Canad. Math. Bull. Vol. **21** (1), 1978

FIRST COUNTABLE SPACES THAT HAVE SPECIAL PSEUDO-BASES

BY

H. E. WHITE, JR.

1. Introduction. Two types of pseudo-bases, σ-disjoint and σ-discrete, are utilized in this note. In the next section, we show that a first countable Hausdorff space has a σ-disjoint pseudo-base if and only if it has a dense metrizable subspace. This result implies that many first countable spaces have dense metrizable subspaces. In section 3, we show that if X is a Hausdorff space that either is quasi-developable or has a base of countable order, then X has a dense metrizable subspace if and only if it has a dense metrizable G_δ subspace. We give an example to show that the conclusion of this theorem is false for semi-metrizable spaces. Finally, in the last section, we investigate when a quasi-developable (resp. semi-metrizable) space can be embedded as a dense subspace of a quasi-developable (resp. semi-metrizable) Baire space.

The author wishes to thank the referee for: noting that 2.6 implies that every screenable, semi-metrizable Haudsdorff space has a dense metrizable subspace; bringing several papers in this area to the author's attention; making several suggestions which improved the presentation of the results.

2. Dense metrizable subspaces. In the past few years, several topologists have shown that certain first countable spaces have dense metrizable subspaces [1, 4, 5, 10, 11, 13, 14, 15, 18]. We mention only the following results.

2.1 [Younglove, 1959] Every complete Moore space has a dense metrizable G_δ subspace.

2.2 [Fitzpatrick, 1965] Every completable Moore space has a dense metrizable G_δ subspace.

2.3 [Fitzpatrick, 1967] Every normal, collectionwise Haudsdorff Moore space has a dense metrizable G_δ subspace.

2.4 [Proctor, 1968] Every locally connected, locally peripherally separable Moore space has a dense metrizable G_δ subspace.

2.5 [Reed, 1972] Every normal, collectionwise Hausdorff semi-metrizable space has a dense metrizable subspace.

In this section, we give a simple criterion for determining when a Hausdorff first countable space has a dense metrizable subspace. We use this to show,

This work was supported, in part, by the Institute for Medicine and Mathematics, Ohio University.

Received last revised version by the editors February 24, 1977.

https://doi.org/10.4153/CMB-1978-016-5 Published online by Cambridge University Press

among other things, that 2.4 and 2.5 can be generalized somewhat (see 2.11 and 2.13). First, however, we must discuss some preliminaries.

The set of positive integers will be denoted by N; the set of real numbers will be denoted by R. A sequence $(A_n)_{n\in N}$ will usually be denoted by (A_n); the set $\cup\{A_n : n \in N\}$ (resp. $\cap\{A_n : n \in N\}$) will be denoted by $\bigcup_n A_n$(resp. $\bigcap_n A_n$). As usual, $\bigcup A_n = \bigcup \{B : B \in A_n\}$. If $\mathcal{F}$ is a collection of subsets of a set X and $A \subset X$, the collection $\{F \cap A : F \in \mathcal{F}\}$ will be denoted by $\mathcal{F} \cap A$. And $\mathcal{F}^*$ will denote $\mathcal{F} \sim \{\phi\}$.

Suppose $(X, \mathcal{T})$ is a topological space. If $A \subset X$, we denote the closure of A, the interior of A, and the boundary of A by cl A, int A, and b dry A, respectively. A pseudo-base for $\mathcal{T}$ is a subset $\mathcal{P}$ of $\mathcal{T}^*$ such that every element of $\mathcal{T}^*$ contains an element of $\mathcal{P}$. A pseudo-base $\mathcal{P}$ is called σ-disjoint if it is the union of a sequence $(\mathcal{P}_n)$ of disjoint collections. A subset A of X is called discrete (resp. very discrete) if there is a subcollection (resp. disjoint subcollection) $\{U_x : x \in A\}$ of $\mathcal{T}$ such that $U_x \cap A = \{x\}$ for all x in A. A subset K of X is called σ-discrete if K is the union of a sequence (K_n) of discrete sets.

In this note, all hypothesized spaces except those in the remark after corollary 2.7, theorem 2.12, and the remarks after theorem 3.1 are assumed to be Hausdorff and to satisfy the first axiom of countability.

2.6 THEOREM. *The space $(X, \mathcal{T})$ has a dense metrizable subspace if and only if $\mathcal{T}$ has a σ-disjoint pseudo-base.*

Proof. Let $S: X \times N \to \mathcal{T}$ be a function such that for each x in X, $(S(x, n))$ is a non-increasing local base for $\mathcal{T}$ at x.

Suppose that D is a dense metrizable subset of X. Then there is a sequence (K_n) of discrete subsets of D such that $\bigcup_n K_n$ is dense in D. For each n in N there is a function $\varepsilon_n : K_n \to N$ such that $\{S(x, \varepsilon_n(x)) : x \in K_n\}$ is a disjoint collection. Then

$$\{S(x, j) : n \in N, x \in K_n, j \geq \varepsilon_n(x)\}$$

is a σ-disjoint pseudo-base for $\mathcal{T}$.

Now suppose $(\mathcal{P}_n)$ is a sequence of disjoint subcollections of $\mathcal{T}^*$ such that $\mathcal{P} = \bigcup_n \mathcal{P}_n$ is a pseudo-base for $\mathcal{T}$. We may, and do, assume that for each n in N, $\cup \mathcal{P}_n$ is dense in X and $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$.

Define, by induction, a sequence $(\mathcal{U}_n, D_n, \varepsilon_n)$ where for each n in N, D_n is a subset of X, ε_n is a function from D_n into N, and $\mathcal{U}_n = \{S(x, \varepsilon_n(x)) : x \in D_n\}$ are such that, for each n in N: $\mathcal{U}_n$ is a disjoint collection; $\bigcup \mathcal{U}_n$ is dense in X; $\mathcal{U}_{n+1}$ refines $\mathcal{U}_n$; $\varepsilon_n(x) \geq n$ for all x in D_n; $D_n \subset D_{n+1}$; $(*)$ $\{S(x, \varepsilon_{n+1}(x)) : x \in D_{n+1} \sim D_n\}$ refines p_{n+1}.

Let $D = \bigcup D_n$. Then $(D, \mathcal{T} \cap D)$ is metrizable, because $\bigcup [\mathcal{U}_n \cap D]$ is a base for $\mathcal{T} \cap D$ and each $\mathcal{U}_n \cap D$ is discrete in D.

We now show that $\operatorname{cl} D = X$. If $P \in \mathcal{P}_n$, then there is x in D_n such that $P \cap S(x, \varepsilon_n(x)) \neq \phi$. If $x \notin P$ then, because $\bigcap\{\operatorname{cl} S(x, \varepsilon_j(x)) : j \geq n\} = \{x\}$, there is m in N such that $m \geq n$ and

$$W = P \cap [S(x, \varepsilon_m(x)) \sim \operatorname{cl} S(x, \varepsilon_{m+1}(x))] \neq \phi.$$

It follows from (*) that there is y in D_{m+1} such that $S(y, \varepsilon_{m+1}(y)) \subset W$. Hence $y \in P \cap D_{m+1}$; therefore D is dense in X. ■

2.7 COROLLARY. *If X has a dense metrizable subspace, then every dense subspace of X has a dense metrizable subspace.*

REMARK. Corollary 2.7 is false if X is not required to be first countable. If X denotes the Čech-Stone compactification [6, p. 86] of the space Q of rational numbers, then $X \sim Q$ is a dense subspace of X that does not contain a dense metrizable subspace even though it has a countable pseudo-base [16, p. 457].

2.8 COROLLARY. *If X is separable, then it has a dense metrizable subspace.*

The space $(X, \mathcal{T})$ is called collectionwise Hausdorff (resp. σ-collectionwise Hausdorff) if every closed discrete subset of X is very discrete (resp. the union of a countable collection of very discrete subsets). And $(X, \mathcal{T})$ is called perfect if every element of $\mathcal{T}$ is an F_σ set.

2.9 PROPOSITION. (1) *If X is screenable* [13, *p.* 164], *then X is σ-collectionwise Hausdorff.*

(2) *X is hereditarily σ-collectionwise Hausdorff if and only if every discrete subset of X is the union of a countable collection of very discrete subsets.*

(3) *If X is perfect and σ-collectionwise Hausdorff, then X is hereditarily σ-collectionwise Hausdorff.*

2.10 COROLLARY. *If $(X, \mathcal{T})$ is hereditarily σ-collectionwise Hausdorff, then the following three statements are equivalent.* (1) *X has a dense metrizable subspace.* (2) *X has a desnse σ-discrete subset.* (3) *There is a sequence (K_n) of metrizable subspaces such that $\bigcup_n K_n$ is dense in X.*

2.11 COROLLARY. *Suppose $(X, \mathcal{T})$ is hereditarily σ-collectionwise Hausdorff. If any of the following statements is true, then X has a dense metrizable subspace.* (*a*) *X is semi-metrizable* [8, *p.* 315]. (*b*) *X is quasi-developable* [14, *p.* 679]. (*c*) *X has a base of countable order* [14, *p.* 679].

Proof. This follows from 2.10 because if any one of the three statements holds, then X has a dense σ-discrete subset [13, theorem 1.10; 14, theorems 5, 6]. ■

We include a simple proof of theorem 1.10 of [13].

2.12 THEOREM. *If $(X, \mathcal{T})$ is a semi-metrizable T_1 space, then X has a dense, σ-discrete developable subspace.*

https://doi.org/10.4153/CMB-1978-016-5 Published online by Cambridge University Press

NOTATION. In this note, whenever d is a semi-metric on a set X which is compatible with a semi-metrizable topology $\mathcal{T}$ on X, then for x in X and n in N, S_d (x, n) denotes $\text{int}\{y \in X : d(x, y) < n^{-1}\}$.

Proof of 2.12. Suppose d is a semi-metric compatible with $\mathcal{T}$. Define, by induction, a sequence (D_n) of subsets such that D_n is maximal with respect to the following properties: if $n > 1$, then $D_{n-1} \subset D_n$; if $x, y \in D_n$ and $x \neq y$, then $d(x, y) \geq n^{-1}$. Let $D = \bigcup_n D_n$; clearly, D is a dense σ-discrete subset of X. For each n in N, let $\mathcal{D}_n$ be the union of $\{S_d(x, n) : x \in D_n\}$ and $\{S_d(x.m) : m \in N, m > n, x \in D_m \sim D_n\}$. Then $(\mathcal{D}_n \cap D)$ is a development for $(D, \mathcal{T} \cap D)$, because if $n \in N$ and $x \in D_n$, then the only element of $\mathcal{D}_n$ that contains x is $S_d(x, n)$. ■

2.13 COROLLARY. *Suppose X has a dense subset of the first category and $\mathcal{T}$ has a pseudo-base $\mathcal{Q}$ of connected sets. If either of the following statements is true, then X has a dense metrizable subspace. (a) $\mathcal{T}$ has a pseudo-base $\mathcal{P}$ each element of which has separable boundary. (b) X is hereditarily σ-collectionwise Hausdorff and $\mathcal{T}$ has a pseudo-base $\mathcal{P}$ such that the boundary of each element of $\mathcal{P}$ contains a dense metrizable subset.*

REMARK. If X has a dense σ-discrete subset, then $X \sim \text{cl}\{x : \{x\} \in \mathcal{T}\}$ has a dense subset of the first category.

Proof of 2.13 Suppose (b) holds (The proof when (a) holds is similar). There is a sequence $(\mathcal{C}_n)$ of disjoint subcollections of $\mathcal{P}$ such that: for each n in N, $\bigcup \mathcal{C}_n$ is dense in X; if, for each n in N, $C_n \in \mathcal{C}_n$, then $\text{int}[\bigcap_n C_n] = \phi$. For each C in $\mathcal{C} = \bigcup_n \mathcal{C}_n$, let $M(C)$ be a metrizable subspace of b dry C which is dense in dry C. Because $\mathcal{Q}$ is a pseudo-base for $\mathcal{T}$, $M = \bigcup\{M(C) \colon C \in \mathcal{C}\}$ is dense in X. It is now easy to show that X has a dense σ-discrete subset; hence the desired conclusion follows from 2.10. ■

3. Dense metrizable G_δ subspaces. We start with the positive results.

3.1 THEOREM. *Suppose $(X, \mathcal{T})$ is of the first category.*

3.2 *If X has a dense metrizable subspace, then X has a dense metrizable G_δ subspace.*

Proof. We shall use the construction and notation used in the proof of 2.6. If X has a dense metrizable subspace, then $\mathcal{T}$ has a σ-disjoint pseudo-base $\mathcal{P} = \bigcup_n \mathcal{P}_n$. Because X is of the first category, the $\mathcal{P}_n$ can be chosen so as to satisfy the conditions they did in the proof of 2.6 and such that $\bigcap_n[\bigcup \mathcal{P}_n] = \phi$. Then $D = \bigcap_n[\bigcup \mathcal{U}_n]$; thus D is a G_δ set. ■

REMARKS. (1) Statement 3.1 is true if X is an arbitrary, not necessarily first countable, regular Hausdorff space.

(2) A regular Hausdorff, not necessarily first countable, Baire space [9, p. 157] $(X, \mathcal{T})$ has a dense metrizable G_δ subspace if and only if $\mathcal{T}$ has a pseudo-base $\mathcal{P}$ that is "of countable order" in the following sense: if $x \in \bigcap_n P_n$

where for each n in N, $P_{n+1} \subset P_n \in \mathcal{P}$ and $P_{n+1} \neq P_n$, then $\{P_n : n \in N\}$ is a local base for $\mathcal{T}$ at x.

3.3 Corollary. *If X has a dense metrizable G_δ subspace, then every dense subspace of X contains a dense metrizable G_δ subspace.*

3.4 Corollary. *Suppose* **C** *is a class of first countable Hausdorff spaces such that if $(X, \mathcal{T}) \in \mathbf{C}$ and $U \in \mathcal{T}$, then $(U, \mathcal{T} \cap U) \in \mathbf{C}$. Then 3.2 holds for all X in* **C** *if and only if it holds for all elements of* **C** *that are Baire spaces.*

3.5 Corollary. *If either (a) X has a base of countable order or (b) X is quasi-developable, then 3.2 holds.*

Proof. This follows from 3.4, because if X is a Baire space that satisfies either (a) or (b), then X has a dense metrizable G_δ subspace [14, theorem 9; 1, 3.3.2, lemma 1]. ■

3.6 Theorem. *Suppose $(X, \mathcal{T})$ is a regular semi-metrizable space which has a dense subset M that is a metrizable Baire space. Then X has a dense metrizable G_δ subspace.*

Proof. Using a metric on M, define a sequence $(\mathcal{P}_n)$ of disjoint subcollections of $\mathcal{T}^*$ such that: for each n in N, $\bigcup \mathcal{P}_n$ is dense in X and $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$; if (P_n) is such that $P_n \in \mathcal{P}_n$ for all n and $[\bigcap_n P_n] \cap M \neq \phi$, then there is x in M such that $\{P_n : n \in N\}$ is a local base for $\mathcal{T}$ at x. Let $M_0 = \bigcap_n [\bigcup \mathcal{P}_n] \cap M$, $\mathcal{P} = \bigcup_n \mathcal{P}_n$, and suppose d is a semi-metric compatible with $\mathcal{T}$. For each n in N, let

$$\mathcal{Q}_n = \{P \in \mathcal{P} : P \subset \text{cl}\{x \in P \cap M_0 : P \subset S_d(x, n)\}\}.$$

Because
$M_0 = \bigcup_k \{x \in M_0$; There is P in $\mathcal{P}_k$ such that $P \subset S_d(x, n)\}$, $\mathcal{Q}_n$ is a pseudo-base for $\mathcal{T}$.

Now define a sequence $(\mathcal{C}_n)$ such that for each n in N: $\mathcal{C}_n$ is a disjoint subcollection of $\mathcal{Q}_n$; $\cup \mathcal{C}_n$ is dense in X; $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$. Then $D = \bigcap_n [\bigcup \mathcal{C}_n]$ is metrizable, because if $x \in D$, then $\{C \in \bigcup_n \mathcal{C}_n : x \in C\}$ is a local base for $\mathcal{T}$ at x. ■

We now give an example which shows that 3.2 does not hold for all semi-metrizable spaces.

3.7 Example. A separable, completely regular, Hausdorff, semi-metrizable Baire space such that every dense metrizable subspace is of the first category.

Let $\mathcal{E}$ denote the Euclidean topology on R. For each n in N, let

$$A_n = \{(0, 0)\} \cup \{(x, y) \in R \times R : n^2|y| < n|x| < 1\}$$

https://doi.org/10.4153/CMB-1978-016-5 Published online by Cambridge University Press

and let $\mathcal{B}=\{p+A_n : P\in R^2,\ n\in N\}$, where "+" denotes vector addition in $R^2=R\times R$. Then $\mathcal{B}$ is a base for a completely regular Hausdorff topology $\mathcal{U}$ on R^2. It follows from 4.7 that $(R^2, \mathcal{U})$ is semi-metrizable. Because $\mathcal{E}^2$, the Euclidean topology on R^2, is a pseudo-base for $\mathcal{U}$, $(R^2, \mathcal{U})$ is a Baire space. It contains a dense metrizable subspace because it is separable. (More generally, every semi-metrizable Baire space contains a dense metrizable subspace [16, 1.10 (2)]).

Now suppose Y is a dense subset of $(R^2, \mathcal{U})$ such that $(Y, \mathcal{U}\cap Y)$ is a metrizable Baire space. Let d denote a metric for $(Y, \mathcal{U}\cap Y)$. For each n in N, let $\mathcal{V}_n=\{V\in\mathcal{E}^2 : d(V\cap Y)\le n^{-1}\}$; then each $\mathcal{V}_n$ is a pseudo-base for $\mathcal{U}$. Let $D=Y\cap\bigcap_n[\bigcup\mathcal{V}_n]$; then D is a dense subset of Y and $\mathcal{U}\cap D=\mathcal{E}^2\cap D$. This is a contradiction, because whenever $p\in D$, $(p+A_n)\cap D\notin\mathcal{E}^2\cap D$. ■

REMARKS. (1) The space $(R^2, \mathcal{U})$ is essentially the one discussed in [7, example 2.2].

(2) The fact that $(R^2, \mathcal{U})$ does not have a dense metrizable G_δ subspace was discovered independently by three people: D. K. Burke; E. K. van Douwen; the author. In [2], it is shown that every metrizable subspace of $(R^2, \mathcal{U})$ is countable. It is then shown that $(R^2, \mathcal{U})$ is not a σ-space [13, p. 161].

(3) It can be shown (using [9, (2.1)]) that any dense G_δ subspace of $(R^2, \mathcal{U})$ contains an uncountable closed discrete set; therefore such a subspace is neither normal nor collectionwise Hausdorff.

(4) If $(X, \mathcal{T})$ is α-favorable (see § 4 for definition) and either has a base of countable order or is quasi-developable, then X has a dense G_δ subspace that is metrically topologically complete [17, 3(11)]. It is easily checked that $(R^2, \mathcal{U})$ is α-favorable, because $\mathcal{E}^2$ is a pseudo-base for $\mathcal{U}$; hence an α-favorable semi-metrizable space need not have even a dense metrizable G_δ subspace.

4. **Baire space extensions,** A topological space $(X, \mathcal{T})$ is called quasi-regular [9, p. 164] if every element of $\mathcal{T}^*$ contains the closure of an element of $\mathcal{T}^*$. In this section, any hypothesized space is assumed to be quasi-regular, Hausdorff, and first countable. A space $(Y, \mathcal{U})$ is called a Baire space extension of $(X, \mathcal{T})$ if Y is a Baire space and $(X, \mathcal{T})$ is a dense subspace of $(Y, \mathcal{U})$.

In [12, theorem 9], G. M. Reed proved the following result.

4.1 THEOREM. *A developable* [13, *p.* 161] *space* $(X, \mathcal{T})$ *has a developable Baire space extension if and only if* $\mathcal{T}$ *has a* σ*-discrete pseudo-base.*

Having a σ-discrete pseudo-base is a stronger condition than having a σ-disjoint pseudo-base; in [13, p. 162] there is an example of a hereditarily paracompact quasi-developable space that has a dense, open, discrete subspace, but which does not have a σ-discrete pseudo-base. However, if $(X, \mathcal{T})$ is a perfect space which either is normal or is a Baire space, then $\mathcal{T}$ has a σ-disjoint pseudo-base if and only if it has a σ-discrete pseudo-base. Therefore a Moore space [13, p. 161] that has a σ-disjoint pseudo-base but does not have a σ-discrete pseudo-base can be neither normal nor a Baire space. In [11, p. 234], G. M. Reed gave an example of such a space. Here is another such example.

4.2 Let $Q=\{q_n : n \in N\}$ denote the set of all rational numbers in (0, 1), endowed with the usual topology, and let $\{U_n : n \in N\}$ denote a pseudo-base for Q. Let (K_n) be a sequence of open and closed subsets of Q such that, for each n in N: if $1 \le j \le n$, then $U_j \cap K_n \ne \phi$;
(*) $K_n \cap \{q_1, \ldots, q_n\} = \phi$.

Using the maximal principle, we obtain a collection $\mathcal{F}$ such that: if $F \in \mathcal{F}$, then $F \subset R$ and $|F| = \aleph_0$; *if* F, $F' \in \mathcal{F}$ and $F \ne F'$, then $F \cap F'$ is finite;
($**$) if $A \subset R$ and $|A| = \aleph_0$, then there is F in $\mathcal{F}$ such that $|F \cap A| = \aleph_0$.
Then there is a one-one mapping φ of R onto $\mathcal{F}$. For each r in R, let φ_r denote a one-one mapping of N onto $\varphi(r)$.

Let $X = R \times [Q \cup \{0\}]$. For n in N and $p = (x, y)$ in X, let

$$V_n(p) = \{x\} \times [(y - n^{-1}, y + n^{-1}) \cap Q] \quad \text{if} \quad y > 0,$$
$$V_n(p) = \{p\} \cup \cup\{\{\varphi_x(j)\} \times K_j : j \ge n\} \quad \text{if} \quad y = 0.$$

If $\mathcal{B} = \{V_n(p)\colon p \in X, n \in N\}$, then $\mathcal{B}$ is a base for a completely regular, Hausdorff (0-dimensional) topology $\mathcal{T}$ on X such that $(R \times Q, \mathcal{T} \cap (R \times Q))$ is a dense, open, metrizable subspace. If $\mathcal{D}_n = \{V_n(p) : p \in X\}$ for each n in N, then, because ($*$) holds, $(\mathcal{D}_n)$ is a development for X.

We shall show that $\mathcal{T}$ does not have a σ-discrete pseudo-base by verifying that (1) every pseudo-base for $\mathcal{T}$ is uncountable and (2) every discrete subcollection of $\mathcal{T}$ is countable. Statement (1) is true because $\{\{r\} \times Q : r \in R\}$ is a disjoint subcollection of $\mathcal{T}$.

Proof of (2). Because $\{\{r\} \times U_n : r \in R, n \in N\}$ is a pseudo-base for $\mathcal{T}$, it suffices to show that if $n \in N$ and $\mathcal{C} = \{\{r\} \times U_n : r \in A\}$ is discrete, then A is finite. To prove this latter statement, suppose A is infinite. By ($**$), there is F in $\mathcal{F}$ such that $|F \cap A| = \aleph_0$.
Then $\varphi^{-1}(F) \in \mathrm{cl}[\cup\mathcal{C}] \sim \cup\{\mathrm{cl}\, C : C \in \mathcal{C}\}$; hence $\mathcal{C}$ is not discrete. ■

Remark. A simple modification of the construction used in 4.2 can be used to prove the following statement. If X is a Moore space of the first category which has a σ-discrete pseudo-base and a is a cardinal number greater than the cellular number of X, then there is a Moore space Y which has a dense open subset homeomorphic with the sum of m copies of X, but which does not have a σ-discrete pseudo-base. (If X is completely regular, then so is Y).

We shall now describe a simple method of constructing Baire space extensions; then we shall use this method to prove the following statements.

4.3 Theorem. *A quasi-developable space* $(X, \mathcal{T})$ *has a quasi-developable*

https://doi.org/10.4153/CMB-1978-016-5 Published online by Cambridge University Press

Baire space extension if and only if $\mathcal{T}$ has a σ-disjoint pseudo-base.

4.4 THEOREM. *A semi-metrizable space $(X, \mathcal{T})$ has a semi-metrizable Baire space extension if and only if $\mathcal{T}$ has a σ-discrete pseudo-base.*

We recall that a space $(Y, \mathcal{U})$ is called α-favorable [3, p. 116] if there is a function $\varphi:\mathcal{U}^*\to\mathcal{U}^*$ such that: $\varphi(U)\subset U$ for all U in $\mathcal{U}^*$; if λ is a sequence such that for all n in N, $\lambda(n)\in\mathcal{U}^*$ and $\lambda(n+1)\subset\varphi(\lambda(n))$, then $\bigcap_n \lambda(n)\neq\phi$. An α-favourable space is necessarily a Baire space.

Suppose $(X, \mathcal{T})$ is a topological space and $\mathcal{P}$ is a subset of $\mathcal{T}^*$ such that: $\mathcal{P}$ is a pseudo-base for $\bigcup\mathcal{P}$;
(***)$\mathcal{P}=\bigcup_n\mathcal{P}_n$ where for each n in N, $\{\text{cl}P: P\in\mathcal{P}_n\}$ is a disjoint collection which, if $n>1$, refines $\mathcal{P}_{n-1}$.

Let L denote the set of all sequences θ such that: for all n in N, $\theta(n+1)\subset\theta(n)\in\mathcal{P}_n$; $\bigcap_n\theta(n)=\phi$. For U in $\mathcal{T}$, let

$$U^+=U\cup\{\theta\in L\colon \text{There is } n \text{ such that } \theta(n)\subset U\};$$

for $\mathcal{S}$ contained $\mathcal{T}$, let $\mathcal{S}^+=\{U^+: U\in\mathcal{S}\}$. Because $(U\cap V)^+=U^+\cap V^+$ for all U, V in $\mathcal{T}$, $\mathcal{T}^+$ is a base for a topology $\mathcal{U}$ on $Y=X\cup L$. We shall denote $(Y, \mathcal{U})$ by $[(X, \mathcal{T}); \mathcal{P}]$.

4.5 PROPOSITION (1) *$(X, \mathcal{T})$ is a dense subspace of the quasi-regular, Hausdorff first countable space $(Y, \mathcal{U})$.*

(2) *$\mathcal{P}^+$ is a σ-disjoint pseudo-base for W^+, where $W=\cup\mathcal{P}$.*

(3) *If $\mathcal{P}$ is σ-discrete in X, then $\mathcal{P}^+$ is σ-discrete in Y.*

(4) *The space $(W^+, \mathcal{U}\cap W^+)$ is α-favorable.*

4.6 COROLLARY. *If $(X, \mathcal{T})$ has a σ-disjoint pseudo-base (resp. base), then it has a Baire space extension which has a σ-disjoint pseudo-base (resp. base).*

Proof of 4.5. The proofs of (1), (2), and (3) are easy, and are omitted. To show that W^+ is α-favorable, because of (2), it suffices to show that: if λ is a sequence such that for each n in N, $\lambda(n+1)\subset\lambda(n)\in\mathcal{P}_n^+$, then $\bigcap_n\lambda(n)\neq\phi$. This last statement is obviously true. ∎

Proof of 4.3. The necessity of the condition follows from 2.6 and [1, 3.3.2, lemma 1].

Suppose, conversely, that $\mathcal{T}$ has a pseudo-base $\mathcal{P}$ that satisfies (***). Then $[(X, \mathcal{T}); \mathcal{P}]$ is quasi-developable, because if $(\mathcal{D}_n)$ is a quasi-development for X and, for each n in N, $\mathcal{C}_{2n}=\mathcal{D}_n^+$ and $\mathcal{C}_{2n-1}=\mathcal{P}_n^+$, then $(\mathcal{C}_n)$ is a quasi-development for Y. ∎

The following result will be useful in proving 4.4.

4.7 PROPOSITION. [7, *theorem* 3.2] *A space $(X, \mathcal{T})$ is semi-metrizable if* *and only if there is a function* $S: X\times N\to \mathcal{T}$ *such that* (*a*) *for each* x *in* X, $(S(x, n))$ *is a non-increasing local base for* $\mathcal{T}$ *at* x; (*b*) *if* (x_n) *is a sequence in* X *and* $x\in\bigcap_n S(x_n, n)$, *then* (x_n) *converges to* x.

Proof of 4.4. The necessity of the condition follows from 2.6 and [16, 1.10(2)].

Now suppose $\mathcal{T}$ has a σ-discrete pseudo-base. By the Banch category theorem, there are X_1, X_2 in $\mathcal{T}$ such that X_1 is of the first category, X_2 is a Baire space, and cl $(X_1\cup X_2)=X$. We assume $X_1\neq\phi$. Then there is a pseudo-base $\mathcal{P}$ for X_1 such that: (***) holds; $\mathcal{P}$ is σ-discrete in X; $\bigcap_n[\bigcup\mathcal{P}_n]=\phi$. Using the argument in the proofs of [13, lemmata 1.3, 1.4], we define $S: X\times N\to\mathcal{T}$ so that S satisfies (a) and (b) of 4.7 and the following condition holds. For each n in N, let $\mathcal{C}_n=\{S(x, n): x\in X\}$. If $n\in N$, $P\in\mathcal{P}_{n+1}$, $Q\in\mathcal{P}_n$, $P\subset Q$, then there is j in N such that

$$\mathrm{St}(P, \mathcal{C}_j)=\cup\{U\in\mathcal{C}_j : U\cap P\neq\phi\}\subset Q.$$

Let $(Y, \mathcal{U})=[(X, \mathcal{T}); \mathcal{P}]$. Then X_1^+ is α-favorable and $X_2^+=X_2$; hence Y is a Baire space. Define $S^+: Y\times N\to\mathcal{U}$ by letting

$$S^+(y, n)=[S(y, n)]^+ \quad \text{if} \quad y\in X,$$
$$S^+(y, n)=[y(n)]^+ \quad \text{if} \quad y\in L.$$

We shall show that S^+ satisfies 4.7(b). Suppose (y_n) is a sequence in Y such that $y\in\cap S^+(y_n, n)$. It suffices to consider the following two cases.

CASE 1. Suppose $\{y_n : n\in N\}\subset X$ and $y\in L$. If $n\in N$, choose j so $\mathrm{St}(y(n+1), \mathcal{C}_j)\subset y(n)$. Then $S(y_m, m)\subset y(n)$ whenever $m\geq j$; hence $y=\lim y_n$.

CASE 2. Suppose $\{y_n : n\in N\}\subset L$. Then $y\in L$, because

$$X\cap[\bigcap_n S^+(y_n, n)]=\bigcap_n y_n(n)\subset\bigcap_n[\cup\mathcal{P}_n]=\phi.$$

So $y(n)=y_n(n)$ for all n; hence (y_n) converges to y. ∎

REMARK. If d is a semi-metric compatible with $\mathcal{T}$, we say a function $f: X\to R$ is d-uniformly continuous if for every positive number ε, there is a positive number δ such that $|f(x)-f(y)|\leq\varepsilon$ whenever $d(x, y)\leq\delta$. In the proof of 4.4, if we choose $\mathcal{P}$ so that, for each n in N and P in $\mathcal{P}_n$, there is an x in X such that $P\subset S_d(x, n)$, then every d-uniformly continuous real valued function defined on X can be extended to a continuous real valued function defined on Y.

Finally, we note that a slight variant of the proof of 4.4 can be used to prove the following statement.

4.8 PROPOSITION If $(X, \mathcal{T})$ has a base of countable order and a σ-discrete pseudo-base, then there is a Baire space extension of X which has the same two properties.

REFERENCES

1. J. M. Aarts and D. J. Lutzer, *Completeness properties designed for recognizing Baire spaces*, Dissertationes Math (Rozprawy Mat), **116** (1974), 1–48.
2. D. K. Burke and E. K. van Douwen, *No nice G_δ-subspaces in semi-metrizable Baire spaces* (Preprint).
3. G. Choquet, Lectures on analysis, Vol. I: *Integration and topological vector spaces*, Benjamin, New York, 1969.
4. B. Fitzpatrick, *On dense subspaces of Moore spaces*, Proc. Amer. Math. Soc., **16** (1965), 1324–1328.
5. ——, *On dense subspaces of Moore spaces II*, Fund. Math. **61** (1967), 91–92.
6. L. Gillman and M. Jerison, *Rings of continuous functions*, Van Nostrand, Princeton, N. J., 1960.
7. R. W. Heath, *Arcwise connectedness in semi-metric spaces*, Pac. J. Math, **12** (1963), 1301–1319.
8. L. F. McAuley, *A relationship between perfect separability, completeness, and normality in semi-metric spaces*, Pac. J. Math, **6** (1956), 315–326.
9. J. C. Oxtoby, *Cartesian products of Baire spaces*, Fund Math, **49** (1960/1961), 157–166.
10. C. W. Procter, *Metrizable subsets of Moore spaces*, Fund Math, **66** (1969), 85–93.
11. G. M. Reed, *Concerning normality, metrizability and the Souslin property in subspaces of Moore spaces*, Gen. Top, **1** (1971), 223–246.
12. ——, *On completeness conditions and the Baire property in Moore spaces*, Topo-72-General topology and its applications, Springer-Verlag, 1974, 368–384.
13. ——, *Concerning first countable spaces*, Fund Math, **74** (1972), 161–169.
14. ——, *Concerning first countable spaces II*, Duke Math J, **40** (1973), 677–682.
15. ——, *Concerning first countable spaces III*, Trans. Amer. Math Soc, **210** (1975), 169–177.
16. H. E. White, Jr. *Topological spaces in which Blumberg's theorem holds*, Proc. Math. Soc., **44** (1974), 454–462.
17. ——, *Topological spaces that are α-favorable for a player with perfect information*, Proc. Amer. Math. Soc., **50** (1975), 477–482.
18. J. N. Younglove, *Concerning metric subspaces of non-metric spaces*, Fund. Math., **48** (1959), 15–25.

671 EUREKA AVENUE
COLUMBUS, OHIO 43204