result of [H13].

THEOREM 5.1. *The measures*

$$\mu_R = h e^{-hR} \sum_{\gamma \in \Gamma, \ell(\gamma) \leq R} \delta_\gamma$$

converge as $R \to \infty$ weakly to the Masur–Veech measure on $\mathcal{Q}$.

Note that in the formula in [H13], the factor h is erroneously missing.

As Q is not compact, this does not immediately imply that $\lim_{R\to\infty}\mu_R(Q) = 1$, nor does it determine the asymptotic growth rate of the number of periodic orbits on Q. However, a formula for this asymptotic growth rate follows from Theorem 5.1 and the main result of [**EMR19**] (which rules out escape of mass) as formulated in the corollary in the introduction of [**H13**].

THEOREM 5.2. $\sharp\{\gamma \in \Gamma \mid \ell(\gamma) \le R\} hRe^{-hR} \to 1$ *as* $R \to \infty$.

COROLLARY 5.3. $\lim_{R\to\infty}\mu_R(Q) = 1$.

Proof. That the statement in the corollary follows from Theorem 5.2 (and, under the assumption of Theorem 5.1, is in fact equivalent to Theorem 5.2) has been used many times in the literature; see e.g. [**EM11**, **EMR19**, **Ma04**]. Differentiate the function $f(t) = (e^{ht}/ht)$ to find $f'(t) = (e^{ht}/t) - (e^{ht}/ht^2)$. Then note that under the assumption that the asymptotic formula in Theorem 5.2 holds true, as $R \to \infty$ the value $\mu_R(Q)$ is close to

$$he^{-hR}\int_1^R f'(t)t\,dt = he^{-hR}\int_1^R e^{ht}(1 - 1/ht)\,dt,$$

which converges to one as $R \to \infty$. □

For $R_1 < R_2$, let $\Gamma(R_1, R_2) \subset \Gamma$ be the set of all periodic orbits for Φ^t of prime period contained in the interval $(R_1, R_2]$ (asking for prime period means that we do not consider multiply covered orbits). For $R > 0$, $0 < \sigma < R$, define a measure

$$\nu_{R,\sigma} = he^{-hR}(1 - e^{-h\sigma})^{-1}\sum_{\gamma\in\Gamma(R-\sigma,R)}\delta_\gamma.$$

As an immediate consequence of Corollary 5.3, we observe the following result.

COROLLARY 5.4. *For every* $\sigma > 0$*, the measures* $\nu_{R,\sigma}$ *converge as* $R \to \infty$ *weakly to the Masur–Veech measure on* Q*, and* $\lim_{R\to\infty}\nu_{R,\sigma}(Q) = 1$.

Proof. By definition, we have (as signed measures)

$$(1 - e^{-h\sigma})\nu_{R,\sigma} = \mu_R - e^{-h\sigma}\mu_{R-\sigma}.$$

Thus, the corollary follows from Theorem 5.1 and Corollary 5.3. □

Let $\mathcal{P}$ be a property for periodic orbits $\gamma \in \Gamma$. For a periodic orbit $\gamma \in \Gamma$, write $\chi_\mathcal{P}(\gamma) = 1$ if $\gamma \in \mathcal{P}$ and write $\chi_\mathcal{P}(\gamma) = 0$ otherwise. We call $\mathcal{P}$ *typical* if as $R \to \infty$, the number of all $\gamma \in \Gamma$ with $\ell(\gamma) \le R$ and $\chi_\mathcal{P}(\gamma) = 1$ is asymptotic to e^{hR}/hR.

For two finite Borel measures ν_1, ν_2 on Q, write $\nu_1 \le \nu_2$ if $\nu_1(A) \le \nu_2(A)$ for all Borel subsets A of Q. For $R > 0$, $0 < \sigma < R$, let

$$\nu_{R,\sigma,\mathcal{P}} = he^{-hR}(1 - e^{-h\sigma})^{-1}\sum_{\gamma\in\Gamma(R-\sigma,R)}\chi_\mathcal{P}(\gamma)\delta_\gamma.$$

Clearly, $\nu_{R,\sigma,\mathcal{P}} \le \nu_{R,\sigma}$ for all R, σ.

The next proposition is our main tool for showing that a property $\mathcal{P}$ for periodic orbits in Q is typical.

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PROPOSITION 5.5. *A property $\mathcal{P}$ for Γ is typical if, for all $\delta > 0$, there exists a number $\sigma > 0$ such that*

$$\lim\inf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(Q) \geq 1-\delta.$$

Proof. Assume that the condition in the proposition is fulfilled. It suffices to show that for every $\alpha > 0$, there exists a number $R = R(\alpha) > 0$ such that for all $R \geq R(\alpha)$, the number of periodic orbits $\gamma \in \mathcal{P}$ with $\ell(\gamma) \leq R$ is at least $(1-\alpha)e^{hR}/hR$.

To this end, let $\delta > 0$ be sufficiently small such that $(1-\delta)^5 \geq 1-\alpha$. For this number δ, let $\sigma > 0$ be such that the condition in the proposition holds true. By Corollary 5.4, there exists a number $R_0 = R_0(\sigma,\delta) > 0$ such that $\nu_{R,\sigma}(Q) \leq (1-\delta)^{-1}$ for all $R \geq R_0$. By perhaps increasing R_0, we may assume that $\sigma/R \leq \delta$ for all $R \geq R_0$.

Since $\lim\inf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(Q) \geq 1-\delta$, for sufficiently large R, say for all $R \geq R_1 \geq R_0$, we have $\nu_{R,\sigma,\mathcal{P}}(Q) \geq (1-\delta)^2$. Let $R \geq R_1$. As each orbit $\gamma \in \mathcal{P}$ with $R-\sigma < \ell(\gamma) \leq R$ contributes to the total mass of $\nu_{R,\sigma,\mathcal{P}}$ with a weight contained in the interval

$$[he^{-hR}(1-e^{-h\sigma})^{-1}(R-\sigma), he^{-hR}(1-e^{-h\sigma})^{-1}R],$$

there are at least $(1-\delta)^2 e^{hR}(1-e^{-h\sigma})/hR$ periodic orbits $\gamma \in \mathcal{P}$ of period in $(R-\sigma, R]$.

Similarly, each orbit $\gamma \in \Gamma(R-\sigma, R)$ contributes at least $he^{-hR}(1-e^{-h\sigma})^{-1}(R-\sigma)$ to the total mass of $\nu_{R,\sigma}$. Now $\nu_{R,\sigma}(Q) \leq (1-\delta)^{-1}$ and therefore there are at most $(1-\delta)^{-1}e^{hR}(1-e^{-h\sigma})/h(R-\sigma)$ periodic orbits $\gamma \in \Gamma(R-\sigma, R)$. Since $(R-\sigma)/R \geq 1-\delta$ by assumption, we conclude that the proportion of the number of periodic orbits in $\Gamma(R-\sigma, R)$ which are contained in $\mathcal{P}$ is at least $(1-\delta)^4$.

As the number of periodic orbits in Γ of length R growths exponentially with R, we can choose $m_0 > 0$ sufficiently large that the number of periodic orbits in Γ of length at most R_1 is smaller than δ times the number of periodic orbits in $\Gamma(R_1, R_1 + m_0\sigma)$. For $m \geq m_0$, summing the above estimate for the cardinalities of $\mathcal{P} \cap \Gamma(R_1 + (j-1)\sigma, R_1 + j\sigma)(j = 1, \ldots, m)$ and for $\Gamma(R_1 + (j-1)\sigma, R_1 + j\sigma)(j = 1, \ldots, m)$ yields that the proportion of periodic orbits in $\mathcal{P}$ of period at most $R_1 + m\sigma$ among all periodic orbits of period at most $R_1 + m\sigma$ is at least $(1-\delta)^5 \geq 1-\alpha$. As $m \geq m_0$ was arbitrary, this shows the required estimate. □

Remark 5.6. Although we did not check this, we believe that the condition in Proposition 5.5 is not necessary for a property $\mathcal{P}$ to be typical.

COROLLARY 5.7. *Suppose that for every $\delta > 0$, there exist an open relative compact subset U of Q with $\lambda(\partial U) = 0$ and a number $\sigma > 0$ such that*

$$\lim\inf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(U) \geq (1-\delta)\lambda(U).$$

Then $\mathcal{P}$ is typical.

Proof. By Proposition 5.5, it suffices to show that under the assumption of the corollary, for every $\delta > 0$ there exists some $\sigma > 0$ such that

$$\lim\inf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(Q) \geq 1-\delta.$$

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To this end, choose an open set $U \subset \mathcal{Q}$ and a number $\sigma > 0$ with the properties stated in the proposition, that is, such that $\lambda(\partial U) = 0$ and $\liminf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(U) \geq (1-\delta)\lambda(U)$.

Note that for all $R > 0$, the measure $\nu_{R,\sigma,\mathcal{P}}$ is invariant under the Teichmüller flow Φ^t and we have $\nu_{R,\sigma,\mathcal{P}} \leq \nu_{R,\sigma}$. As $\nu_{R,\sigma} \to \lambda$ weakly as $R \to \infty$, we conclude that any weak limit ν of a sequence of measures $\nu_{R_i,\sigma,\mathcal{P}}$ with $R_i \to \infty$ $(i \to \infty)$ is Φ^t-invariant and satisfies $\nu \leq \lambda$. Then $\nu = f\lambda$ for a Φ^t-invariant Borel function $f : \mathcal{Q} \to [0, 1]$. As λ is ergodic [**M82**, **V86**], the function f is constant almost everywhere and $\nu = c\lambda$ for a number $c \in [0, 1]$. But $\nu(U) \geq (1-\delta)\lambda(U)$ by assumption, and $\lambda(U) > 0$ as λ is of full support. This shows that $c \geq 1-\delta$ and hence $\liminf_{R\to\infty} \nu_{R,\sigma,\mathcal{P}}(\mathcal{Q}) \geq 1-\delta$. The corollary follows. □

6. *Lyapunov exponents*

In this section we apply the criterion established in Corollary 5.7 to prove the main theorem from the introduction. The only tools we are going to use are Proposition 4.6, Corollary 5.7 and mixing of the Teichmüller flow. As these tools are available for Anosov flows on compact manifolds and the unique measure of maximal entropy, the main result (Theorem 6.1) is valid for cocycles over such flows as well.

Let as before $\mathcal{Q}$ be a component of a stratum of abelian or quadratic differentials. Consider a flat rank $n \geq 1$ vector bundle $\mathcal{N} \to \mathcal{Q}$ (here flat is taken in the orbifold sense). Note that this implies that the restriction of $\mathcal{N}$ to $\mathcal{Q}_{\text{good}}$ is a smooth vector bundle in the usual sense. Parallel transport for the flat connection then defines a cocycle Θ^t over the Teichmüller flow Φ^t. We assume that the bundle $\mathcal{N}$ is equipped with a continuous norm $\| \ \|$ and that with respect to this norm, this cocycle is integrable with respect to the Masur–Veech measure λ on $\mathcal{Q}$. Then its *Lyapunov exponents* are defined. These exponents measure the asymptotic growth rate of vectors along orbits of Φ^t which are generic for λ. Let

$$\kappa_1 \geq \cdots \geq \kappa_n \tag{25}$$

be the n Lyapunov exponents.

Let $\gamma \in \Gamma$ be a periodic orbit for Φ^t contained in $\mathcal{Q}_{\text{good}}$. Then parallel transport of $\mathcal{N}$ along γ determines a holonomy transformation whose conjugacy class does not depend on choices. If we define

$$\alpha_1(\gamma) \geq \cdots \geq \alpha_n(\gamma) \geq 0$$

to be the quotients by the length $\ell(\gamma)$ of γ of the logarithms of the n absolute values of the eigenvalues of the holonomy transformation along γ, ordered in decreasing order and counted with multiplicities, then the numbers $\alpha_i(\gamma)$ depend only on γ and not on any choices made.

Let $\epsilon > 0$. For $\gamma \in \Gamma$, define $\chi_\epsilon(\gamma) = 1$ if $|\alpha_i(\gamma) - \kappa_i| < \epsilon$ for every $i \in \{1, \ldots, n\}$ and define $\chi_\epsilon(\gamma) = 0$ otherwise. Thus, for a periodic orbit γ with $\chi_\epsilon(\gamma) = 1$, the vector of absolute values of the normalized eigenvalues for the return map $A(\gamma)$ of the holonomy along γ at any choice of a point $p \in \gamma$, ordered in descending order, is ϵ-close to the vector given by the Lyapunov exponents for the cocycle. Recall that this property is independent of the choice of the point p on γ. We refer to §2 for more details.

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Our goal is to show that periodic orbits $\gamma \in \Gamma$ with $\chi_\epsilon(\gamma) = 1$ are typical in the sense described in the introduction. The strategy is to find for every $\delta > 0$ an open relatively compact subset U of $\mathcal{Q}$ and a number $\sigma > 0$ with the property stated in Corollary 5.7, where $\mathcal{P} = \{\gamma \in \Gamma \mid \chi_\epsilon(\gamma) = 1\}$.

THEOREM 6.1. *For every* $\epsilon > 0$*, the set*

$$\{\gamma \in \Gamma \mid |\alpha_i(\gamma) - \kappa_i| \le \epsilon\}$$

is typical.

Proof. Let $V \subset \mathcal{Q}_{\text{good}}$ be an open relatively compact contractible neighborhood of a recurrent point $q \in \mathcal{Q}_{\text{good}}$.

Let $\|\cdot\|$ be a continuous riemannian norm on the vector bundle $\mathcal{N} \to \mathcal{Q}$ so that the cocycle defined by parallel transport is integrable for this norm and the Masur–Veech measure λ on $\mathcal{Q}$. Let $\epsilon > 0$.

Adjust the riemannian norm $\|\cdot\|$ on V so that it is invariant under parallel transport along curves in V and it is unchanged outside a small contractible compact neighborhood V' of V. As V' is compact and since every orbit which intersects V' also exits V', this does not change Lyapunov exponents.

Let Θ^t be the lift of the Teichmüller flow on $\mathcal{Q}_{\text{good}}$ to a flow on $\mathcal{N}$ defined by parallel transport for the flat connection. For $z \in \mathcal{Q}_{\text{good}}$, let $\mathcal{N}_z$ be the fibre of $\mathcal{N}$ at z. For $1 \le i \le n$ and for $t > 0$, let

$$\zeta_i(t, z)$$

be the infimum of the operator norms for $\|\cdot\|$ of the restriction of $\Theta^t(z)$ to a subspace of $\mathcal{N}_z$ of codimension $i - 1$. Define

$$\kappa_i(t, z) = \frac{1}{t} \log \zeta_i(t, z).$$

As the Teichmüller flow is ergodic for the Masur–Veech measure λ [**M82**, **V86**], the Oseledets multiplicative ergodic theorem [**O68**] states that for λ-almost every point $z \in \mathcal{Q}$, the numbers $\kappa_i(R, z)$ converge as $R \to \infty$ to the ith Lyapunov exponent κ_i of the cocycle defined by the flow Θ^t.

Let $\delta > 0$. As $\lim_{\sigma \searrow 0} h\sigma(1 - e^{-h\sigma})^{-1} \to 1$, for sufficiently small $\sigma > 0$, say for all $\sigma < \sigma_0$, we have $h(1 - e^{-h\sigma})^{-1} \in [\sigma^{-1}(1 - \delta), \sigma^{-1}(1 - \delta)^{-1}]$; furthermore, we may assume that $e^{h\sigma} \le (1 - \delta)^{-1}$. Using the notation from §5, for $\mathcal{P} = \{\gamma \mid \chi_\epsilon(\gamma) = 1\}$ and a number $\sigma < \sigma_0/2$ to be determined below, define $\nu_{R,\sigma,\epsilon} = \nu_{R,\sigma,\mathcal{P}}$. Our goal is to find an open set $U \subset V$ in $\mathcal{Q}_{\text{good}}$ such that

$$\lim \inf_{R \to \infty} \nu_{R,\sigma,\epsilon}(U) \ge \lambda(U)(1 - \delta)^7.$$

By Corollary 5.7, this is sufficient for the proof of the theorem.

To this end, let $m > 1/\delta$ be sufficiently large such that $((m - 2)/m) > 1 - \delta$. Assume that this holds true for all $m > (1/\delta')$, where $\delta' < \delta$. Consider neighborhoods $Z_1 \subset Z_2$ and $Z_3 \subset Z_4 \subset V$ as in Proposition 4.6, constructed from V, the volume control constant δ' and the length control constant $\eta = \sigma_0/2$. As we require $m > 1/\delta'$, we have $((m - 2)/m) > 1 - \delta$.

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Let $\sigma = s_1 \in (0, \sigma_0/2)$ be as in property (b) in the proof of Proposition 4.6 and put $t_0 = \sigma/2m$. Let $Z_0 \subset Z_1$ and $R_0 > 0$ be as in Proposition 4.6. Then there are a number $R(\epsilon) > R_0$ and a Borel subset $E \subset Z_0$ of measure

$$\lambda(E) > \lambda(Z_0)(1-\delta) \geq (1-\delta)^2\lambda(Z_2) \geq (1-\delta)^2\lambda(Z_1)$$

with the following property. Let $u \in E$ and let $R > R(\epsilon)$; then $|\kappa_i(R, u) - \kappa_i| \leq \epsilon/2$.

Let $R \geq R(\epsilon)$ and let $z \in E$ be such that $\Phi^R z \in Z_3$. By Proposition 4.6, the pseudo-orbit (z, T) with end points in Z_3 determines a subarc $\gamma(z, R)$ of a periodic orbit γ for Φ^t passing through Z_4. The period of γ is contained in $[R-\sigma, R+\sigma]$. Moreover, there exists a set $B(z) \subset Z_2$ with $\Phi^R B(z) \subset Z_4$ and

$$\lambda(B(z)) \in [e^{-hR}\lambda(Z_1)(1-\delta)^2, e^{-hR}\lambda(Z_1)(1-\delta)^{-2}] \tag{26}$$

consisting of points with the same characteristic curve as the pseudo-orbit (z, R).

There may however be a small shift in length, that is, the period of γ may differ from the return time R by an additive constant of absolute value up to σ. Now, as the Lyapunov exponents are uniformly bounded, there exists a number $R_1 > R(\epsilon)$ such that for all $t \geq R_1$, all i, all $|s| \leq \sigma$ and all $z \in E$, we have

$$\left|\frac{1}{t+s}\log\zeta_i(t, z) - \frac{1}{t}\log\zeta_i(t, z)\right| < \epsilon/2.$$

By the assumption that the norm is invariant under parallel transport along curves in the set V, we conclude that for $R > R_1$ and all i, we have

$$\left|\frac{1}{\ell(\gamma)}\log\zeta_i(\ell(\gamma), \gamma(0)) - \kappa_i\right| < \epsilon,$$

where, as before, $\ell(\gamma)$ is the length of the periodic orbit γ. This shows that $\chi_\epsilon(\gamma) = 1$. Recall that furthermore if $u \in E - B(z)$ and if $\Phi^R u \in Z_3$, then the set $B(u)$ is disjoint from $B(z)$, and the intersection arc $\eta(u, R)$ with Z_4 of the corresponding periodic orbit η is distinct from $\gamma(z, R)$.

Now the Teichmüller flow is mixing for λ [**M82**, **V86**] and consequently there is a number $R_2 > R_1$ such that

$$\lambda(\Phi^R E \cap Z_3) \geq \lambda(E)\lambda(Z_3)(1-\delta) \tag{27}$$

for all $R \geq R_2$.

By (27) and the estimate (26), for large enough $R > R_2$, the number of components of the intersection of the set Z_4 with periodic orbits of period in the interval $[R-\sigma, R+\sigma]$ which are induced by the characteristic curve of a point $z \in E$ recurring to Z_3 after time R is at least

$$\lambda(E)\lambda(Z_3)(1-\delta)e^{hR}\lambda(Z_1)^{-1}(1-\delta)^2 \geq e^{hR}(1-\delta)^5\lambda(Z_3) \geq e^{hR}(1-\delta)^6\lambda(Z_4).$$

Each of these intersection components is an arc of length 2σ and hence it deposits the mass 2σ on Z_4.

Now $h(1-e^{-h2\sigma})^{-1} \in [(2\sigma)^{-1}(1-\delta), (2\sigma)^{-1}(1-\delta)^{-1}]$ and therefore multiplying the above estimate with $h(1-e^{-h2\sigma})^{-1}$ yields that for $R > R_2 + mt_0$, we have

$$\nu_{R+\sigma,2\sigma,\epsilon}(Z_4) \geq (1-\delta)^7\lambda(Z_4).$$

In other words, the set Z_4 fulfills the estimate in Corollary 5.7 for the constant $(1-\delta)^7$. Since $\delta > 0$ was arbitrary, the theorem follows. □

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