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On the oscillation of certain second-order linear differential equations

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This paper consists of three parts: First, letting $b_1(z)$, $b_2(z)$, $p_1(z)$ and $p_2(z)$ be nonzero polynomials such that $p_1(z)$ and $p_2(z)$ have the same degree $k \geqslant 1$ and distinct leading coefficients 1 and α, respectively, we solve entire solutions of the Tumura–Clunie type differential equation $f^n + P(z, f) = b_1(z)e^{p_1(z)} + b_2(z)e^{p_2(z)}$, where $n \geqslant 2$ is an integer, $P(z, f)$ is a differential polynomial in f of degree $\leqslant n-1$ with coefficients having polynomial growth. Second, we study the oscillation of the second-order differential equation $f'' - [b_1(z)e^{p_1(z)} + b_2(z)e^{p_2(z)}]f = 0$ and prove that $\alpha = [2(m+1)-1]/[2(m+1)]$ for some integer $m \geqslant 0$ if this equation admits a nontrivial solution such that $\lambda(f) < \infty$. This partially answers a question of Ishizaki. Finally, letting $b_2 \neq 0$ and b_3 be constants and l and s be relatively prime integers such that $l > s \geqslant 1$, we prove that $l = 2$ if the equation $f'' - (e^{lz} + b_2e^{sz} + b_3)f = 0$ admits two linearly independent solutions f_1 and f_2 such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. In particular, we precisely characterize all solutions such that $\lambda(f) < \infty$ when $l = 2$ and $l = 4$.

Keywords: Nevanlinna theory; differential equation; entire solutions; oscillation

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1. Introduction

In the last several decades, the growth and value distribution of meromorphic solutions of complex differential equations have attracted much interest; see [**23**] and references therein. One of the main tools in this subject is Nevanlinna theory; see, e.g., [**14**, **23**] for the standard notation and basic results of Nevanlinna theory. Bank and Laine [**2**, **3**] initiated the study on the oscillation of the second-order linear differential equation

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is an entire function. It is well-known that all solutions of equation (1.1) are entire. For an entire function f, denote by $\sigma(f)$ the *order* of f which is defined as

$$\sigma(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r} = \limsup_{r\to\infty} \frac{\log\log M(r, f)}{\log r},$$

where $M(r, f)$ is the maximum modulus of f on the circle $|z| = r$. When A is transcendental, an application of the lemma on the logarithmic derivative easily yields that all nontrivial solutions of (1.1) satisfy $\sigma(f) = \infty$. Denote by $\lambda(f)$ the *exponent of convergence of zeros* of f which is defined as

$$\lambda(f) = \limsup_{r\to\infty} \frac{\log n(r, f)}{\log r},$$

where $n(r, f)$ denotes the number of zeros of f in the disc $\{z : |z| < r\}$. Concerning the zero distribution of solutions of equation (1.1), Bank and Laine [**2**, **3**] proved: Let f_1 and f_2 be two linearly independent solutions of (1.1). If $\sigma(A)$ is not an integer, then $\max\{\lambda(f_1), \lambda(f_2)\} \geqslant \sigma(A)$; if $\sigma(A) < 1/2$, then $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$. Later, Shen [**29**] and Rossi [**28**] relaxed the condition $\sigma(A) < 1/2$ to the case $\sigma(A) = 1/2$. Based on these results, Bank and Laine conjectured that $\max\{\lambda(f_1), \lambda(f_2)\} = \infty$ whenever $\sigma(A)$ is not an integer. This conjecture is known as *the Bank–Laine conjecture* and has attracted much interest; see the surveys [**13**, **24**] and references therein. Recently, this conjecture was disproved by Bergweiler and Eremenko [**7**, **8**]. They constructed counterexamples for the coefficient A such that $\sigma(A)$ is not an integer and equation (1.1) admits two linearly independent solutions such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. In particular, one of the solutions is free of zeros. In their constructions, they used the solutions of (1.1) with A being a polynomial of e^z of degree 2, namely $A(z) = a_1e^{2z} + a_2e^z + a_3$ with certain coefficients a_1, a_2 and a_3.

On the other hand, it is natural to give explicit solutions of (1.1) such that $\lambda(f) < \infty$ when A is a periodic entire function of the form

$$A(z) = B(e^z), \quad B(\zeta) = b_{-k}\zeta^{-k} + \cdots + b_0 + \cdots + b_l\zeta^l, \quad b_{-k}b_l \neq 0. \tag{1.2}$$

For such solutions, a remarkable result in [**4**, **9**] states that there exist complex constants c, c_j and a polynomial $P(z)$ with simple roots only such that if l is an odd positive integer, then

$$f = P(e^{z/2}) \exp\left(\sum_{j=0}^{l} c_j e^{(l-j)z/2} + cz\right), \tag{1.3}$$

where $c_j = 0$ whenever j is even; while if l is an even positive integer, then

$$f = P(e^z) \exp\left(\sum_{j=0}^{l/2} c_j e^{(l/2-j)z} + cz\right). \tag{1.4}$$

However, it seems difficult to determine explicitly c_j and also the polynomial $P(z)$ in the above two expressions and, until now, they are only known in some special cases. For example, Bank and Laine [**4**] gave a precise characterization of all nontrivial solutions such that $\lambda(f) < \infty$ of (1.1) when $A(z) = e^z - b$ for some constant b; see also [**23**, theorem 5.22]. Bank and Laine [**4**] also characterized entire solutions such that $\lambda(f) < \infty$ of equation (1.1) when $A(z) = -(1/4)e^{-2z} + (1/2)e^{-z} + b$ for some constant b. For these two coefficients, Chiang and Ismail [**10**] expressed all solutions of (1.1) in terms of some special functions and give a complete characterization of the zero distribution of these solutions.

In [**1**], Bank developed a method to find entire solutions such that $\lambda(f) < \infty$ of equation (1.1), but the manipulation of this method seems complicated. One of the main purposes of this paper is to give a more precise description of the oscillation of equation (1.1) when $A(z)$ contains two exponential terms, i.e.,

$$A(z) = B(e^z), \quad B(\zeta) = b_{-k}\zeta^{-k} + b_0 + b_l\zeta^l, \quad b_{-k}b_l \neq 0, \tag{1.5}$$

or

$$A(z) = B(e^z), \quad B(\zeta) = b_0 + b_s\zeta^s + b_l\zeta^l, \quad b_s b_l \neq 0. \tag{1.6}$$

In particular, this provides a different approach from that in [**10**] and also leads to a complete characterization of all solutions such that $\lambda(f) < \infty$ of (1.1) when $A(z)$ is an arbitrary polynomial in e^z of degree 2; see theorem 4.4 in § 4. This work is a continuation of [**33**], where the present author found all nontrivial solutions such that $\lambda(f) < k$ of the differential equation

$$f'' - \left[b_1(z)e^{p_1(z)} + b_2(z)e^{p_2(z)} + b_3(z)\right] f = 0, \tag{1.7}$$

where $b_1(z)$, $b_2(z)$ and $b_3(z)$ are three polynomials such that $b_1(z)b_2(z) \not\equiv 0$ and $p_1(z)$ and $p_2(z)$ are two polynomials of the same degree $k \geqslant 1$ with distinct leading coefficients 1 and α, respectively.

Theorem 1.1 see [**33**]. *Let b_1, b_2 and b_3 be polynomials such that $b_1b_2 \not\equiv 0$ and p_1, p_2 be two polynomials of degree $k \geqslant 1$ with distinct leading coefficients* 1 *and α, respectively, and $p_1(0) = p_2(0) = 0$. Suppose that* (1.7) *admits a nontrivial solution such that $\lambda(f) < k$. Then $\alpha = 1/2$ or $\alpha = 3/4$. Moreover,*

(1) *if $\alpha = 1/2$, then $p_2 = p_1/2$, $f = \kappa e^h$, where κ is a polynomial with simple roots only and h satisfies $h' = \gamma_1 e^{p_1/2} + \gamma$ with γ_1 and γ being two polynomials such that $\gamma_1^2 = b_1$, $2\gamma_1\gamma + \gamma_1' + \gamma_1 p_1'/2 + 2\kappa'/\kappa\gamma_1 = b_2$ and $\gamma^2 + \gamma' + 2\gamma\kappa'/\kappa + \kappa''/\kappa = b_3$;*

(2) *if $\alpha = 3/4$, then $p_1 = z$, $p_2 = 3z/4$ and $f = e^h$, where h satisfies $h' = -4c^2e^{z/2} + ce^{z/4} - 1/8$ and $A = -(16c^2e^z - 8c^3e^{3z/4} + 1/64)$, where c is a nonzero constant.*

The proof of theorem 1.1 is based on a development of the Tumura–Clunie method; see [**14**, chapter 4]. Define a *differential polynomial* $P(z, g)$ in g to be a finite sum of monomials in g and its derivatives of the form $P(z, g) = \sum_{l=1}^{m} a_l g^{n_{l0}}(g')^{n_{l1}} \cdots (g^{(s)})^{n_{ls}}$, where $n_{l0}, \cdots, n_{ls} \in \mathbb{N}$ and the coefficients a_l are meromorphic functions of order less than $\sigma(g)$. Define the *degree* of $P(z, g)$ to be the greatest integer of $d_l := \sum_{t=0}^{s} n_{lt}$, $l = 1, \cdots, m$, and denote it by $\deg_g(P(z, g))$. Consider the equation

$$g^n + P(z, g) = b_1e^{p_1} + b_2e^{p_2}, \tag{1.8}$$

where $n \geqslant 2$ and $P(z, g)$ is a differential polynomial in g of degree $\leqslant n - 1$ with meromorphic functions of order less than k as coefficients. If equation (1.7) admits an entire solution such that $\lambda(f) < k$, then equation (1.7) reduces to an equation of the form in (1.8) with $n = 2$. It is shown in [**33**, theorem 2.1] that if equation (1.8) admits an entire solution, then either $\alpha = -1$ or α is positive rational number and in either case g is a linear combination of certain exponential functions plus some function of order less than k. However, to solve entire solutions of (1.7) such that $\lambda(f) < \infty$, [**33**, theorem 2.1] fails to work since in this case the coefficients of $P(z, g)$ shall contain some logarithmic derivatives which have order no less than k.

The remainder of this paper is organized in the following way. Denote by $\mathcal{R}$ the set of rational functions and by $\mathcal{L}$ the set of functions $a(z)$ such that $a(z) = h^{(l)}(z)/h(z)$, $l \geqslant 1$, for some meromorphic function $h(z)$ of finite order, respectively. In § 2, we further develop the Tumura–Clunie method by solving entire solutions of equation (1.8), where $P(z, g)$ is now a differential polynomial in g with coefficients that are combinations of functions in the set $\mathcal{S} = \mathcal{R} \cup \mathcal{L}$. For equation (1.8) with such coefficients, we can also write the entire solution as a linear combination of exponential functions with certain constant coefficients, but unlike in [**33**, theorem 2.1], it is impossible to determine whether α is a rational number; see theorem 2.1. In § 3, we apply our results on equation (1.8) to study the oscillation of equation (1.7) and prove that $\alpha = [2(m + 1) - 1]/[2(m + 1)]$ for some integer $m \geqslant 0$ provided that equation (1.7) with $b_3 \equiv 0$ admits a nontrivial solution such that $\lambda(f) < \infty$; see theorem 3.1. This gives a partial answer to a question of Ishizaki [**19**]. In § 4, we consider the equation $f'' - (b_1e^{lz} + b_2e^{sz} + b_3)f = 0$, where l, s are relatively prime integers such that $l > s \geqslant 1$ and b_i are constants such that $b_1b_2 \neq 0$. We prove that $l = 2$ if this equation admits two linearly independent solutions f_1 and f_2 such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. In particular, when $l = 2$ or $l = 4$, we determine the polynomial $P(z)$ and the coefficients c_j and c in (1.4) precisely. Finally, in § 5, we give some remarks on our results.

2. Tumura–clunie differential equations

Let $b_1(z)$ and $b_2(z)$ be two nonzero polynomials and $p_1(z)$ and $p_2(z)$ be two polynomials of the same degree $k \geqslant 1$ with distinct leading coefficients 1 and α, respectively, and $p_1(0) = p_2(0) = 0$. Without loss of generality, we may suppose that $0 < |\alpha| \leqslant 1$. In this section, we solve entire solutions of the differential equation

$$f^n + P(z, f) = b_1e^{p_1} + b_2e^{p_2}, \tag{2.1}$$

where $n \geqslant 2$ and $P(z, f)$ is a differential polynomial in f of degree $\leqslant n - 1$ with coefficients being combinations of functions in $\mathcal{S}$. In the following, a differential polynomial in f will always have coefficients which are combinations of functions in $\mathcal{S}$ and thus we will omit mentioning this from now on.

To state our results, we first set up some notation: Let $p(z)$ be a polynomial of degree $k \geqslant 1$. We write $p(z) = (a + ib)z^k + q(z)$, where a, b are real and $a + ib \neq 0$ and $q(z)$ is a polynomial of degree at most $k - 1$. Denote

$$\delta(p, \theta) = a\cos k\theta - b\sin k\theta, \quad \theta \in [0, 2\pi). \tag{2.2}$$

Then on the ray $z = re^{i\theta}$, $r \geqslant 0$, from [**6**] (or [**23**, lemma 5.14]) we know that:

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1. if $\delta(p, \theta) > 0$, then there exists an $r_0 = r_0(\theta)$ such that $\log|e^{p(z)}|$ is increasing on $[r_0, \infty)$ and $|e^{p(z)}| \geqslant e^{\delta(p,\theta)r^n/2}$ there;

2. if $\delta(p, \theta) < 0$, then there exists an $r_0 = r_0(\theta)$ such that $\log|e^{p(z)}|$ is decreasing on $[r_0, \infty)$ and $|e^{p(z)}| \leqslant e^{\delta(p,\theta)r^n/2}$ there.

Let $\theta_1, \theta_2, \cdots, \theta_{2k} \in [0, 2\pi)$ be such that $\delta(p, \theta_j) = 0$, $j = 1, 2, \cdots, 2k$. We may suppose that $\theta_1 < \pi$ and $\theta_j = \theta_1 + (j-1)\pi/k$. Denoting $\theta_{2k+1} = \theta_1 + 2\pi$, then θ_1, $\theta_2, \cdots, \theta_{2k}$ divides the complex plane $\mathbb{C}$ into $2k$ sectors S_j, namely

$$S_j = \left\{re^{i\theta} : 0 \leqslant r < \infty, \quad \theta_j < \theta < \theta_{j+1}\right\}, \quad j = 1, 2, \cdots, 2k. \tag{2.3}$$

Throughout this paper, we let $\epsilon > 0$ be an arbitrary constant. We also denote

$$S_{j,\epsilon} = \left\{re^{i\theta} : 0 \leqslant r < \infty, \quad \theta_j + \epsilon < \theta < \theta_{j+1} - \epsilon\right\}, \quad j = 1, 2, \cdots, 2k. \tag{2.4}$$

Denote by $\overline{S}_j$ and $\overline{S}_{j,\epsilon}$ the closure of S_j and $S_{j,\epsilon}$, respectively. For p_1 in (2.1), we choose $\theta_1 = -\pi/(2k)$ and thus $\delta(p_1, \theta) > 0$ in the sectors S_j when j is odd, and $\delta(p_1, \theta) < 0$ in the sectors S_j when j is even. Denote by J_1 and J_2 the subsets of odd and even integers in the set $J = \{1, 2 \cdots, 2k\}$, respectively, i.e., $J_1 = \{1, 3, \cdots, 2k-1\}$ and $J_2 = \{2, 4, \cdots, 2k\}$. We prove the following

THEOREM 2.1. *Let $n \geqslant 2$ be an integer and $P(z, f)$ be a differential polynomial in f of degree $\leqslant n-1$. Suppose that (2.1) admits an entire solution f. Then α is real. Moreover,*

(1) *if $-1 \leqslant \alpha < 0$, then $f = \gamma_1 e^{p_1/n} + \gamma_2 e^{p_2/n} + \eta$, where γ_1, γ_2 are two polynomials such that $\gamma_1^n = b_1$, $\gamma_2^n = b_2$ and η is an entire function such that $\eta = (\mu_{1,j} - 1)\gamma_1 e^{p_1/n} + (\mu_{2,j} - 1)\gamma_2 e^{p_2/n} + \eta_j$, where $\mu_{1,j}$ and $\mu_{2,j}$ are the n-th roots of 1 such that $\mu_{1,j} = 1$ when $j \in \{1\} \cup J_2$ and $\mu_{2,j} = 1$ when $j \in \{2\} \cup J_1$, and there is an integer N such that $|\eta_j| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$; in particular, when $k = 1$, η is a polynomial;*

(2) *if $0 < \alpha < 1$, letting m be the smallest integer such that $\alpha \leqslant [(m+1)n - 1]/[(m+1)n]$, then $f = \gamma_1 \sum_{j=0}^{m} c_j (b_2/b_1)^j e^{[jn(\alpha-1)+1]p_1/n} + \eta$, where γ_1 is a polynomial such that $\gamma_1^n = b_1$ and $c_0, \cdots, c_m$ are constants such that $c_0^n = 1$ when $m = 0$, and $c_0^n = nc_0^{n-1}c_1 = 1$ when $m = 1$, and $c_0^n = nc_0^{n-1}c_1 = 1$ and $\sum_{\substack{j_0+\cdots+j_m=n,\\ j_1+\cdots+mj_m=k_0}} \frac{n!}{j_0!j_1!\cdots j_m!} c_0^{j_0} c_1^{j_1} \cdots c_m^{j_m} = 0$, $k_0 = 2, \cdots, m$, when $m \geqslant 2$, and η is a meromorphic function with at most finitely many poles such that $\eta = \gamma_1 \sum_{l=0}^{m} (\mu_j - 1) c_j (b_2/b_1)^j e^{[jn(\alpha-1)+1]p_1/n} + \eta_j$, where μ_j are the n-th roots of 1 such that $\mu_j = 1$ when $j \in \{1\} \cup J_2$, and there is an integer N such that $|\eta_j| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$; moreover, we have $p_2 = \alpha p_1$ when $m \geqslant 1$; in particular, when $k = 1$, η is a rational function.*

In theorem 2.1, if all coefficients of the monomials in $P(z, f)$ of degree $n-1$ are rational functions, then we may use the method in the proof of [**33**, theorem 2.1] to show that η is a polynomial or a rational function. We also remark that, by using the method in the proof of theorem 2.1 for the case $-1 \leqslant \alpha < 0$ together with the method in [**34**], we may extend [**33**, theorem 2.1] to the case $P(z, f)$ is a delay–differential polynomial in f with meromorphic functions of order less than k as coefficients; see [34] for the definition of a delay–differential polynomial.

As in the proof of theorems [34, theorem 1.1] and [33, theorem 2.1], we also start from analysing first-order linear differential equation $f' - uf = w$, where u is a nonzero polynomial and w is a meromorphic function with at most finitely many poles. Let $p(z)$ be a primitive function of u and suppose that $\deg(p(z)) = k \geqslant 1$. If f is meromorphic, then there is a rational function $v(z)$ such that $v(z) \to 0$ as $z \to \infty$ and $h(z) = f(z) - v(z)$ is entire. It follows that $f(z) = h(z) + v(z)$ and h satisfies $h' - uh = w - (v' - uv)$ and $w - (v' - uv)$ is an entire function. By elementary integration, the meromorphic solutions of $f' - uf = w$ are $f = ce^{p(z)} + H(z)$, where

$$H(z) = e^{p(z)} \int_0^z w(t) e^{-p(t)} \, \mathrm{d}t. \tag{2.5}$$

To study the growth behaviour of this function, a useful tool is the Phragmén–Lindelöf theorem (see [18, theorem 7.3]): Let $f(z)$ be an analytic function, regular in a region D between two straight lines making an angle π/τ_1 at the origin, and on the lines themselves. Suppose that $|f(z)| \leqslant M$ on the line, and that, as $r \to \infty$ $|f(z)| = O(e^{r^{\tau_2}})$, where $\tau_2 < \tau_1$, uniformly in the angle. Then actually $|f(z)| \leqslant M$ holds throughout the region. Moreover, if $f(z) \to c_1$ and $f(z) \to c_2$ as $z \to \infty$ along the two lines, respectively, then $c_1 = c_2$ and $f(z) \to c_1$ uniformly as $z \to \infty$ in D. Using the Phragmén–Lindelöf theorem, the present author proved the following

Lemma 2.2 see [33, 34]. *Let $p(z)$ be a polynomial with degree $k \geqslant 1$ and w be a nonzero polynomial. Then there is an integer N such that for each S_j where $\delta(p, \theta) > 0$, there is a constant a_j such that $|H(re^{i\theta}) - a_j e^{p(re^{i\theta})}| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$, and for each S_j where $\delta(p, \theta) < 0$ and any constant a, $|H(re^{i\theta}) - ae^{p(re^{i\theta})}| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$.*

Most arguments we use below are the same as that in the proof of [33, theorem 2.1]. We also first introduce the definition of *R–set*: An R–set in the complex plane is a countable union of discs whose radii have finite sum. Let $f(z)$ be an entire solution of (2.1). We denote the union of all R–sets associated with $f(z)$ and each coefficient of $P(z, f)$ by $\tilde{R}$ from now on. In the proof of theorem 2.1, after taking the derivatives on both sides of equation (2.1), there may be some new coefficients appearing in the resulting equations. We will always assume that $\tilde{R}$ also contains those R-sets associated with these new coefficients.

As in the proof of [33, theorem 2.1], we first reduce (2.1) into a non-homogeneous linear differential equation with rational coefficients. Now, with all coefficients of $P(z, f)$ being combinations of functions in $\mathcal{S}$, the key lemma for this aim is the following

Lemma 2.3. *Under the assumptions of theorem 2.1, $\sigma(f) = k$ and α is real. Moreover, for any $\theta \in [0, 2\pi)$ such that the ray $z = re^{i\theta}$ meets finitely discs in $\tilde{R}$,*

(1) *when $-1 \leqslant \alpha < 0$, if $\delta(p_1, \theta) > 0$, then $|f(re^{i\theta})^n| = (1 + o(1))|b_1(re^{i\theta})e^{p_1(re^{i\theta})}|$, $r \to \infty$; if $\delta(p_2, \theta) > 0$, then $|f(re^{i\theta})^n| = (1 + o(1))|b_2(re^{i\theta})e^{p_2(re^{i\theta})}|$, $r \to \infty$;*

(2) *when* $0<\alpha<1$, *if* $\delta(p_1,\,\theta)>0$, *then* $|f(re^{i\theta})^n|=(1+o(1))|b_1(re^{i\theta})e^{p_1(re^{i\theta})}|$, $r\to\infty$; *if* $\delta(p_1,\,\theta)<0$, *then there is an integer* N *such that* $|f(re^{i\theta})|\leqslant r^N$ *for all large* r.

Proof of lemma 2.3. Since $\alpha\neq 1$, then by Steinmetz's result [**30**] for exponential polynomials, we have $T(r,\,b_1e^{p_1}+b_2e^{p_2})=K(1+o(1))r^k$ for some nonzero constant K depending only on α. Recall that the coefficients of equation (2.1) are combinations of functions in $\mathcal{S}$. By the lemma on the logarithmic derivative, we deduce from equation (2.1) that

$$\begin{aligned} T\left(r,b_1e^{p_1}+b_2e^{p_2}\right) &= m\left(r,b_1e^{p_1}+b_2e^{p_2}\right)\\ &= m\left(r,f^n+P(z,f)\right)\leqslant nm(r,f)+O(\log r). \end{aligned} \tag{2.6}$$

Therefore, f is transcendental and $T(r,\,f)\geqslant K_1r^k$ for some positive constant K_1. On the other hand, by the lemma on the logarithmic derivative we also have from equation (2.1) that

$$\begin{aligned} nT(r,f)=T\left(r,f^n\right)=m\left(r,f^n\right)&=m\left(r,b_1e^{p_1}+b_2e^{p_2}-P(z,f)\right)\\ &\leqslant m\left(r,b_1e^{p_1}+b_2e^{p_2}\right)+m\left(r,P(z,f)\right)+O(1)\\ &\leqslant K(1+o(1))r^k+(n-1)m(r,f)+O(\log r), \end{aligned} \tag{2.7}$$

which yields that $T(r,\,f)\leqslant K_2r^k$ for some positive constant K_2. This together with $T(r,\,f)\geqslant K_1r^k$ yields $\sigma(f)=k$. Then by definition of $\mathcal{S}$ and looking at the proof of [**33**, theorem 2.1], we see that α is real. Now, $-1\leqslant\alpha<0$ or $0<\alpha<1$.

Recall that $\theta_1=-\pi/(2k)$ and from (2.2) that $\delta(p_1,\,\theta)=\cos k\theta$ and $\delta(p_2,\,\theta)=\alpha\cos k\theta$. When $\alpha<0$, we see that $\delta(p_1,\,\theta)$ and $\delta(p_2,\,\theta)$ have opposite signs for each θ in the sectors S_j defined in (2.3) for p_1 and $\delta(p_1,\,\theta)>0$ for θ in the sectors S_j where $j\in J_1$; when $\alpha>0$, we see that $\delta(p_1,\,\theta)>0$ and $\delta(p_2,\,\theta)>0$ simultaneously for each θ in the sectors S_j where $j\in J_1$ and $\delta(p_1,\,\theta)<0$ and $\delta(p_2,\,\theta)<0$ simultaneously for each θ in the sectors S_j where $j\in J_2$. Then we see that the assertion (1) and the assertion (2) for the case that $\delta(p_1,\,\theta)>0$ can be obtained by directly following the proof of [**34**, lemma 2.5].

Now we consider the growth behaviour of $f(z)$ along the ray $z=re^{i\theta}$ such that $\delta(p_1,\,\theta)<0$ when $0<\alpha<1$. Let $\varepsilon>0$ be given. By [**12**, corollary 1], there exists a constant $r_0=r_0(\theta)>1$ such that for all z on the ray $z=re^{i\theta}$ which does not meet $\tilde{R}$ when $r\geqslant r_0$, and for all positive integers j,

$$\left|\frac{f^{(j)}(re^{i\theta})}{f(re^{i\theta})}\right|\leqslant r^{j(k-1+\varepsilon)}. \tag{2.8}$$

Since all coefficients of $P(z,\,f)$ are combinations of functions in $\mathcal{S}$, then for each coefficient of $P(z,\,f)$, say a_l, by [**12**, corollary 1], we also have, along the ray $z=re^{i\theta}$, that

$$\left|a_l(re^{i\theta})\right|\leqslant r^M, \tag{2.9}$$

for sufficiently large r and some large integer M. Recalling from the introduction that $P(z,\,f)=\sum_{l=1}^m a_lf^{n_{l0}}(f')^{n_{l1}}\cdots(f^{(s)})^{n_{ls}}$, where m is an integer and

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$n_{l0}+n_{l1}+\cdots n_{ls}\leqslant n-1$, we may write

$$P(z,f)=\sum_{l=1}^{m}\hat{a}_l f^{n_{l0}+n_{l1}+\cdots+n_{ls}}, \tag{2.10}$$

with the new coefficients $\hat{a}_l=a_l(f'/f)^{n_{l1}}\cdots(f^{(s)}/f)^{n_{ls}}$, where $n_{l0},\cdots,n_{ls}$ are nonnegative integers. Note that the greatest order of the derivatives of f in $P(z,f)$ is equal to $s\geqslant 0$. Suppose now that $|f(r_je^{i\theta})|\geqslant r_j^N$ for some infinite sequence $z_j=r_je^{i\theta}$ and some large $N\geqslant M+s(k-1+\varepsilon)$. Then, from (2.1), (2.8), (2.9) and (2.10) we have

$$\begin{aligned}&\left|b_1(r_je^{i\theta})e^{p_1(r_je^{i\theta})}+b_2(r_je^{i\theta})e^{p_2(r_je^{i\theta})}\right|\\&\quad=\left|f(r_je^{i\theta})^n\right|\left|1+\frac{P(r_je^{i\theta},f(r_je^{i\theta}))}{f(r_je^{i\theta})^n}\right|\geqslant(1-o(1))r^{nN},\end{aligned} \tag{2.11}$$

which is impossible when r_j is large since $b_1(r_je^{i\theta})e^{p_1(r_je^{i\theta})}+b_2(r_je^{i\theta})e^{p_2(r_je^{i\theta})}\to 0$ as $z_j\to\infty$. Therefore, along the ray $z=re^{i\theta}$ such that $\delta(p_1,\theta)<0$ we must have $|f(re^{i\theta})|\leqslant r^N$ for all large r and some integer N. Thus our second assertion follows. □

Now we begin to prove theorem 2.1.

Proof of theorem 2.1. For simplicity, we denote $P=P(z,f)$. By taking the derivatives on both sides of (2.1) and eliminating e^{p_2} and e^{p_1} from (2.1) and the resulting equation, respectively, we get the following two equations:

$$b_2B_2f^n-nb_2f^{n-1}f'+b_2B_2P-b_2P'=A_1e^{p_1}, \tag{2.12}$$

$$b_1B_1f^n-nb_1f^{n-1}f'+b_1B_1P-b_1P'=-A_1e^{p_2}, \tag{2.13}$$

where $B_1=b_1'/b_1+p_1'$, $B_2=b_2'/b_2+p_2'$ and $A_1=b_1b_2(B_2-B_1)$. Note that $B_1B_2A_1\not\equiv 0$. By differentiating on both sides of (2.12) and then eliminating e^{p_1} from (2.12) and the resulting equation, we get

$$h_1f^n+h_2f^{n-1}f'+h_3f^{n-2}(f')^2+h_4f^{n-1}f''+P_1=0, \tag{2.14}$$

where $h_1=b_2B_2(A_1'+p_1'A_1)-(b_2B_2)'A_1$, $h_2=-nb_2A_1(p_1'+p_2')-nb_2A_1'$, $h_3=n(n-1)b_2A_1$, $h_4=nb_2A_1$, and $P_1=(A_1'+p_1'A_1)(b_2B_2P-b_2P')-A_1(b_2B_2P-b_2P')'$ is a differential polynomial in f of degree $\leqslant n-1$. By lemma 2.3 and our assumption, α is a nonzero real number such that $-1\leqslant\alpha<1$. Below we consider the two cases where $-1\leqslant\alpha<0$ and $0<\alpha<1$, respectively.

Case 1: $-1\leqslant\alpha<0$. We multiply both sides of equations (2.12) and (2.13) and obtain

$$g_1f^{2n}+g_2f^{2n-1}f'+g_3f^{2n-2}(f')^2+P_2=-A_1^2e^{p_1+p_2}, \tag{2.15}$$

where $g_1=b_1b_2B_1B_2$, $g_2=-nb_1b_2(B_1+B_2)$, $g_3=n^2b_1b_2$ and $P_2=b_1b_2(B_2f^n-nf^{n-1}f')(B_1P-P')+b_1b_2(B_1f^n-nf^{n-1}f')(B_2P-P')+b_1b_2(B_1P-P')(B_2P-$

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P') is a differential polynomial in f of degree $\leqslant 2n-1$. By eliminating $(f')^2$ from (2.14) and (2.15), we get

$$f^{2n-1}\left[(g_3h_1-h_3g_1)f+(g_3h_2-h_3g_2)f'+g_3h_4f''\right]+P_3=h_3A_1^2e^{p_1+p_2}, \quad (2.16)$$

where $P_3=g_3f^nP_1-h_3P_2$ is a differential polynomial in f of degree $\leqslant 2n-1$. For simplicity, we denote

$$\varphi=\frac{h_3A_1^2}{g_3h_4}\frac{e^{p_1+p_2}}{f^{2n-1}}-\frac{1}{g_3h_4}\frac{P_3}{f^{2n-1}}. \quad (2.17)$$

Recalling $B_1=b_1'/b_1+p_1'$ and $B_2=b_2'/b_2+p_2'$, we get from equation (2.16) that

$$f''+H_1f'+H_2f=\varphi, \quad (2.18)$$

where

$$\begin{aligned} H_1&=\frac{h_2}{h_4}-\frac{g_2h_3}{g_3h_4}=-\left[\frac{1}{n}(p_1'+p_2')-\frac{n-1}{n}\left(\frac{b_1'}{b_1}+\frac{b_2'}{b_2}\right)+\frac{A_1'}{A_1}\right],\\ H_2&=\frac{h_1}{h_4}-\frac{g_1h_3}{g_3h_4}=\frac{1}{n}\left[B_2\left(\frac{A_1'}{A_1}-\frac{b_1'}{b_1}\right)-\frac{(b_2B_2)'}{b_2}\right]+\frac{1}{n^2}B_1B_2. \end{aligned} \quad (2.19)$$

Now we prove that φ is a rational function. Recall that b_1, b_2, p_1, p_2 are all polynomials and $B_1=b_1'/b_1+p_1'$, $B_2=b_2'/b_2+p_2'$ and $A_1=b_1b_2(B_2-B_1)$. Since f is entire, we see that φ has only finitely many poles. By lemma 2.3, $\sigma(f)=k$. By the lemma on the logarithmic derivative, we deduce from (2.18) that

$$T(r,\varphi)=m(r,\varphi)+O(\log r)\leqslant m(r,f)+O(\log r)=T(r,f)+O(\log r). \quad (2.20)$$

Therefore, $\sigma(\varphi)\leqslant k$. Now let $\theta\in[0,2\pi)$ be such that $\delta(p_1,\theta)\neq 0$ and $z=re^{i\theta}$ is a ray that meets only finitely discs in $\tilde{R}$. Since $\alpha<0$, then by lemma 2.3 (1) we see that in both cases that $\delta(p_1,\theta)>0$ and $\delta(p_1,\theta)<0$ we always have $|e^{p_1(re^{i\theta})+p_2(re^{i\theta})}/f(re^{i\theta})^{2n-1}|\to 0$ as $r\to\infty$ along the ray $z=re^{i\theta}$. Together with [**12**, corollary 1] we see from (2.17) that there is some integer N such that $|\varphi(re^{i\theta})|\leqslant r^N$ for all large r. Then by the Phragmén–Lindelöf theorem we see that $|\varphi|\leqslant r^N$ uniformly in each $\overline{S}_{j,\epsilon}$, $j=1,2,\cdots,2k$, for some integer $N=N(j)$. Since ϵ can be arbitrarily small, then by the Phragmén–Lindelöf theorem again we conclude that φ is a rational function. From now on we fix one large N.

Recall that $B_2=b_2'/b_2+p_2'$. Denote $F_1=f'-(B_1/n)f$. Then by simple computations we obtain from (2.18) that

$$F_1'-\left(\frac{1}{n}p_2'-\frac{b_1'}{b_1}-\frac{n-1}{n}\frac{b_2'}{b_2}+\frac{A_1'}{A_1}\right)F_1=\varphi. \quad (2.21)$$

Denote $\xi_1=p_2'/n-b_1'/b_1-(n-1)b_2'/nb_2+A_1'/A_1$. Then the general solution of the homogeneous equation $F_1'-\xi_1F_1=0$ is defined on a finite-sheeted Riemann surface and is of the form $F_1=C_2b_2^{1/n}A_1/(b_1b_2)e^{p_2/n}$, where C_2 is a constant and $b_2^{1/n}$ is in general an algebraic function (see [**21**] for the theory of algebroid functions). Suppose that Γ_2 is a particular solution of $F_1'-\xi_1F_1=\varphi$. We may write

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the meromorphic solution of this equation as $F_1 = C_2 b_2^{1/n} A_1/(b_1 b_2) e^{p_2/n} + \Gamma_2$. By an elementary series expansion analysis around the zeros of b_2, we conclude that $\Gamma_2/b_2^{1/n}$ is a meromorphic function. This implies that b_2 is an n-square of some polynomial. Then by lemma 2.2 we integrate the equation (2.21) along the ray $z = re^{i\theta}$ in S_2 such that $\delta(p_2, \theta) > 0$ and obtain

$$F_1 = f' - \frac{1}{n} B_1 f = \frac{c_2}{n} \frac{b_2^{1/n} A_1}{b_1 b_2} e^{p_2/n} + \Gamma_2, \tag{2.22}$$

where

$$\Gamma_2 = \frac{A_1 b_2^{1/n}}{b_1 b_2} e^{p_2/n} \int_0^z e^{-p_2/n} \frac{b_1 b_2}{A_1 b_2^{1/n}} \varphi \, \mathrm{d}t - a_{2,2} \frac{A_1 b_2^{1/n}}{b_1 b_2} e^{p_2/n}, \tag{2.23}$$

where $a_{2,2} = a_{2,2}(\theta)$ is a constant such that $|\Gamma_2| = O(r^N)$ along the ray $z = re^{i\theta}$ in S_2. Now, for $z \in S_{j,\epsilon}$ where $j \in J_2$, we have $\delta(p_2, \theta) > 0$ and so $\Gamma_2 = (c_2 d_{2,j}/n) b_2^{1/n} A_1/(b_1 b_2) e^{p_2/n} + \gamma_{2,j}$, where $d_{2,j}$ are some constants related to a sector $S_{j,\epsilon}$ and $|\gamma_{2,j}| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$. Of course, for $j = 2$, we have $d_{2,2} = 0$. Furthermore, $|\Gamma_2| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$ where $j \in J_1$. We then define $d_{2,j} = 0$ for $j \in J_1$.

Similarly, denoting that $\xi_2 = p_1'/n - b_2'/b_2 - (n-1)b_1'/nb_1 + A_1'/A_1$ we also have $F_2' - \xi_2 F_2 = \varphi$ and it follows by integration that $F_2 = -(c_1/n) b_1^{1/n} A_1/(b_1 b_2) e^{p_1/n} + \Gamma_1$, where $\Gamma_1 = -(c_1 d_{1,j}/n) b_1^{1/n} A_1/(b_1 b_2) e^{p_1/n} + \gamma_{1,j}$, where $d_{l,j}$ are some constants related to a sector $S_{j,\epsilon}$ and $|\gamma_{1,j}| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$ for $j \in J_1$. Of course, for $j = 1$, we have $d_{1,1} = 0$. Furthermore, $|\Gamma_1| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$ where $j \in J_2$. We then define $d_{1,j} = 0$ for $j \in J_2$.

Denoting $B = n/(B_2 - B_1)$, we have $f = B(F_1 - F_2)$. Together with the relation $A_1 = b_1 b_2 (B_2 - B_1)$, we have $f = c_1 b_1^{1/n} e^{p_1/n} + c_2 b_2^{1/n} e^{p_2/n} + \eta$ with an entire function $\eta = B(\Gamma_2 - \Gamma_1)$. We see that $\eta = c_2 d_{2,j} b_2^{1/n} e^{p_2/n} + B(\gamma_{2,j} - \gamma_{1,j})$ when $j \in J_1$ and $\eta = c_1 d_{1,j} b_1^{1/n} e^{p_1/n} + B(\gamma_{2,j} - \gamma_{1,j})$ when $j \in J_2$.

Now we determine $d_{1,j}$ and $d_{2,j}$. By [**12**, corollary 1], we may suppose that along the ray $z = re^{i\theta}$ we have $|f^{(j)}(re^{i\theta})/f(re^{i\theta})| = r^{j(k-1+\varepsilon)}$ for all $j > 0$ for all sufficiently large r and thus write P in the form in (2.10) with the new coefficients $\hat{a}_l = a_l (f'/f)^{n_{l1}} \cdots (f^{(s)}/f)^{n_{ls}}$, where $n_{l1}, \cdots, n_{ls}$ are nonnegative integers. For simplicity, denote $D_{1,j} = c_1 + c_1 d_{1,j}$. By substituting $f = c_1 b_1^{1/n} e^{p_1/n} + c_2 b_2^{1/n} e^{p_2/n} + \eta$ into (2.1), we obtain, for $z = re^{i\theta}$ for a θ in S_j and $j \in J_1$,

$$\begin{aligned} &\left(D_{1,j}^n - 1\right) b_1 e^{p_1} + \sum_{k_0=1}^{n-1} \binom{n}{k_0} \left(D_{1,j} b_1^{1/n}\right)^{n-k_0} (c_2 b_2^{1/n})^{k_0} e^{[(n-k_0)p_1 + k_0 p_2]/n} \\ &\quad + (c_2^n - 1) b_2 e^{p_2} + \sum_{s=1}^{n} \sum_{k_s=0}^{n-s} \alpha_{s,k_s} e^{[(n-s-k_s)p_1 + k_s p_2]/n} = 0, \end{aligned} \tag{2.24}$$

where α_{s,k_s}, $s = 1, \cdots, n$, $k_s = 0, \cdots, n-s$, are functions satisfying $|\alpha_{s,k_s}(re^{i\theta})| = O(r^N)$ along the ray $z = re^{i\theta}$. By letting $r \to \infty$ along the above ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$ and comparing the growth on both sides of the above equation we conclude that $c_1^n(1+d_{1,j})^n = 1$. Since $d_{1,1} = 0$, we have $c_1^n = 1$ and $d_{1,j} = \mu_{1,j} - 1$ for some $\mu_{1,j}$ such that $\mu_{1,j}^n = 1$. Similarly, we can prove that $d_{2,j} = \mu_{2,j} - 1$ for some $\mu_{2,j}$ such that $\mu_{2,j}^n = 1$. In particular, when $k = 1$, since $d_{1,1} = d_{2,2} = 0$ and $|\eta_j| = O(r^N)$ uniformly in the sectors $\overline{S}_{j,\epsilon}$, $j = 1, 2$ and since ϵ can be arbitrarily small, by the Phragmén–Lindelöf theorem we conclude that η is a polynomial. Thus our first assertion follows.

Case 2: $0 < \alpha < 1$.

As in the proof of [**33**, theorem 2.1], we first define some functions in the following way: We let m be the smallest integer such that $\alpha \leqslant [(m+1)n-1]/[(m+1)n]$ and $\iota_0, \cdots, \iota_m$ be a finite sequence of functions such that

$$\begin{aligned} \iota_0 &= \frac{A_1}{nb_1}, \\ \iota_j &= (-1)^j \left(\frac{A_1}{nb_1}\right)^{j+1} (jn-1)\cdots(n-1), \quad j = 1, 2, \cdots, m. \end{aligned} \tag{2.25}$$

Recall that $B_1 = b_1'/b_1 + p_1'$. We also let $\kappa_0, \cdots, \kappa_m$ be a finite sequence of functions defined in the following way:

$$\begin{aligned} \kappa_0 &= \frac{1}{n}\frac{b_1'}{b_1} + \frac{1}{n}p_1', \\ \kappa_j &= \frac{\iota_{j-1}'}{\iota_{j-1}} - \frac{jn-1}{n}\frac{b_1'}{b_1} + \left[j(\alpha-1) + \frac{1}{n}\right]p_1', \quad j = 1, 2, \cdots, m. \end{aligned} \tag{2.26}$$

Then we define $m+1$ functions $G_0, G_1, \cdots, G_m$ in the way that $G_0 = f' - \kappa_0 f$, $G_1 = G_0' - \kappa_1 G_0, \cdots, G_m = G_{m-1}' - \kappa_m G_{m-1}$. Now we have equation (2.13) and it follows that

$$G_0 = f' - \kappa_0 f = \iota_0 \frac{e^{p_2}}{f^{n-1}} + W_0, \tag{2.27}$$

where $W_0 = -(B_1 P - P')/(nf^{n-1})$. Moreover, when $m \geqslant 1$, by simple computations we obtain

$$G_1 = G_0' - \kappa_1 G_0 = \iota_1 \frac{e^{2p_2}}{f^{2n-1}} + W_1,$$

$$W_1 = W_0' - \kappa_1 W_0 - (n-1)\iota_0 \frac{e^{p_2}}{f^n} W_0,$$

and by induction we obtain

$$G_j = G_{j-1}' - \kappa_j G_{j-1} = \iota_j \frac{e^{(j+1)p_2}}{f^{(j+1)n-1}} + W_j, \quad j = 1, \cdots, m, \tag{2.28}$$

$$W_j = W_{j-1}' - \kappa_j W_{j-1} - (jn-1)\iota_{j-1} \frac{e^{jp_2}}{f^{jn}} W_0, \quad j = 1, \cdots, m. \tag{2.29}$$

For an integer $l \geqslant 0$, by elementary computations it is easy to show that $W_0^{(l)} = W_{0l}/f^{n+l-1}$, where $W_{0l} = W_{0l}(z, f)$ is a differential polynomial in f of degree

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$\leqslant n+l-1$, and also that $(e^{p_2}/f^n)^{(l)} = e^{p_2}W_{1l}/f^{n+l}$, where $W_{1l} = W_{1l}(z, f)$ is a differential polynomial in f of degree $\leqslant n+l$. We see that W_j, $1 \leqslant j \leqslant m$, is formulated in terms of W_0 and e^{p_2}/f^n and their derivatives. We may write

$$G_m = \iota_m \frac{e^{(m+1)p_2}}{f^{(m+1)n-1}} + F(W_0, e^{p_2}/f^n), \tag{2.30}$$

where $F(W_0, e^{p_2}/f^n)$ is a combination of W_0 and e^{p_2}/f^n and their derivatives with functions being combinations of functions in $\mathcal{S}$. Moreover, from the recursion formula $G_j = G'_{j-1} - \kappa_j G_{j-1}$, $j \geqslant 1$, and $G_0 = f' - \kappa_0 f$, we easily deduce that f satisfies the linear differential equation

$$f^{(m+1)} - \hat{t}_m f^{(m)} + \cdots + (-1)^{m+1}\hat{t}_0 f = G_m, \tag{2.31}$$

where $\hat{t}_m$, $\hat{t}_{m-1}$, $\cdots$, $\hat{t}_0$ are functions formulated in terms of $\kappa_0, \cdots, \kappa_m$ and their derivatives.

Now we prove that G_m is a rational function. Recall that b_1, b_2, p_1, p_2 are all polynomials. Since f is entire, then by the definitions of κ_0 and κ_j in (2.26), we see that G_m has only finitely many poles. With an application of the lemma on the logarithmic derivative as in previous case, we deduce from (2.31) that $\sigma(G_m) \leqslant \sigma(f) = k$. Now let $\theta \in [0, 2\pi)$ be such that $\delta(p_1, \theta) \neq 0$ and $z = re^{i\theta}$ be a ray that meets only finitely may discs in $\tilde{R}$. By [**12**, corollary 1] and lemma 2.3 (2), we see from (2.31) that there is some integer N such that $|G_m(re^{i\theta})| \leqslant r^N$ for all large r along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) < 0$. On the other hand, by lemma 2.3 (2) there is some integer N such that

(1) if $\alpha < [(m+1)n-1]/[(m+1)n]$, then $|e^{(m+1)p_2(re^{i\theta})}/f(re^{i\theta})^{(m+1)n-1}| \to 0$ as $r \to \infty$ along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$;

(2) if $\alpha = [(m+1)n-1]/[(m+1)n]$, then $|e^{(m+1)p_2(re^{i\theta})}/f(re^{i\theta})^{(m+1)n-1}| \leqslant e^{Nr^{k-1}}$ for all large r along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$.

Note that $e^{p_2(re^{i\theta})}/f(re^{i\theta})^n \to 0$ as $r \to \infty$ along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$. In case (1), together with [**12**, corollary 1] we see from (2.30) that $|G_m(re^{i\theta})| \leqslant r^N$ for all large r and thus by the Phragmén–Lindelöf theorem we see that $|G_m| \leqslant r^N$ uniformly in each $\overline{S}_{j,\epsilon}$, $j \in J_2$, for some integer $N = N(j)$; in case (2), together with [**12**, corollary 1] we see from (2.30) that $|G_m(re^{i\theta})| \leqslant e^{Nr^{k-1}}$ for all large r and, since the set of rays $z = re^{i\theta}$ meeting infinitely many discs in $\tilde{R}$ has zero linear measure, then by the Phragmén–Lindelöf theorem we see that $|G_m| \leqslant e^{Nr^{k-1}}$ uniformly in each $\overline{S}_{j,\epsilon}$, $j \in J_2$, for some integer $N = N(j)$. Since ϵ can be arbitrarily small, then in either case of (1) and (2) by the Phragmén–Lindelöf theorem again we conclude that G_m is a rational function. From now on we fix one large N.

We denote $D_0 = b_1^{1/n}$ and $D_j = \iota_{j-1}b_1^{-j}b_1^{1/n}$, $j = 1, \cdots, m$. Now we choose one θ such that $\delta(p_1, \theta) > 0$ and let $z = re^{i\theta} \in S_1$. Let $t_0 = 1/n$, $t_1 = (\alpha-1) + 1/n$, $\cdots$, $t_m = m(\alpha-1) + 1/n$. Similarly as in the proof of [**33**, theorem 2.1], we may use lemma 2.2 to integrate the recursion formulas $G_j = G'_{j-1} - \kappa_j G_{j-1}$ from $j = m$ to

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$j = 1$ along the above ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$ inductively and finally integrating $G_0 = f' - \kappa_0 f$ along this ray $z = re^{i\theta}$ to obtain

$$f = b_1^{1/n} \sum_{i=0}^{m} c_i \left(\frac{b_2}{b_1}\right)^i e^{t_i p_1} + H_0, \tag{2.32}$$

where $c_0, \cdots, c_m$ are constants and

$$H_0 = b_1^{1/n} e^{t_0 p_1} \int_0^z b_1^{-1/n} e^{-t_0 p_1} H_1 ds - a_0 b_1^{1/n} e^{t_0 p_1}, \tag{2.33}$$

where $a_0 = a_0(\theta)$ is a constant such that $|H_0| = O(r^N)$ along the ray $z = re^{i\theta}$.

As is shown in the proof of [**33**, theorem 2.1], b_1 is an n-square of some polynomial and we may write the entire solution of (2.1) as $f = \gamma_1 \sum_{j=0}^m c_j (b_2/b_1)^j e^{t_j p_1} + \eta$, where γ_1 is a polynomial such that $\gamma_1^n = b_1$ and η is a meromorphic function with at most finitely many poles. Then we can integrate $G_j = G'_{j-1} - \kappa_j G_{j-1}$ from $j = m$ to $j = 1$ inductively and finally integrate $G_0 = f' - \kappa_0 f$ to obtain that H_0 is a meromorphic function with at most finitely many poles. We choose $\eta = H_0$. Recall that along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) > 0$ and $z = re^{i\theta} \in S_1$, we have $|H_0| = O(r^N)$. Denote $g = b_1^{1/n} \sum_{i=0}^m c_i (b_2/b_1)^i e^{t_i p_1}$. Then

$$g^n = b_1 \sum_{k_0=0}^{mn} C_{k_0} \left(\frac{b_2}{b_1}\right)^{k_0} e^{(k_0 t - k_0 + 1) p_1}, \tag{2.34}$$

where

$$C_{k_0} = \sum_{\substack{j_0 + \cdots + j_m = n, \\ j_1 + \cdots + m j_m = k_0}} \frac{n!}{j_0! j_1! \cdots j_m!} c_0^{j_0} c_1^{j_1} \cdots c_m^{j_m}, \quad k_0 = 0, 1, \cdots, mn. \tag{2.35}$$

By [**12**, corollary 1], we may suppose that along the ray $z = re^{i\theta}$ we have $|f(re^{i\theta})^{(j)}/f(re^{i\theta})| = r^{j(k-1+\varepsilon)}$ for all $j > 0$ and all sufficiently large r. By writing P in the form in (2.10) with the new coefficients $\hat{a}_l = a_l (f'/f)^{n_{l1}} \cdots (f^{(s)}/f)^{n_{ls}}$, where $n_{l1}, \cdots, n_{ls}$ are nonnegative integers, and using [**12**, corollary 1], we see that each term in $P(z, f)$ of degree $n - j$, $1 \leqslant j \leqslant n - 1$, equals a linear combination of exponential functions of the form $e^{[nk_j(\alpha-1)+n-j]p_1/n}$, $0 \leqslant k_j \leqslant (n - j)m$, with coefficients β_j having polynomial growth along the ray $z = re^{i\theta}$. Therefore, by substituting $f = g + H_0$ into (2.1) we obtain by the same arguments in the proof of [**33**, theorem 2.1] that $c_0^n = 1$ when $m = 0$, and $c_0^n = 1$, $nc_0^{n-1} c_1 = 1$ and $p_2 = \alpha p_1$ when $m = 1$ and further that $C_{k_0} \equiv 0$ for all $2 \leqslant k_0 \leqslant m$ when $m \geqslant 2$.

Now, $b_1^{1/n}$ denotes a polynomial. By the definition of ι_j and D_j, we see that D_j are rational functions. Recall that G_m is a rational function. By lemma 2.2 and looking at the calculations to obtain H_0 in (2.33), we have, for $z \in S_{j,\epsilon}$, $j \in J_1$, such that $\delta(p_1, \theta) > 0$, $H_0 = \gamma_1 \sum_{l=0}^m d_{l,j} (b_2/b_1)^j e^{t_j p_1} + \eta_j$, where $d_{l,j}$, $l = 0, \cdots, m$, are some constants related to a sector $S_{j,\epsilon}$ and $|\eta_j| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$, $j \in J_1$. Of course, for $j = 1$, we have $d_{l,1} = 0$ for all l. Since $c_0^n = 1$ when $m = 0$, $c_0^n = nc_0^{n-1} c_1 = 1$ when $m = 1$, and $c_0^n = nc_0^{n-1} c_1 = 1$ and $C_{k_0} = 0$ for all $2 \leqslant k_0 \leqslant m$ when $m \geqslant 2$, then by simple computations, we deduce that $c_j = s_j c_0$ for some

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nonzero rational numbers s_j, $j = 0, 1, \cdots, m$. Therefore, by considering the growth of f along the ray $z = re^{i\theta}$ such that $z \in S_{j,\epsilon}$, $j \in J_1$, as for the ray $z = re^{i\theta} \in S_1$, we have $(c_0 + d_{0,j})^n = 1$ when $m = 0$, $(c_0 + d_{0,j})^n = n(c_0 + d_{0,j})^{n-1}(c_1 + d_{1,j}) = 1$ when $m = 1$ and further that $\hat{C}_{k_0} = \sum_{\substack{j_0+\cdots+j_m=n,\\ j_1+\cdots+mj_m=k_0}} \frac{n!}{j_0!j_1!\cdots j_m!}(c_0 + d_{0,j})^{j_0}(c_1 + d_{1,j})^{j_1} \cdots (c_m + d_{m,j})^{j_m} = 0$ for $k_0 = 2, \cdots, m$ when $m \geqslant 2$. Therefore, for each $j \in J_1$, there is a μ_j satisfying $\mu_j^n = 1$ such that $c_l + a_{l,j} = \mu_j c_l$ for all l. Note that $\mu_1 = 1$. Also, we have $|\eta| = O(r^N)$ uniformly in the sectors $\overline{S}_{j,\epsilon}$, $j \in J_2$. In conclusion, we may write $\eta = \gamma_1 \sum_{l=0}^{m}(\mu_j - 1)c_j(b_2/b_1)^j e^{[jn(\alpha-1)+1]p_1/n} + \eta_j$, where μ_j are the n-th roots of 1 such that $\mu_j = 1$, $j = \{1\} \cup \in J_2$, and $|\eta_j| = O(r^N)$ uniformly in the sector $\overline{S}_{j,\epsilon}$. In particular, when $k = 1$, since ϵ can be arbitrarily small, then by the Phragmén–Lindelöf theorem we conclude that η is a rational function. This completes the proof. □

3. An oscillation question of Ishizaki

Let $b_1(z)$, $b_2(z)$ and $b_3(z)$ be three polynomials such that $b_1 b_2 \not\equiv 0$ and $p_1(z)$ and $p_2(z)$ be two polynomials of the same degree $k \geqslant 1$ with distinct leading coefficients 1 and α, respectively, and $p_1(0) = p_2(0) = 0$. In this section, we use theorem 2.1 to investigate the oscillation of the second-order linear differential equation:

$$f'' - \left[b_1(z)e^{p_1(z)} + b_2(z)e^{p_2(z)} + b_3(z)\right] f = 0. \tag{3.1}$$

There have been several results about the oscillation of equation (3.1) and recently second-order linear differential equations with exponential polynomials are taken into more consideration in [**15**, **16**]. The results of Bank, Laine and langely [**5**], Ishizaki and Kazuya [**20**] and Ishizaki [**19**] can be summarized as follows:

(1) if α is non-real, then all nontrivial solutions of (3.1) satisfy $\lambda(f) = \infty$;

(2) if α is negative, then all nontrivial solutions of (3.1) satisfy $\lambda(f) = \infty$;

(3) if $0 < \alpha < 1/2$ or if $b_3 \equiv 0$ and $3/4 < \alpha < 1$, then all nontrivial solutions of (3.1) satisfy $\lambda(f) \geqslant k$.

Theorem 1.1 shows that the condition $b_3 \equiv 0$ in the third result can be removed. Ishizaki [**19**] asked if the third result $\lambda(f) \geqslant k$ above can be replaced by $\lambda(f) = \infty$. With theorem 2.1 at our disposal, we are able to answer this question partially. We prove the following

Theorem 3.1. *Let $0 < \alpha < 1$ and m be the smallest integer such that $\alpha \leqslant [2(m+1) - 1]/[2(m+1)]$. Suppose that $b_3 \equiv 0$ in (3.1). If (3.1) admits a nontrivial solution f such that $\lambda(f) < \infty$, then $\alpha = [2(m+1) - 1]/[2(m+1)]$ and $p_2 = \alpha p_1$.*

We will mainly use the techniques in [**6**] (see also [**23**, theorem 5.7]) to prove theorem 3.1. Since α is a positive number, we have $\int_1^{\infty} r|A(re^{i\theta})|dr < \infty$ along the ray $z = re^{i\theta}$ such that $\delta(p_1, \theta) < 0$. The following lemma can be proved similarly as in [**23**, lemma 5.16] by using Gronwall's lemma (see [**23**, p. 86]).

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LEMMA 3.2. *Under the assumptions of theorem 3.1, all solutions of equation (3.1) satisfy* $|f(re^{i\theta})| = O(r)$ *as* $r \to \infty$ *along the ray* $z = re^{i\theta}$ *such that* $\delta(p_1, \theta) < 0$.

Now we begin to prove theorem 3.1.

Proof of theorem 3.1. Let f be a nontrivial solution of equation (3.1) such that $\lambda(f) < \infty$. By Hadamard's factorization theorem we may write $f = \kappa e^h$, where h is an entire function and κ is the canonical product from the zeros of f satisfying $\rho(\kappa) = \lambda(\kappa) < \infty$. Denoting $g = h'$, then from (3.1) we have

$$g^2 + g' + 2\frac{\kappa'}{\kappa}g + \frac{\kappa''}{\kappa} = b_1(z)e^{p_1(z)} + b_2(z)e^{p_2(z)}. \tag{3.2}$$

Below we consider the two cases where $0 < \alpha \leqslant 1/2$ and $(2m-1)/(2m) < \alpha \leqslant [2(m+1)-1]/[2(m+1)]$, $m \geqslant 1$, respectively.

Case 1: $0 < \alpha \leqslant 1/2$.

By theorem 2.1, we may write $g = \gamma_1 e^{p_1/2} + \eta$, where γ_1 is a polynomial such that $\gamma_1^2 = b_1$ and η is an entire function such that $|\eta| = O(r^N)$ uniformly in $\overline{S}_{1,\epsilon}$ and $\overline{S}_{2,\epsilon}$. By substituting this expression into equation (3.2), we obtain

$$2\gamma_1\left(\frac{\kappa'}{\kappa} + \frac{1}{2}\frac{\gamma_1'}{\gamma_1} + \frac{p_1'}{4} + \eta\right)e^{p_1/2} - b_2e^{p_2} + \frac{\kappa''}{\kappa} + 2\eta\frac{\kappa'}{\kappa} + \eta^2 + \eta' = 0. \tag{3.3}$$

Suppose that $0 < \alpha < 1/2$. We define

$$w = \kappa\gamma_1^{1/2}e^{p_1/4 + \int_{z_0}^{z}\eta\,\mathrm{d}t}, \tag{3.4}$$

where z_0 is chosen so that $|z_0|$ is large. Then w is analytic outside a finite disc centred at 0 and satisfy

$$\frac{w'}{w} = \frac{\kappa'}{\kappa} + \frac{1}{2}\frac{\gamma_1'}{\gamma_1} + \frac{p_1'}{4} + \eta. \tag{3.5}$$

Dividing by $2\gamma_1 e^{p_1/2}$ on both sides of equation (3.3) and then considering the growth of w'/w along the ray $z = re^{i(\theta_2 - \epsilon)}$ such that w has no zero around the neighbourhood of the ray $z = re^{i(\theta_2-\epsilon)}$, we have by [**12**, corollary 1] that $|w'(re^{i\theta})/w(re^{i\theta})| = O(r^{-2})$ as $r \to \infty$. By integration, we obtain that $w(re^{i(\theta_2-\epsilon)}) \to a$ as $r \to \infty$ along the ray $z = re^{i(\theta_2-\epsilon)}$ for some nonzero constant $a = a(\theta_2, \epsilon)$. On the other hand, by applying lemma 3.2 to equation (3.1) we have $|f(re^{i(\theta_2+\epsilon)})| = O(r)$ along the ray $z = re^{i(\theta_2+\epsilon)}$. Recalling that $f = \kappa e^h$ and $g = h' = \gamma_1 e^{p_1/2} + \eta$, we may write

$$w = fe^{-h}\gamma_1^{1/2}e^{p_1/4 + \int \eta dz} = f\gamma_1^{1/2}e^{p_1/4 - \int_{z_0}^{z}\gamma_1 e^{p_1/2}\,\mathrm{d}t}. \tag{3.6}$$

Since $\delta(p_1, \theta_2 + \epsilon) < 0$ and thus along the ray $z = re^{i(\theta_2+\epsilon)}$ we have $\int_{z_0}^{z}\gamma_1 e^{p_1/2}\,\mathrm{d}t \to c$ for some constant $c = c(\theta_2, \epsilon)$, we see from (3.6) that w defined in (3.4) satisfies

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$w(re^{i(\theta+\epsilon)}) \to 0$ as $r \to \infty$. Denote

$$S_\epsilon = \{re^{i\theta} : \theta_2 - \epsilon \leqslant \theta \leqslant \theta_2 + \epsilon\}. \tag{3.7}$$

By choosing ϵ to be small and applying the Phragmén–Lindelöf theorem to w defined in (3.4) in the sector in (3.7), we get $a = 0$, a contradiction. Therefore, we must have $\alpha = 1/2$ when $b_3 \equiv 0$.

Now, if $k = 1$, then obviously $p_2 = p_1/2$ since we have assumed $p_1(0) = p_2(0) = 0$. If $k > 1$, then by theorem 2.1 we have $g = \mu_j\gamma_1 e^{p_1/2} + \eta_j$, where $\mu_j^n = 1$, γ_1 is a polynomial such that $\gamma_1^2 = b_1$ and η_j is an entire function such that $|\eta_j| = O(r^N)$ uniformly in $\overline{S}_{j,\epsilon}$. Note that η_j has finite order. Denoting $p_3 = p_2 - p_1/2$, we rewrite equation (3.3) as

$$\left[b_2 e^{p_3} - 2\mu_j\gamma_1\left(\frac{\kappa'}{\kappa} + \frac{1}{2}\frac{\gamma_1'}{\gamma_1} + \frac{p_1'}{4} + \eta_j\right)\right]e^{p_1/2} = \frac{\kappa''}{\kappa} + 2\eta_j\frac{\kappa'}{\kappa} + \eta_j^2 + \eta_j'. \tag{3.8}$$

If $p_2 \not\equiv p_1/2$, then p_3 is a nonconstant polynomial such that $\deg(p_3) \leqslant \deg(p_2) - 1$. By the definition of S_j in (2.3), we may choose a $\theta \in [0, 2\pi)$ so that the ray $z = re^{i\theta}$ meets only finitely discs in $\tilde{R}$ and also that $\log|e^{p_1/2}|$ and $\log|e^{p_2 - p_1/2}|$ both increase along the ray $z = re^{i\theta}$. By [**12**, corollary 1] we see that $\kappa'/\kappa + \gamma_1'/2\gamma_1 + p_1'/4 + \eta_j$ and $\kappa''/\kappa + 2\eta_j\kappa'/\kappa + \eta_j^2 + \eta_j' - b_3$ both have polynomial growth along the ray $z = re^{i\theta}$. Then by comparing the growth on both sides of equation (3.8) along the ray $z = re^{i\theta}$, we get a contradiction. Therefore, we must have $p_2 \equiv p_1/2$ when $\alpha = 1/2$.

Case 2: $(2m-1)/(2m) < \alpha \leqslant [2(m+1)-1]/[2(m+1)]$, $m \geqslant 1$.

In this case, by theorem 2.1 we already have $p_2 = \alpha p_1$ and we may write $g = \gamma_1\sum_{j=0}^m c_j(b_2/b_1)^j e^{[2j(\alpha-1)+1]p_1/2} + \eta$, where $m \geqslant 1$, γ_1 is a polynomial such that $\gamma_1^2 = b_1$ and η is a mermorphic function with at most finitely many poles such that $|\eta| = O(r^N)$ uniformly in $\overline{S}_{1,\epsilon}$ and $\overline{S}_{2,\epsilon}$. By substituting this expression into equation (3.2), we obtain

$$\begin{aligned} &2\gamma_1\sum_{j=0}^m c_j\left(\frac{b_2}{b_1}\right)^j\left[\frac{\kappa'}{\kappa} + \frac{1}{2}\frac{\gamma_1'}{\gamma_1} + j\frac{(b_2/b_1)'}{b_2/b_1} + \frac{2j(\alpha-1)+1}{4}p_1' + \eta\right]e^{L_j p_1} \\ &\gamma_1^2\sum_{k_0=m+1}^{2m} C_{k_0}\left(\frac{b_2}{b_1}\right)^{k_0} e^{M_{k_0}p_1} + \frac{\kappa''}{\kappa} + 2\eta\frac{\kappa'}{\kappa} + \eta^2 + \eta' = 0, \end{aligned} \tag{3.9}$$

where $L_j = [2j(\alpha-1)+1]/2$, $M_{k_0} = k_0\alpha - k_0 + 1$ and the coefficients $C_{k_0} = \sum_{\substack{j_0+\cdots+j_m=2,\\ j_1+\cdots+mj_m=k_0}} \frac{2!}{j_0!j_1!\cdots j_m!}c_0^{j_0}c_1^{j_1}\cdots c_m^{j_m}$, $k_0 = m+1, \cdots, 2m$. Suppose that $\alpha < [2(m+1)-1]/2(m+1)$. Then, for $k_0 = m+1+j$, $j = 0, 1, \cdots, m-1$, we have $L_{j+1} < k_0\alpha - \alpha + 1 < L_j$. As in previous case, we also define the function w in (3.4), where z_0 is chosen so that $|z_0|$ is large and w is analytic outside a finite disc centred at 0. It follows that w'/w has the form in (3.5). Similarly as in previous case, we first divide by $2c_0\gamma_1 e^{p_1/2}$ on both sides of equation (3.9) and conclude that $w(re^{i(\theta_2-\epsilon)}) \to a$ as $r \to \infty$ along the ray $z = re^{i(\theta_2-\epsilon)}$ for some nonzero constant $a = a(\theta_2, \epsilon)$; then we use the expression $g = h' = \gamma_1\sum_{j=0}^m c_j(b_2/b_1)^j e^{[2j(\alpha-1)+1]p_1/2} + \eta$ to derive from (3.4) that $w(re^{i(\theta+\epsilon)}) \to 0$

as $r \to \infty$ along the ray $z = re^{i(\theta_2+\epsilon)}$. An application of the Phragmén–Lindelöf theorem to w in the sector in (3.7) then yields a contradiction. We omit those details. Therefore, we must have $\alpha = [2(m+1)-1]/2(m+1)$. We complete the proof. □

4. Equation (1.1) with periodic coefficients in (1.6)

As mentioned in the introduction, all nontrivial solutions of the second-order linear differential equation $f'' + (e^z - b)f = 0$ such that $\lambda(f) < \infty$ are given explicit expressions. In this section we solve nontrivial solutions such that $\lambda(f) < \infty$ of the second-order linear differential equation:

$$f'' - \left(e^{lz} + b_2 e^{sz} + b_3\right) f = 0, \tag{4.1}$$

where l and s are relatively prime integers such that $l > s \geqslant 1$, b_2 and b_3 are constants and $b_2 \neq 0$. We remark that by using the method in [**25**], we may prove that all nontrivial solutions of equation (4.1) satisfy $\lambda(f) = \infty$ when b_3 is replaced by a nonconstant polynomial.

Suppose that equation (4.1) admits a nontrivial solution such that $\lambda(f) < \infty$. Then f has the form in (1.3) or (1.4). Also, we may write $f = \kappa e^h$, where h is an entire function and κ is the canonical product from the zeros of f satisfying $\sigma(\kappa) = \lambda(\kappa) < \infty$. Thus we may suppose that κ equals a polynomial in $e^{z/2}$ or e^z and h' equals a polynomial in $e^{z/2}$ or e^z. By denoting $g = h'$, from (4.1) we have

$$g^2 + g' + 2\frac{\kappa'}{\kappa}g + \frac{\kappa''}{\kappa} = e^{lz} + b_2 e^{sz} + b_3. \tag{4.2}$$

By theorem 2.1, we may determine the coefficients c_j in (1.3) or (1.4) from equation (4.2). Our main result is the following

Theorem 4.1. *Let b_2 and b_3 be constants such that $b_2 \neq 0$ and l, s be relatively prime integers such that $l > s \geqslant 1$. Suppose that (4.1) admits two linearly independent solutions f_1 and f_2 such that* $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. *Then $s = 1$ and $l = 2$.*

Recall the following well-known result due to Wittich [**32**]. We say that a function f is *subnormal* if $\limsup_{r\to\infty} \log T(r, f)/r = 0$. This lemma gives the form of subnormal solutions of second-order linear differential equations with certain periodic functions as coefficients.

Lemma 4.2. *Let $P(z)$ and $Q(z)$ be polynomials in z and not both constants. If $w \not\equiv 0$ is a subnormal solution of equation*

$$w'' + P(e^z)w' + Q(e^z)w = 0, \tag{4.3}$$

then w must have the form $w = e^{cz}(a_0 + a_1 e^z + \cdots + a_k e^{kz})$, where $k \geqslant 0$ is an integer and c, $a_0, \cdots, a_k$ are constants with $a_0 \neq 0$ and $a_k \neq 0$. Moreover, we have $c^2 + cP(0) + Q(0) = 0$.

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Proof of lemma 4.2. By Wittich [32], we have $w = e^{cz}(a_0 + a_1 e^z + \cdots + a_k e^{kz})$. By taking the derivatives of w and then dividing w' and w'' by w, respectively, we get

$$\frac{w'}{w} = \frac{\sum_{j=0}^{k}(c+j)a_j e^{jz}}{\sum_{j=0}^{k} a_j e^{jz}}, \tag{4.4}$$

$$\frac{w''}{w} = \frac{\sum_{j=0}^{k}(c^2+2jc+j^2)a_j e^{jz}}{\sum_{j=0}^{k} a_j e^{jz}}. \tag{4.5}$$

We write equation (4.3) as $Q(e^z) = -w''/w - P(e^z)w'/w$. Since w is of finite order, then an application of the lemma on the logarithmic derivative yields $\deg(Q(z))m(r, e^z) \leqslant \deg(P(z))m(r, e^z) + O(\log r)$, i.e., $[\deg(Q(z)) - \deg(P(z))]T(r, e^z) \leqslant O(\log r)$. Therefore, $\deg(Q(z)) \leqslant \deg(P(z))$ and thus $P(z)$ is nonconstant. Together with equations (4.4) and (4.5), we rewrite equation (4.3) as

$$\frac{\sum_{j=0}^{k}(c^2+2jc+j^2)a_j e^{jz}}{\sum_{j=0}^{k} a_j e^{jz}} + \frac{\sum_{j=0}^{k}(c+j)a_j e^{jz}}{\sum_{j=0}^{k} a_j e^{jz}}P(e^z) + Q(e^z) = 0. \tag{4.6}$$

Since along a ray $z = re^{i\theta}$ such that $\cos\theta < 0$, we have $e^z \to 0$ as $r \to \infty$, then by letting $r \to \infty$ along the ray $z = re^{i\theta}$, we obtain from equation (4.6) that $c^2 + cP(0) + Q(0) = 0$. This completes the proof. □

Unlike in § 2 and 3 where Nevanlinna theory plays the central role in proving theorems 2.1 and 3.1, the proof of theorem 4.1 will, however, mainly rely on the *Lommel transformation* for the *generalized Bessel equation*:

$$x^2y'' + xy' + \left(\sum_{-n'}^{n} d_j x^j\right) y = 0. \tag{4.7}$$

Recall the *Bessel equation*: $x^2y'' + xy' + (x^2 - \nu^2)y = 0$, where ν is a nonzero constant. Lommel [26] and Pearson [27] independently (see also [31]) studied the following transformation given by:

$$x = \alpha t^{\beta}, \quad y(x) = t^{\gamma}u(t), \tag{4.8}$$

where α, β and γ are constants and applied to the Bessel equation. By using the above transformation to equation (4.7) and by computing the derivatives of x and y, we get

$$t^2u''(t) + (2\gamma+1)tu'(t) + \left(\gamma^2 + \beta^2 \sum_{-n'}^{n} \alpha^j d_j t^{\beta j}\right) = 0. \tag{4.9}$$

A further change of variable such that

$$t = e^{pz}, \quad f(z) = u(t), \tag{4.10}$$

leads to an equation of the form

$$f'' + 2\gamma p f' + p^2\left(\gamma^2 + \beta^2 \sum_{-n'}^{n} \alpha^j d_j e^{\beta p j z}\right) = 0. \tag{4.11}$$

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In the case of equation (4.1), by Lommel's transformation we have

$$x^2y'' + xy' - \left(d_1x^l + d_2x^s + d_3\right)y = 0, \tag{4.12}$$

where d_1, d_2 and d_3 are some constants. By comparing the coefficients of equation (4.1) and (4.11), we deduce that $2\gamma p = 0$, $\beta p = 1$, $\alpha^l d_1 = 1$, $\alpha^s d_2 = b_2$ and $d_3 = b_3$. Further, for equation (4.12), it is well-known that the transformation $y = x^{-1/2}u$ leads to an equation of the form

$$u'' - \left[\frac{1}{\alpha^l}x^{l-2} + \frac{b_2}{\alpha^s}x^{s-2} + \left(b_3 - \frac{1}{4}\right)\frac{1}{x^2}\right]u = 0. \tag{4.13}$$

In the case $l = 4$, it has been shown by Chiang and Yu [**11**] that there is a full correspondence between solutions of (4.1) such that $\lambda(f) < \infty$ and Liouvillian solutions of (4.13). The only possible singular point of equation (4.13) is $x = 0$. Concerning the local solutions around a singular point of a second-order linear differential equation, we have the following elementary lemma 4.3; see [**17**] or in [**23**, lemma 6.6].

Lemma 4.3 [**17**, **23**]. *Suppose that h is analytic in $|z| < R$, $R > 0$, and consider the differential equation*

$$u'' + \frac{h(z)}{z^2}u = 0 \tag{4.14}$$

in the disc $|z| < R$. Let ρ_1 and ρ_2 be the roots of

$$\rho(\rho - 1) + h(0) = 0. \tag{4.15}$$

Denote by $D = D(r)$ the slit disc $D := \{z : |z| < r\} \setminus \{t \mid 0 \leqslant t < r\}$. Then

(1) *if $\rho_1 - \rho_2 \in \mathbb{Z} \setminus \{0\}$, then equation (4.14) admits in some slit disc $D = D(r)$, $r \leqslant R$, two linearly independent solutions u_1 and u_2 of the form:*

$$\begin{aligned} u_1(z) &= z^{\rho_1}\sum_{i=0}^{\infty} a_i z^i, \quad a_0 \neq 0, \\ u_2(z) &= u_1(z)d\log z + z^{\rho_2}\sum_{i=0}^{\infty} b_i z^i, \end{aligned} \tag{4.16}$$

where $d = 0$ or $d = 1$;

(2) *if $\rho_1 - \rho_2 \notin \mathbb{Z}$, then equation (4.14) admits in some slit disc $D = D(r)$, $r \leqslant R$, two linearly independent solutions u_1 and u_2 of the form:*

$$\begin{aligned} u_1(z) &= z^{\rho_1}\sum_{i=0}^{\infty} a_i z^i, \quad a_0 \neq 0, \\ u_2(z) &= z^{\rho_2}\sum_{i=0}^{\infty} b_i z^i, \quad b_0 \neq 0; \end{aligned} \tag{4.17}$$

(3) *if* $\rho_1 - \rho_2 = 0$, *then equation* (4.14) *admits in some slit disc* $D = D(r)$, $r \leqslant R$, *two linearly independent solutions* u_1 *and* u_2 *of the form:*

$$u_1(z) = z^{\rho_1} \sum_{i=0}^{\infty} a_i z^i, \quad a_0 \neq 0,$$
$$u_2(z) = u_1(z) \log z + z^{\rho_2} \sum_{i=0}^{\infty} b_i z^i. \tag{4.18}$$

For the solution u_2 in (4.16), if $d = 0$, then from the proof of [**23**, lemma 6.6] we know that $b_0 \neq 0$.

Now, by elementary theory of ordinary differential equation (see [**17**]), lemma 4.3 shows that equation (4.13) admits two linearly independent solutions u_1 and u_2 in the broken plane $\mathbb{C}^- = \mathbb{C} \setminus \{x \mid 0 \leqslant x < \infty\}$. When $p = 1$ in (4.10), by the Lommel transformation and analytic continuation principle, the general solution of (4.1) is thus given by

$$f = (\alpha e^z)^{-1/2} [E_1 u_1(\alpha e^z) + E_2 u_2(\alpha e^z)], \tag{4.19}$$

where E_1 and E_2 are two arbitrary constants. Note that the above solution is independent from the choice of the branches of u_1 and u_2 in lemma 4.3. This is the key observation for the proof of theorem 4.1.

Now we begin to prove theorem 4.1.

Proof of theorem 4.1. We first suppose that f is a nontrivial solution such that $\lambda(f) < \infty$ of equation (4.1) and use the expressions in (1.3) and (1.4) to write $f(z) = \Psi(x) = x^c \psi(x) e^{\chi(x)}$, where $x = e^{z/h}$, $h = 1$ or $h = 2$. From the proof of [**11**, theorem 1.2], we know that in the broken plane $\mathbb{C}^-$ equation (4.13) admits a solution of the form

$$u = \exp\left(\int \omega \mathrm{d}x\right), \tag{4.20}$$

where $\omega := \chi' + \psi'/\psi + (2hc + 1)/(2x)$ is rational function in the complex plane $\mathbb{C}$. By using Kovacic's algorithm in [**22**] and giving the same discussions as in the proof of [**11**, theorem 3.1] to equation (4.13) for the two cases $b_3 \neq 1/4$ and $b_3 = 1/4$, respectively, we conclude that l must be even. It follows that f has the form in (1.4) and thus $p = 1$ in (4.10) and $\alpha = 1$ in (4.8). We write $f = \kappa e^h$, where κ and h' are both polynomials in $\zeta = e^z$ such that $\kappa(0) \neq 0$. We may also write $f = \kappa_c e^{h_c}$, where $\kappa_c = \kappa e^{cz}$ and $h_c' = h' - c$. Then, denoting $g_c = h_c'$, we have from (4.1) that

$$g_c^2 + g_c' + 2\frac{\kappa_c'}{\kappa_c} g_c + \frac{\kappa_c''}{\kappa_c} = e^{lz} + b_2 e^{sz} + b_3. \tag{4.21}$$

By lemma 4.2 and the expression in (1.4), we see that the constant c in (1.4) satisfies $c^2 = b_3$.

Now, for the solution in (4.19), by lemma 4.3 we have $\rho_1 + \rho_2 = 1$ and $\rho_1 \rho_2 = 1/4 - b_3$, which yield $(\rho_1 - \rho_2)^2 = 4b_3$. Then $\rho_1 - \rho_2 = -2c$ and it follows that $\rho_1 =$

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$(1-2c)/2$ and $\rho_2 = (1+2c)/2$. Thus the solutions in (4.19) can be written as

$$f = (e^z)^c \left[(E_1 + E_2 d \log e^z)\,(e^z)^{-2c} \sum_{j=0}^{\infty} a_j (e^z)^j + E_2 \sum_{j=0}^{\infty} b_j (e^z)^j \right], \tag{4.22}$$

when $\rho_1 - \rho_2 \neq 0$, or

$$f = (E_1 + E_2 \log e^z) \sum_{j=0}^{\infty} a_j (e^z)^j + E_2 \sum_{j=0}^{\infty} b_j (e^z)^j, \tag{4.23}$$

when $\rho_1 - \rho_2 = 0$. Note that $d = 0$ in (4.22) when $\rho_1 - \rho_2$ is not an integer. On the other hand, for the solution $f = \kappa e^h$, we may write the expression in (1.4) in the form $f = e^{cz} \sum_{j=0}^{\infty} d_j e^{jz}$. By comparing this series with the one in (4.22) or in (4.23), we conclude that the logarithmic term in u_2 does not occur. This implies that $d = 0$ or $E_2 = 0$ in (4.22) and $E_2 = 0$ in (4.23) when $f = \kappa e^h$.

With these preparations, we now suppose that f_1 and f_2 are two linearly independent solutions of (4.1) such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. We write $l = 2(m+1)$ for some integer $m \geqslant 0$ and also write $s = 2(m+1) - t$ for some integer $t \geqslant 1$.

Let q be the smallest integer such that $s/l \leqslant [2(q+1)-1]/[2(q+1)]$. Since l and s are relatively prime, we see that the equality holds only when $q = m$. For each of f_1 and f_2, denoted by f, we may write $f = \kappa_c e^{h_c}$, where $\kappa_c = \kappa e^{cz}$. Then, denoting $g_c = h_c'$, we have equation (4.21). By theorem 2.1, $g_c = h_c' = \sum_{j=0}^{q} c_j e^{(m+1-jt)z}$, where $c_0, c_1, \cdots, c_q$ are constants such that $c_0^2 = 1$ and $c_1, \cdots, c_q$ satisfy certain relations. In both of the two cases where $q = 1$ and $q \geqslant 2$, we have $2c_0 c_1 = b_2$ and, by simple computations, that,

$$c_j = \frac{t_j c_1^j}{(-2c_0)^{j-1}}, \quad j = 1, \cdots, q, \tag{4.24}$$

where t_j are positive integers such that $t_1 < \cdots < t_q$, and further that

$$C_{q+j} = \sum_{\substack{j_0+\cdots+j_q=2,\\ j_1+\cdots+qj_q=q+j}} \frac{2}{j_0! \cdots j_q!} c_0^{j_0} \cdots c_q^{j_q} = \frac{T_{q+j} c_1^{q+j}}{(-2c_0)^{q+j-2}}, \quad j = 1, \cdots, q, \tag{4.25}$$

where T_{q+j} are positive integers such that $T_{q+1} < \cdots < T_{2q}$. By substituting $g_c = \sum_{j=0}^{q} c_j e^{(m+1-jt)z}$ into (4.21) together with theorem 2.1, we get

$$\frac{\kappa_c''}{\kappa_c} + 2c_0 e^{(m+1)z} \frac{\kappa_c'}{\kappa_c} + c_0(m+1)e^{(m+1)z} - b_2 e^{sz} - b_3 = 0, \tag{4.26}$$

when $q = 0$, and

$$\begin{aligned} &\frac{\kappa_c''}{\kappa_c} + 2\left(\sum_{j=0}^{q} c_j e^{(m+1-jt)z}\right) \frac{\kappa_c'}{\kappa_c} - b_3 \\ &\quad + \sum_{j=0}^{q} \left[C_{k_j} e^{[m+1-(q+1)t]z} + (m+1-jt)c_j \right] e^{(m+1-jt)z} = 0, \end{aligned} \tag{4.27}$$

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when $q \geqslant 1$, where $C_{k_j} = C_{q+1+j}$ and $C_{2q+1} = 0$. By substituting equations (4.4) and (4.5) for $\kappa_c = e^{cz}(\sum_{i=1}^{k} a_i e^{iz})$, $a_0 a_k \neq 0$, into (4.26) or (4.27) and noting that $b_3 = c^2$, we finally get

$$\sum_{i=0}^{k} \left[(2ic + i^2) a_i e^{iz} + c_0(2c + 2i + m + 1) a_i e^{(m+1+i)z} - b_2 a_i e^{(i+s)z} \right] = 0, \quad (4.28)$$

when $q = 0$, and

$$\sum_{i=0}^{k} (2ic + i^2) a_i e^{iz} + 2 \left(\sum_{i=0}^{k} (c + i) a_i e^{iz} \right) \left(\sum_{j=0}^{q} c_j e^{(m+1-jt)z} \right) + \left(\sum_{i=0}^{k} a_i e^{iz} \right) \left\{ \sum_{j=0}^{q} [C_{k_j} e^{[m+1-(q+1)t]z} + (m + 1 - jt) c_j] e^{(m+1-jt)z} \right\} = 0, \quad (4.29)$$

when $q \geqslant 1$. Note that the inequality $(2q-1)/(2q) < s/l \leqslant [2(q+1)-1]/[2(q+1)]$ implies $qt < m + 1 \leqslant (q + 1)t$, where the equality holds when $q = m$. The left-hand side of equations (4.28) and (4.29) are polynomials in e^z of degree $k + m + 1$ and thus all coefficients of these two polynomials vanish. When $s = 2(m + 1) - t < 2m + 1$, we have $q < m$ and $t \geqslant 2$. By looking at the highest-degree term in the resulting polynomial and noting that $a_k \neq 0$, we find

$$m + 2c + 2k + 1 = 0. \quad (4.30)$$

Similarly, when $s = 2m + 1$, we have $q = m$ and $t = 1$ and find

$$C_{m+1} + (m + 2c + 2k + 1) c_0 = 0. \quad (4.31)$$

Let c_+ or c_- be any square root of b_3. We may write $f_1 = \kappa_{1c} e^{h_{1c}}$ and $f_2 = \kappa_{2c} e^{h_{2c}}$, where $\kappa_{1c} = \kappa_1 e^{c_+ z}$ and $\kappa_{2c} = \kappa_2 e^{c_- z}$ and κ_1 and κ_2 are two polynomials in e^z of degrees k_1 and k_2, respectively. Moreover, h_{1c} satisfies $h'_{1c} = \sum_{j=0}^{q} c_j e^{(m+1-jt)z}$. Since $c_0^2 = 1$ and since $2c_0 c_1 = b_2$ when $q \geqslant 1$, we easily deduce from (4.24) that $h_{2c} = \pm h_{1c}$. Recall the elementary *Wronskian determinant*: $f'_1 f_2 - f_1 f'_2 = D$, where D is a nonzero constant (see [**23**]). Then we have $(f_1/f_2)' = D/f_2^2$. If $h_{2c} = h_{1c}$, then f_1/f_2 is of finite order while f_1^2 is of infinite order, a contradiction. Therefore, $h_{2c} = -h_{1c}$. We may suppose that $c_+ = c$. When $s < 2m + 1$, from (4.30) we deduce that $c_+ = c_- = c$, which implies that $-2c = m + 1 + 2k_1$ is a positive integer and hence $k_1 = k_2$; when $s = 2m + 1$, using (4.25) and the relation $2c_0 c_1 = b_2$ we deduce from (4.31) that $c_+ + c_- + m + 1 + k_1 + k_2 = 0$, which implies that $c_+ = c_- = c$ and hence $-2c = m + 1 + k_1 + k_2$ is a positive integer. Now $\rho_1 - \rho_2 = -2c$ is a positive integer. Together with (4.19) and previous preparations, we conclude that equation (4.13) admits in the broken plane $\mathbb{C}^-$ two linearly independent solutions of the form $u_1 = x^{\rho_1} v_1$ and $u_2 = x^{\rho_2} v_2$, where v_1 and v_2 are two entire functions

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such that $v_1(0) \neq 0$ and $v_2(0) \neq 0$, so that

$$\begin{aligned} f_1 &= x^c \kappa_{11} e^{h_{11}} = x^{-1/2}\left(D_1 x^{\rho_1} v_1 + D_2 x^{\rho_2} v_2\right), \\ f_2 &= x^c \kappa_{12} e^{-h_{11}} = x^{-1/2}\left(D_3 x^{\rho_1} v_1 + D_4 x^{\rho_2} v_2\right), \end{aligned} \tag{4.32}$$

where D_j are constants, $h_{11} = \sum_{j=0}^{q} \frac{c_j}{m+1-jt} x^{m+1-jt}$, and κ_{11} and κ_{12} are two polynomials of degrees k_1 and k_2, respectively, such that $\kappa_{11}(0) \neq 0$ and $\kappa_{12}(0) \neq 0$. Noting $\rho_1 = (1-2c)/2$ and $\rho_2 = (1+2c)/2$, we see from (4.22) that $D_2 D_4 \neq 0$. Obviously, $E := D_1 D_4 - D_2 D_3 \neq 0$. From equation (4.32) we get

$$u_1 = x^{\rho_1} v_1 = \frac{1}{E} x^{1/2} x^c \left(D_4 \kappa_{11} e^{h_{11}} - D_2 \kappa_{12} e^{-h_{11}}\right). \tag{4.33}$$

Since v_1 is an entire function with $v_1(0) \neq 0$, we see from (4.33) that the function $w := D_4 \kappa_{11} e^{2h_{11}} - D_2 \kappa_{12}$ has a zero of order $-2c$ at the point $z = 0$ and so $w(0) = w'(0) = \cdots = w^{(-2c-1)}(0) = 0$. Denote

$$\begin{aligned} \kappa_{11} &= a_{1,0} + a_{1,1} x + \cdots + a_{1,k_1} x^{k_1}, \\ \kappa_{12} &= a_{2,0} + a_{2,1} x + \cdots + a_{2,k_2} x^{k_2}, \end{aligned} \tag{4.34}$$

where $a_{1,0}$, $a_{2,0}$, a_{1,k_1}, $a_{1,k_2} \neq 0$. $w(0) = 0$ implies that $D_4 a_{1,0} = D_2 a_{2,0}$. Supposing that $a_{1,0} = a_{2,0} = 1$, we have $D_4 = D_2$. Below we consider the case when $m \geqslant 1$.

Consider first the case when $s/l < 1/2$. Since $m \geqslant 1$, by theorem 1.1 and (4.30), we see that $k_1 = k_2 \geqslant 1$. Now $w^{(m+1)}(0) = 0$ implies that $a_{1,m+1} + 2(m!)c_0 = a_{2,m+1}$. Here $a_{1,m+1} = 0$ if $m+1 > k_1$ and so is for $a_{2,m+1}$. Obviously, $m+1 \leqslant k_1$. For each of f_1 and f_2, denoted by f, we may write $f = \kappa_c e^{h_c}$. Then we have equation (4.28). Note that $1 \leqslant s = 2(m+1) - t \leqslant m$. The left-hand side of equation (4.28) is a polynomial in e^z of degree $m+1+k$ and thus all coefficients of this polynomial vanish. Denoting $a_{-m-1} = \cdots = a_{-2} = a_{-1} = 0$ and $a_{k+1} = \cdots = a_{k+m} = 0$, we obtain from equation (4.28) that $c_0(2c+2k+m+1)a_k = 0$, which yields (4.30), and

$$c_0(2c + 2i - m - 1)a_{i-m-1} = b_2 a_{i-s} - (2ic + i^2)a_i, \quad i = 0, \cdots, k+m. \tag{4.35}$$

By substituting $2c = -2k - m - 1$ into the equations in (4.35) we get

$$2c_0(k - i + m + 1)a_{i-m-1} = -b_2 a_{i-s} + (2ic + i^2)a_i, \quad i = 0, \cdots, k+m. \tag{4.36}$$

Since $1 \leqslant s \leqslant m$, by letting $i = 1, 2, \cdots, m$, we see that $a_1/a_0 = K_1, \cdots, a_m/a_0 = K_m$ for some constants $K_1, \cdots, K_m$ independent from c_0. Then by letting $i = m+1$ in (4.36) together with the relation $a_{1,m+1} + 2(m!)c_0 = a_{2,m+1}$, we get $2k_1/(m+1)(2c+m+1) + 2(m!) = -2k_2/(m+1)(2c+m+1)$. Since $k_1 = k_2$ and $-2c = 2k_1 + m + 1$, we get $(m+1)! = 1$, which is impossible when $m \geqslant 1$.

Consider next the case when $s/l > 1/2$ and $s < 2m+1$. Now $qt < m+1 < (q+1)t$ for some integer $1 \leqslant q < m$. Recalling that $s = 2(m+1) - t$ and l and s are relatively prime, we see that $t \geqslant 3$ is an odd integer. Denoting each of f_1 and f_2 by f, we may write $f = \kappa_c e^{h_c}$. Then we have equation (4.29). Denote $M = m + 1 - qt$ for simplicity. Since $-2c = 2k + m + 1$ and $C_{2q+1} = 0$ and $C_{2q} = c_q^2$, then by looking

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at the coefficient of the term e^{Mz} on the left-hand side of equation (4.29), we find $M(2c+M)a_M + 2ca_0c_q + Ma_0c_q = 0$, which gives $a_0c_q + Ma_M = 0$. Recall that the function $w := D_4\kappa_{11}e^{2h_{11}} - D_2\kappa_{12}$ has a zero of order $-2c$ at the point $z = 0$. Then $w^{(M)}(0) = 0$ implies that $a_{1,M} + 2(M-1)!c_q = a_{2,M}$. Using equation (4.24) together with $2c_0c_1 = b_2$ and $a_0c_q + Ma_M = 0$, we get $M! = 1$, which implies that $M = 1$. It follows that $m = qt$. By looking at the coefficient of the term e^{2z} on the left-hand side of equation (4.29), we find $2(2c+2)a_2 + 2(c+1)a_1c_q + a_0c_q^2 + a_1c_q = 0$, which together with $a_1 = -a_0c_q$ yields $2a_2 = a_0c_q^2$. Then by looking at the coefficient of the term e^{3z} on the left-hand side of equation (4.29), we find $3(2c+3)a_3 + 2(c+2)a_2c_q + a_2c_q + a_1c_q^2 = 0$, i.e., $(2c+3)(6a_3 + c_q^3) = 0$ and thus $6a_3 + c_q^3 = 0$. Now $w^{(3)}(0) = 0$ implies that $a_{1,3} + 6a_{1,2}c_q + 12a_{1,1}c_q^2 + 8c_q^3 = a_{2,3}$, which together with the relation $a_1 = -a_0c_q$ and $2a_2 = a_0c_q^2$ gives $a_{1,3} - c_q^3 = a_{2,3}$. Then using equation (4.24) together with $2c_0c_1 = b_2$ and $c_q^3 + 6a_3 = 0$, we get $c_q^3 = 0$, a contradiction to (4.24).

Finally, we consider the case when $s = 2m+1$. Recall that $q = m$ and $t = 1$ in (4.29). In this case, if $k_1 = k_2$, then using (4.25) and the relation $2c_0c_1 = b_2$ we get from (4.31) that $C_{m+1} = 0$, a contradiction. Therefore, without loss of generality, we may suppose that $k_1 > k_2 \geqslant 0$. If $k_2 = 0$, then by theorem 1.1 have $2c+1 = 0$, which is impossible since $-2c = m+1+k_1+k_2$. Therefore, $k_2 > 0$. Note that $C_{2m} = c_m^2$. Since $-2c = m+1+k_1+k_2$, then by looking at the coefficient of the terms e^z and e^{2z} in equation (4.29), respectively, we find $a_1 + a_0c_m = 0$ and $2(2c+2)a_2 + (2c+3)c_ma_1 + [c_m^2 + (2c+2)c_{m-1}]a_0 = 0$ and so $2a_2 - c_m^2a_0 + c_{m-1}a_0 = 0$. Recall that the function $w := D_4\kappa_{11}e^{2h_{11}} - D_2\kappa_{12}$ has a zero of order $-2c$ at the point $z = 0$. Now $w''(0) = 0$ implies that $a_{1,2} + 4a_{1,1}c_m + 4c_{m-1} + 4c_m^2 = a_{2,2}$. Using equation (4.24) together with $2c_0c_1 = b_2$, we get $-c_{m-1}/2 + 4c_{m-1} = c_{m-1}/2$, which yields $c_{m-1} = 0$, a contradiction to (4.24).

From the above reasoning, we conclude that $l = 2$. We complete the proof. □

In the rest of this section, we use theorem 2.1 to determine precisely all nontrivial solutions such that $\lambda(f) < \infty$ of equation (4.1) for the case $l = 2$ and $l = 4$.

THEOREM 4.4. *Let b_1, b_2 and b_3 be constants such that $b_1b_2 \neq 0$ and s and l be relatively prime integers such that $1 \leqslant s < l \leqslant 4$. Suppose that (4.1) admits a nontrivial solution f such that $\lambda(f) < \infty$. Then*

(1) *if $s = 1$ and $l = 2$, then $f = \kappa e^h$, $\kappa = \sum_{i=-1}^{k} a_ie^{iz}$ and $h = c_0e^z + cz$, where $k \geqslant 0$ is an integer, c_0 and c are constants such that $c_0^2 = 1$, $2c_0(c+k) + c_0 = b_2$ and $c^2 = b_3$, and $a_{-1}, a_0, \cdots, a_k$ are constants such that $a_0a_k \neq 0$, $a_{-1} = 0$ and*

$$2c_0(k+1-i)a_{i-1} = (2ic + i^2)a_i, \quad i = 0, 1, \cdots, k; \tag{4.37}$$

(2) *if $s = 1$ and $l = 4$, then $f = \kappa e^h$, $\kappa = \sum_{i=-2}^{k+1} a_ie^{iz}$ and $h = (c_0/2)e^{2z} + cz$, where $k \geqslant 1$ is an integer, c_0 and c are constants such that $c_0^2 = 1$, $2c + 2k + 2 = 0$ and $c^2 = b_3$, and $a_{-2}, a_{-1}, a_0, \cdots, a_{k+1}$ are constants such that $a_0a_k \neq 0$, $a_{-2} = a_{-1} = a_{k+1} = 0$ and*

$$2c_0(k-i+2)a_{i-2} = -b_2a_{i-1} + (2ic + i^2)a_i, \quad i = 0, 1, \cdots, k+1; \tag{4.38}$$

(3) *if* $s = 3$ *and* $l = 4$, *then* $f = \kappa e^h$, $\kappa = \sum_{i=-2}^{k+1} a_i e^{iz}$ *and* $h = (c_0/2)e^{2z} + c_1 e^z + cz$, *where* $k \geqslant 0$ *is an integer,* c_0, c_1 *and* c *are constants such that* $c_0^2 = 1$, $2c_0c_1 = b_2$, $c^2 = b_3$ *and* $c_1^2 + (2 + 2c + 2k)c_0 = 0$, *and* $a_{-2}, a_{-1}, a_0, \cdots, a_{k+1}$ *are constants such that* $a_0 a_k \neq 0$, $a_{-2} = a_{-1} = a_{k+1} = 0$ *and*

$$(2k - 2i + 4)c_0 a_{i-2} = (2c + 2i - 1)c_1 a_{i-1} + (2ic + i^2)a_i, \quad i = 0, \cdots, k + 1. \tag{4.39}$$

For the convenience to write the recursive formulas in (4.37)–(4.39), we have introduced some extra coefficients a_{-2}, a_{-1}, a_{k+1}, which are all equal to 0.

When $s = 1$ and $l = 4$, since $a_0 \neq 0$ and $a_k \neq 0$ and $2c + 2k + 2 = 0$, the recursive formulas in (4.38) yield a polynomial equation $P(b_2) = 0$ with respect to b_2 with coefficients formulated in terms of c_0 and k. For example, when $k = 1$, we have $0 = -b_2 a_0 + (2c + 1)a_1$ and $2c_0 a_0 = -b_2 a_1$, which together with the equation $2c + 4 = 0$ yield $b_2^2 - 6c_0 = 0$, etc.

When $s = 3$ and $l = 4$, if $2c + 1 \neq 0$, then we may solve from the first $k + 1$ equations in (4.39) that $a_{k-1} = P(c)a_k$ for some polynomial $P(c)$ with respect to c with coefficients formulated in terms of c_0 and c_1. By combining this equation with the equation $2c_0 a_{k-1} = (2c + k)c_1 a_k$ together with the relation $c_1^2 + (2 + 2c + 2k)$ $c_0 = 0$ we may obtain a polynomial equation $P(t) = 0$ with respect to $t = 2c$ with coefficients independent from c_0 and c_1. For example, when $k = 1$, we have $0 = (2c + 1)(c_1 a_0 + a_1)$ and $2c_0 a_0 = (2c + 3)c_1 a_1$, which yield $P(t) = (t + 2)(t + 5) = 0$ and thus $2c = -2$ or $2c = -5$, etc.

Proof of theorem 4.4. Suppose that f is a nontrivial solution such that $\lambda(f) < \infty$ of equation (4.1). Following the proof of theorem 4.1, we may write $f = \kappa_c e^{h_c}$, where $\kappa_c = \kappa e^{cz}$ and $g_c = h_c'$ and then from (4.1) we get equation (4.21). Below we consider three cases: (1) $s = 1$ and $l = 2$; (2) $s = 1$ and $l = 4$; (3) $s = 3$ and $l = 4$.

For the first two cases $s = 1$ and $l = 2$ or $s = 1$ and $s = 4$, we have $\kappa_c = e^{cz}(\sum_{i=0}^{k} a_i e^{iz})$ and $h_c = [c_0/(m + 1)]e^{(m+1)z}$, where $m = 0$ or $m = 1$ and c_0 is a constant such that $c_0^2 = 1$. Moreover, if $m = 1$, then by theorem 1.1 we see $k \geqslant 1$. From the proof of theorem 4.1, we have equations (4.26) and (4.28) with $s = 1$. When $l = 2$, since the left-hand side of equation (4.28) is a polynomial in e^z of degree $1 + k$, all coefficients of this polynomial vanish. Therefore, denoting $a_{-1} = 0$, we obtain from equation (4.28) that $[2c_0(c + k) + c_0 - b_2]a_k = 0$ and

$$[2c_0(c + i - 1) + c_0 - b_2]a_{i-1} + (2ic + i^2)a_i = 0, \quad i = 0, 1, \cdots, k. \tag{4.40}$$

Since $a_k \neq 0$, we have $2c_0(c + k) + c_0 - b_2 = 0$ and then obtain the recursive formulas in (4.37) by substituting $2c_0 c + c_0 - b_2 = -2c_0 k$ into the equations in (4.40). When $m = 1$, we have the recursive formulas in (4.36) with $s = 1$. Denoting $a_{-2} = a_{-1} = 0$ and $a_{k+1} = 0$, we have the recursive formulas in (4.38).

When $s = 3$ and $l = 4$, we have $\kappa_c = e^{cz}(\sum_{i=0}^{k} a_i e^{iz})$ and $h_c = (c_0/2)e^{2z} + c_1 e^z$, where c_0, c_1 are two constants such that $c_0^2 = 1$, $2c_0 c_1 = b_2$. From the proof of theorem 4.1, we get equation (4.29) with $q = m = 1$. Similarly as in previous cases, denoting $a_{-2} = a_{-1} = a_{k+1} = 0$, we finally get the recursive formulas in (4.39). We omit those details. □

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By theorem 4.4, we may give a different formulation from the results in [10, theorem 1.6].

COROLLARY 4.5. *Let $s = 1$ and $l = 2$. Then equation (4.1) admits two linearly independent solutions f_1 and f_2 such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$ if and only if there are two distinct nonnegative integers k_1, k_2 such that $b_2 = \pm(k_1 - k_2)$ and $4b_3 = (k_1 + k_2 + 1)^2$. In particular, it is possible that $\min\{\lambda(f_1), \lambda(f_2)\} = 0$.*

Proof of corollary 4.5. Let f_1 and f_2 be two linearly independent solutions of equation (4.1) such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. Let c_+ or c_- be any square-root of b_3. By theorem 4.4, we may write $f_1 = \kappa_1 e^{h_1}$ and $f_2 = \kappa_2 e^{h_2}$, where $h_1 = c_0 e^z + c_+ z$ and c_0 is a constant such that $c_0^2 = 1$, $h_2 = \pm c_0 e^z + c_- z$, κ_1 and κ_2 are two polynomials in e^z of degrees k_1 and k_2, respectively. From the proof of theorem 4.1 we know that $h_2 = -c_0 e^z + c_- z$. Since $2c_0(c_+ + k_1) + c_0 = -2c_0(c_- + k_2) - c_0 = b_2$, we see that $c_+ = c_-$ for otherwise we have $1 + k_1 + k_2 = 0$, which is impossible. Letting $c_+ = c_- = c$, then we have $2c + k_1 + k_2 + 1 = 0$ and it follows that $b_2 = c_0(k_1 - k_2)$. Since $b_2 \neq 0$ and $b_3 = c^2$, we have $k_1 \neq k_2$ and $4b_3 = (k_1 + k_2 + 1)^2$.

Conversely, we let k_1 and k_2 be two nonnegative integers such that $2c + k_1 + k_2 + 1 = 0$, where c satisfies $c^2 = b_3$. We first define $f_1 = \kappa_1 e^{h_1}$, where $\kappa_1 = \sum_{i=-1}^{k_1} a_i e^{iz}$, $h_1 = c_0 e^z + cz$, $k_1 \geqslant 0$ is an integer, c_0 satisfies $c_0^2 = 1$ and $c_0[2(c + k_1) + 1] = b_2$, and $a_{-1}, a_0, \cdots, a_k$ are constants such that $a_{-1} = 0$ and

$$2c_0(k_1 + 1 - i)a_{i-1} = (2ic + i^2)a_i, \quad i = 0, 1, \cdots, k_1. \tag{4.41}$$

Also, we define $f_2 = \kappa_2 e^{h_2}$, where $\kappa_2 = \sum_{i=-1}^{k_2} \hat{a}_i e^{iz}$ and $h_2 = -c_0 e^z + cz$, $k_2 \geqslant 0$ is an integer, c_0 satisfies $c_0^2 = 1$ and $-c_0[2(c + k_2) + 1] = b_2$, and $\hat{a}_{-1}, \hat{a}_0, \cdots, \hat{a}_k$ are constants such that $\hat{a}_{-1} = 0$ and

$$-2c_0(k_2 + 1 - i)\hat{a}_{i-1} = (2ic + i^2)\hat{a}_i, \quad i = 0, 1, \cdots, k_2. \tag{4.42}$$

Then by theorem 4.4 we see that f_1 and f_2 are two linearly independent solutions of (4.1) such that $\max\{\lambda(f_1), \lambda(f_2)\} < \infty$. Obviously, we may choose one of k_1 and k_2 to be zero and thus $\min\{\lambda(f_1), \lambda(f_2)\} = 0$. We complete the proof. □

5. Concluding remarks

The oscillation of certain second-order linear differential equation (1.1) are investigated in this paper. If equation (1.1) with $A(z)$ being a linear combination of two exponential type functions admits a nontrivial solution such that $\lambda(f) < \infty$, by Hadamard's factorization theorem we obtain a Tumura–Clunie type differential equation with coefficients being combinations of functions in $\mathcal{S}$. In § 2, we give the form of entire solutions of the Tumura–Clunie type differential equations. As an application, in § 3 we give a partial answer to an oscillation question concerning equation (3.1) proposed by Ishizaki [19]. In § 4, we consider equation (1.1) for the case $A(z) = e^{lz} + b_2 e^{sz} + b_3$, where l and s are two relatively prime integers and b_2, b_3 are constants such that $b_2 \neq 0$. The general form of solutions such that $\lambda(f) < \infty$ are known. If there are two linearly independent such solutions, we prove that the only possible case is when $l = 2$.

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By doing straightforward computations, we precisely characterize all solutions such that $\lambda(f) < \infty$ of equation (4.1) for the two cases $l = 2$ and $l = 4$. Unfortunately, we are unable to include or exclude other possibilities. Although, by using theorems 3.1 and 4.1 together with lemma 4.2, when $l \neq 2$, 4, we may also obtain some recursive formulas as in (4.37), (4.38) and (4.39) for the solutions such that $\lambda(f) < \infty$, it is difficult to verify the existence of b_2 and b_3 satisfying these recursive formulas. We conjecture that equation (4.1) can admit a nontrivial solution f such that $\lambda(f) < \infty$ only when $l = 2$ or $l = 4$. We will study this conjecture further.

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