23 The heterotic string

In the Type II theory we have seen that the left and right movers are essentially independent. At the level of the two-dimensional Lagrangian, there is a reflection symmetry between left and right movers; however, this symmetry does not hold sector by sector and is broken by boundary conditions and projectors.

In the *heterotic theory* this independence is taken further, and the degrees of freedom of the left and right movers are taken to be independent – and different. There are two convenient world-sheet realizations of this theory, known as the fermionic and bosonic formulations. In both there are eight left-moving and eight right-moving X^Is, associated with ten flat coordinates in space–time. There are eight right-moving two-dimensional fermions, ψ^I. There is a right-moving supersymmetry but no left-moving supersymmetry. In the fermionic formulation there are, in addition, 32 left-moving fermions which have no obvious connection with space–time, λ^A. In the bosonic description there are an additional 16 left-moving bosons. In other words, there are 24 left-moving bosonic degrees of freedom. There are actually several heterotic string theories in ten dimensions. Rather than attempt a systematic construction, we will describe the two supersymmetric examples. These have gauge groups $O(32)$ and $E_8 \times E_8$. The group E_8, one of the exceptional groups in Cartan's classification, is not very familiar to most physicists. However, it is in this theory that we can most easily find solutions which resemble the Standard Model. We will introduce certain features of E_8 group theory as we need them. More detail can be found in the suggested reading. In this chapter we will work principally in the fermionic formulation. We will develop some features of the bosonic formulation in later chapters, once we have introduced the compactification of strings.

23.1 The $O(32)$ theory

The $O(32)$ ($SO(32)$) theory is somewhat simpler to write down, so we will develop it first. In this theory the 32 λ^A fields are taken to be on an equal footing. The GSO projector, for the right movers, is as in the superstring theory. In the RNS formalism, in the NS sector we keep only states of odd fermion number and similarly in the Ramond sector, where fermion number includes a factor $e^{i\Gamma_{11}}$. For the left movers, the conditions are different. Again, we have a Ramond and an NS sector. In the NS sector we keep only states of even fermion number. In the R sector the ground state is a spinor of $SO(32)$. The spinor representation can be constructed just as we constructed the spinor representation of $O(8)$. Again, there are two inequivalent irreducible representations. There is a chirality, which we can call Γ_{33}.

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The lowest spinor representation of definite chirality is the 32. Again, in the Ramond sector we project (by convention) onto states of even fermion number.

As for the superstring, there is a different light cone Hamiltonian for each sector. The right-moving part is just as in the superstring. The left-moving part includes a contribution from the bosonic operators and a contribution from the fermions, λ^A. As for the superstring, in the Ramond sector the λ^As are integer moded; they are half-integer moded in the NS sector. From our formula, the left-moving normal-ordering constant is -1 in the NS sector and zero in the R sector.

Now, we can consider the spectrum. Take, first, the NS–NS sector, i.e. the sector with NS boundary conditions for both the left and the right movers. The states are space–time bosons. The left-moving normal-ordering constant is -1. Without λ^As, the lowest mass states we can form are

$$\tilde{\alpha}^I_{-1}\psi^J_{-1/2}|0\rangle. \tag{23.1}$$

From our discussion of the normal-ordering constants, we see that these states are massless. They have the quantum numbers of a graviton, antisymmetric tensor and scalar field.

Using the left-moving fermion operators, we can construct additional massless states in this sector:

$$\lambda^A_{-1/2}\lambda^B_{-1/2}\psi^J_{-1/2}|0\rangle. \tag{23.2}$$

These are vectors in space–time. Because the λ^As are fermions, they are antisymmetric under $A \leftrightarrow B$. So, they are naturally identified as gauge bosons of the gauge group $SO(32)$. We will show shortly that they have the couplings of $O(32)$ Yang–Mills theories.

Let's first consider the other sectors. In the NS–R sector, the right-moving states, $\psi^J_{-1/2}|\vec{p}\rangle$, are replaced by the states we labeled $|a\rangle$. Again these must be massless, so we now have particles with the quantum numbers of the gravitino, one additional fermion and the gauginos of $O(32)$. In the NS–R and R–R sectors, however, it turns out that there are no massless states, as can be seen by computing the normal-ordering constants. It is necessary to include the R sector for the left movers. Here the normal-ordering constant is $+1$, and there are no massless states.

23.2 The $E_8 \times E_8$ theory

The E_8 group is unfamiliar to many physicists, and one might wonder how one could obtain two such groups from a string theory. To begin, it is useful to note that E_8 has an $O(16)$ subgroup. Under this group the adjoint of E_8, which is 248-dimensional, decomposes as a 120 – the adjoint of $O(16)$ – and a 128, a spinor of $O(16)$.

In ten dimensions we have seen we can build a sensible string theory with eight left-moving bosons and 32 left-moving fermions. So the strategy is to break the fermions into two groups of 16, λ^A and $\lambda^{\tilde{A}}$, and to treat these as independent. This gives a manifest $O(16) \times O(16)$ symmetry, similar to the symmetry of the $O(32)$ theory. There are now NS and R sectors for each set of fermions separately. The right-moving GSO projectors are

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as before. For the left movers, in each NS sector the action of the left-moving projector is onto states of even fermion number. With a suitable convention for the Γ_{11} chirality, this is also true of the R sectors. So, consider again the spectrum. In the NS–NS–NS sector, just as before, there are a graviton, antisymmetric tensor and scalar field. We can also construct gauge bosons in the adjoint of each of the two $O(16)$s,

$$\lambda^A \lambda^B \psi^J_{-1/2}|0\rangle, \quad \lambda^{\tilde{A}} \lambda^{\tilde{B}} \psi^J_{-1/2}|0\rangle. \tag{23.3}$$

Note that, because of the projectors, there are no massless states carrying quantum numbers of both $O(16)$ groups simultaneously. In the NS–NS–R sector we find the superpartners of these fields.

Now consider the R–NS–NS sector. Here the ground state is a spinor of the first $O(16)$. So now we have a set of gauge bosons in the spinor 128-dimensional representation. Similarly, in the NS–R–NS sector we have a spinor of the other $O(16)$. These are the correct set of states to form the *adjoints* of two E_8s. Again, establishing that the group is actually $E_8 \times E_8$ requires showing that the gauge bosons interact correctly. We will do that in the following section.

Finally, in the R–R–NS and R–R–R sectors there are no massless states.

23.3 Heterotic string interactions

We would like to show that the states we have identified as gauge bosons in the heterotic string interact at low energies, as required by Yang–Mills gauge invariance. To do this we work in the covariant formulation and construct vertex operators corresponding to the various states. Consider the $O(32)$ theory first. With our putative gauge bosons we associate the vertex operators

$$\int d^2z\, V^{AB\mu} = \int d^2z\, \lambda^A(\bar{z})\lambda^B(\bar{z}) \left[\partial_z X^\mu(z) - ik_\nu \psi^\mu \psi^\nu(z)\right] e^{ik\cdot x}. \tag{23.4}$$

For the right movers, as in the Type II theories we have required invariance under the right-moving world-sheet supersymmetry. For the left-moving vertex operators we have simply required that the operators have dimension one, so that overall the vertex operator has dimension one with respect to the left- and right-moving conformal symmetry (the operator is said to be $(1,1)$, just like those of the Type II theory). To determine their interactions, we will study the operator product of two such operators. The left-moving part of the vertex operator is a current,

$$j^{AB}(\bar{z}) = \lambda^A(\bar{z})\lambda^B(\bar{z}). \tag{23.5}$$

The operator product of two of these currents is

$$j^{AB}(\bar{z})j^{CD}(\bar{w}) = \frac{\delta^{AC}\delta^{BD} + \cdots}{(\bar{z}-\bar{w})^2} + \frac{\delta^{AC}\lambda^B(\bar{z})\lambda^D(\bar{w}) + \cdots}{\bar{z}-\bar{w}}. \tag{23.6}$$

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An algebra of currents of this kind is called a *Kac–Moody algebra*. It has the general form

$$j^a(\bar{z})j^b(\bar{w}) = \frac{k\delta^{ab}}{(\bar{z}-\bar{w})^2} + \frac{f^{abc}j^c(\bar{w})}{\bar{z}-\bar{w}}, \tag{23.7}$$

where k is called the central extension of the algebra. In our case $k = 1$. The f^{abc}s are the structure constants of the group. They can be found from Eq. (23.6).

To see the Yang–Mills structure it is helpful to use the general Kac–Moody form, denoting the currents and the corresponding vertex operators by a subscript a. Regarding the operator product, we have seen from our discussion of factorization that the interaction is proportional to the coefficient of $1/|z-w|^2$. In the product $V_a(z)V_b(w)$ the term $1/(\bar{z}-\bar{w})$ is proportional to f_{abc}, just what is needed for the Yang–Mills vertex. The momentum and $g_{\mu\nu}$ contributions arise from the right-moving operator product. In

$$[\partial X^\mu(z) + k_{1\rho}\psi^\rho(z)\psi^\mu(z)]e^{ik_1\cdot X(z)}[\partial X^\nu(w) + k_{2\sigma}\psi^\sigma(w)\psi^\nu(w)]e^{ik_2\cdot X(w)} \tag{23.8}$$

the $1/(z-w)$ terms arise from various sources. One can contract the ∂X factors in each vertex with the exponential factors. This gives

$$V_a^\mu V_b^\nu \sim \frac{f^{abc}V^{c\nu}\left(k_2^\mu - k_1^\mu\right)}{|z-w|^2}. \tag{23.9}$$

Contracting the two ∂X factors with each other gives two factors of $z - w$ in the denominator. These can be compensated by Taylor-expanding $X(z)$ about w. Additional terms arise from contracting the fermions with each other. The details of collecting all the terms and comparing with the three-gauge-boson vertex are left for the exercises.

23.4 A non-supersymmetric heterotic string theory

One can verify the modular invariance of the heterotic string theory, with the GSO projections we have used, in precisely the same way as we did for the superstring theories. This raises the question: are there other ten-dimensional heterotic theories, obtained by combining the partition functions of the separate sectors in different ways? The answer is definitely yes. Several of these have tachyons, but one does not. Its gauge group is $O(16) \times O(16)$. It is most readily described in the Green–Schwarz formalism. It will also provide us with our first example of "modding out", i.e. obtaining a new string theory by making various projections.

On the other hand, in order to obtain the smaller gauge group we need to get rid of the gauge bosons from E_8 which lie in the spinor representation. On the other hand there is no harm in having the corresponding gauginos, if supersymmetry is broken. So we take the original $E_8 \times E_8$ theory and keep only states which are even under the symmetry $(-1)^F$ in space–time and a corresponding symmetry in the gauge group (i.e. spinorial representations are odd, and non-spinorial are even). This immediately gets rid of:

1. the gravitinos, and
2. the gauge bosons which are in spinorial representations of the group.

However, we have seen that, for consistency, it is important that string theories be modular invariant. Simply throwing away states spoils modular invariance; it is necessary to add in additional states. In the present case one has to add a sector with different, twisted, boundary conditions for the fields, as follows:

$$S_a(\sigma + \pi, \tau) = -S_a(\sigma, \tau). \tag{23.10}$$

For the gauge fermions there is a related boundary condition but this is more easily described in the bosonic formulation which we will discuss in Chapter 25 on compactification.

Suggested reading

The original heterotic string papers by Gross *et al.* (1985, 1986) are remarkably clear. Polchinski's book (1998) provides a quite thorough overview of these theories. For example, for those who are not enamored of the Green–Schwarz formalism, it develops the non-supersymmetric $O(32)$ in the RNS formalism in some detail. The absence of global symmetries in the heterotic string is demonstrated in Banks and Dixon (1988).

Exercises

(1) Construct the states corresponding to the gauge bosons of $E_8 \times E_8$. In particular, use the creation–annihilation operator construction of $O(2N)$ spinor representations to build the 128-dimensional representations of $O(16)$.

(2) Verify that the algebra of $O(32)$ currents is of the Kac–Moody form. To work out the structure constants, remember that the generators of O groups are just the antisymmetric matrices

$$(\omega^{AB})_{CD} = \delta^{AC}\delta^{BD} - \delta^{AD}\delta^{BC}. \tag{23.11}$$

(3) Verify that, on-shell, the three-gluon vertex has the correct form. In addition to carefully evaluating the terms in the operator product expansion, it may be necessary to use momentum conservation and the transversality of the polarization vectors.