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RINGS HAVING ZERO-DIVISOR GRAPHS OF SMALL DIAMETER OR LARGE GIRTH

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Let R be a commutative ring possessing (non-zero) zero-divisors. There is a natural graph associated to the set of zero-divisors of R. In this article we present a characterisation of two types of R. Those for which the associated zero-divisor graph has diameter different from 3 and those R for which the associated zero-divisor graph has girth other than 3. Thus, in a sense, for a generic non-domain R the associated zero-divisor graph has diameter 3 as well as girth 3.

Let R be a commutative ring with $1 \neq 0$ and let $Z(R)$ denote the set of non-zero zero-divisors of R. By the *zero-divisor-graph* of R we mean the graph with vertices $Z(R)$ such that there is an (undirected) edge between vertices x, y if and only if $x \neq y$ and $xy = 0$ (see [**1**, **3**, **4**]). Since there is hardly any possibility of confusion, we allow $Z(R)$ to denote the zero-divisor graph of R. Following their introduction in [**3**], zero-divisor graphs have received a good deal of attention. For a more comprehensive list of references the reader is requested to refer to the bibliographies of [**1**, **2**, **4**]. Zero-divisor graphs are highly symmetric and structurally very special; for example, in [**4**] this author has investigated the structure of cycles, the graph-automorphism-group $\Gamma(R)$ and its explicit relationship with the ring-automorphism-group $\mathrm{Aut}(R)$. A sample consequence of interest is: if $\Gamma(R)$ is solvable, so is $\mathrm{Aut}(R)$. For this reason alone it is of interest to understand the nature of zero-divisor graphs. From the available evidence one is tempted to surmise that *generic* zero-divisor graphs may be completely classifiable (in some sense). It has been the experience that whenever one assumes $Z(R)$ to have some special feature, one can narrow down R to a small class of rings. The present article provides an instance of this facet.

We tacitly assume that R has at least 2 non-zero zero-divisors. By declaring the length of each edge to be 1, $Z(R)$ becomes a metric space in which the distance between two vertices is, by definition, the length of a shortest path connecting them. The *diameter* of a metric space is the supremum (possibly ∞) of the distances between pairs of points of the space. With this structure, a zero-divisor graph is known to be a connected graph of diameter at most 3 (for example see [**1**] or [**4**, (1.2)]). The *girth* of a graph is the

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length of a shortest cycle (or equivalently the number of vertices of a least sided polygon) contained in the graph. If $Z(R)$ does not contain a cycle, then its girth is defined to be ∞. Obviously the girth of a graph is at least 3. The girth of $Z(R)$, for an arbitrary R, is known to be either infinite or 3 or 4 (see [4, (1.4)]).

In (1.2) of this article we present a characterisation of the rings R for which the associated zero-divisor graph has diameter at most 2. In (2.3) we identify those R whose associated zero-divisor graph has girth exactly 4. In (2.5) and (2.6) we determine the rings R for which the associated zero-divisor graph has infinite girth. The graph-theoretic counterpart of this has already been dealt with in [4] where the zero-divisor graphs of infinite girth are completely determined. Except for one class of zero-divisor graphs of diameter 2, the nature of zero-divisor graphs having either small diameter or large girth is readily understood from the corresponding ring-theoretic characterisations.

By the *total quotient ring* of R we mean the quotient ring $Q(R) := T^{-1}R$ where T stands for the multiplicative subset of non zero-divisors of R. Since the canonical homomorphism from R to $Q(R)$ is injective, R is thought of as a sub-ring of $Q(R)$. As mentioned above, we shall tacitly assume $Z(R)$ to have at least two elements.

THEOREM 1.1. *The diameter of $Z(Q(R))$ is the same as the diameter of $Z(R)$. The girth of $Z(R)$ is the same as the girth of $Z(Q(R))$.*

PROOF: Observe that the diameter of $Z(Q(R))$ is 1 if and only if the diameter of $Z(R)$ is also 1. Now suppose that the diameter of $Z(Q(R))$ is 2. Then the diameter of $Z(R)$ is at least 2. Consider any $a, b \in Z(R)$ with $a \neq b$ and $ab \neq 0$. By our assumption about the diameter of $Z(Q(R)$, there is a $q \in Z(Q(R))$ such that $a \neq q \neq b$ and $aq = 0 = bq$. Let $q := c/t$ with $c, t \in R$ such that t is a non zero-divisor of R. Then $ac = 0 = bc$. It follows that c is in $Z(R)$ and hence the distance between a, b (when considered as vertices of the zero-divisor graph of R) is 2. Conversely, assuming that the diameter of $Z(R)$ is 2 it is easy to see that the diameter of $Z(Q(R))$ must also be 2. In general, the diameter of any zero-divisor graph is at most 3. Therefore we have established the first assertion.

Since $Z(R)$ is a sub-graph of $Z(Q(R))$, it is clear that the girth of $Z(R)$ is greater than or equal to the girth of $Z(Q(R))$. Earlier we have noted that the girth of any zero-divisor graph, when finite, is either 3 or 4. Suppose $Z(Q(R))$ has girth 3. Then there are distinct elements q_1, q_2, q_3 of $Z(Q(R))$ such that $q_1q_2 = q_2q_3 = q_3q_1 = 0$. For $i = 1, 2, 3$ let $q_i := a_i/t$ with $a_i, t \in R$ and where t is a non zero-divisor of R. Then a_1, a_2, a_3 are distinct elements of $Z(R)$ and since $a_1a_2 = a_2a_3 = a_3a_1 = 0$, they form a triangle in the graph $Z(R)$. Thus $Z(R)$ also has girth 3. ∎

THEOREM 1.2. *Assume that the diameter of $Z(R)$ is $\leqslant 2$. Then exactly one of the following holds.*

(i) *$Z(R) \cup \{0\}$ is a prime ideal of R,*

(ii) *R is a sub-ring of a product of two integral domains.*

If (ii) *holds for (a non-domain) R, then the diameter of $Z(R)$ is at most* 2. *If* (i) *holds for (a non-domain) R and R is Noetherian, then $Z(R)$ is of diameter at most* 2.

PROOF: Suppose $Q(R)$ has two distinct maximal ideals M_1 and M_2. Let $x \in M_1$ and $y \in M_2$ be such that $x + y = 1$. Then x, y are in $Z(Q(R))$ and

$$(0 : x) \cap (0 : y) = 0$$

(considered as ideals of $Q(R)$.) Since, by (1.1), the diameter of $Z(Q(R))$ is also 2, we must have $xy = 0$. But $y = 1 - x$ and hence x is an idempotent of $Q(R)$. It follows that $Q(R)$ is isomorphic to a product of two rings. Say $Q(R) = R_1 \times R_2$. Suppose $Z(R_1)$ is non-empty. Let a be an element of $Z(R_1)$. Then $(a,1), (1,0)$ are elements of $Z(Q(R))$ such that the distance between them is at least 3. This is impossible due to the assumption that $Z(Q(R)$ has diameter 2. Hence $Z(R_1)$ has to be empty. Symmetrically, $Z(R_2)$ must also be empty. Consequently R_1, R_2 are domains (in fact, fields) that is, assertion (ii) holds. It is clear that (i) holds if and only if $Q(R)$ has a unique maximal ideal.

If (ii) holds, then the diameter of $Z(R)$ is easily seen to be either 1 or 2. If (i) holds and R is Noetherian, then $P := Z(R) \cup \{0\}$, being an associated prime of 0, is of the form $(0 : x)$ for some $x \in Z(R)$ and hence the diameter of $Z(R)$ is at most 2. ∎

REMARKS. 1. The diameter of $Z(R)$ is 0 if and only if $Z(R)$ is a singleton set if and only if R is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{F}_2[X]/X^2\mathbb{F}_2[X]$ (or example, see [**4**, (1.1)]).

2. The diameter of $Z(R)$ is 1 (that is, $Z(R)$ is a complete graph) if and only if either R is the product of the field of 2 elements with itself or (i) holds with the added property that $P^2 = 0$ (see [**1**]).

3. Let A be a quasi-local factorial domain of (Krull) dimension at least 2. Let $m(A)$ denote the maximal ideal of A and let $p(A)$ be a set of primes of A such that for each height-one prime ideal P of A there is a unique $p \in p(A)$ with $P = pA$. Let $A[X]$ be the polynomial ring over A in the set of indeterminates $X := \{X_p \mid p \in p(A)\}$. Then $p(A)$ (and hence X) is necessarily infinite. Let J be the ideal of $A[X]$ generated by $\{pX_p \mid p \in p(A)\}$ and $I := J + (XA[X])^2$. Define $R := A[X]/I$ and $M := (m(A) + XA[X])/I$. Then M is a maximal ideal of R whose elements constitute the zero-divisors of R. Thus (i) holds for R. It is straightforward to verify that A is (naturally) a sub-ring of R and given two distinct members p, q of $p(A)$ and an $r \in R$ with $pr = 0 = qr$ we must have $r = 0$. Consequently, $Z(R)$ has diameter 3.

LEMMA 2.1. *Assume that $Z(R)$ has girth 4. Then R has at most one non-zero nilpotent. Furthermore, if a is the non-zero nilpotent of R, then $(0 : a)$ is a maximal ideal having $\mathbb{F}_2$ as its residue field.*

PROOF: Assume R has non-zero nilpotents. Let a be a non-zero nilpotent of R such that $a^2 = 0$. If there are 4 distinct elements in the ring $R/(0 : a)$, then there are

3 distinct elements of $Z(R)$ of the form xa, ya, za (with $x, y, z \in R$) which constitute a 3-cycle in $Z(R)$. Thus, in view of our hypothesis, it follows that $R/(0:a)$ is a ring of cardinality at most 3. It is straightforward to verify that the zero-divisor graph of a ring of cardinality < 9 has girth 3. Hence $(0:a)$ has cardinality at least 4. If $aR \neq \{0, a\}$, then for any $b \in aR \setminus \{0, a\}$ and any $x \in (0:a) \setminus \{0, a, b\}$ the elements a, x, b of $Z(R)$ form a 3-cycle, contrary to our hypothesis. Therefore $aR = \{0, a\}$. It now follows that $(0:a)$ is a maximal ideal having the field of 2 elements as its residue field. If there is some non-zero $x \in (0:a)$ for which $(0:x) \cap (0:a)$ is not a subset of $\{0, a, x\}$, then for any $y \in (0:x) \cap (0:a) \setminus \{0, a, x\}$, elements a, y, x form a 3-cycle. Hence $(0:x) \cap (0:a)$ is a subset of $\{0, a, x\}$ for all non-zero x in $(0:a)$.

Let y be a nilpotent of R such that $y^{n+1} = 0$ but $y^n \neq 0$ for a positive integer n. Clearly $n \leqslant 3$, otherwise, y^n, y^{n-1}, y^{n-2} would be distinct elements of $Z(R)$ forming a 3-cycle. In other words $y^4 = 0$ for every nilpotent y of R. Suppose there is a nilpotent y in R with $y^3 \neq 0$. If $(0:y^2) \neq \{0, y^2, y^3\}$, then for any $z \in (0:y^2) \setminus \{0, y^2, y^3\}$ elements y^2, z, y^3 constitute a 3-cycle. On the other hand, if $(0:y^2) = \{0, y^2, y^3\}$, then since $R/(0:y^2)$ has cardinality 2 by the above argument, R would be a ring of cardinality at most 6 and hence $Z(R)$ can not possibly have girth 4. Summarising, we must have $y^3 = 0$ for every nilpotent y of R. Next suppose there is nilpotent y of R with $y^2 \neq 0$. If $(0:y) \neq \{0, y^2\}$, then y, x, y^2 forms a 3-cycle of $Z(R)$ for any $x \in (0:y) \setminus \{0, y^2\}$. So $(0:y) = \{0, y^2\}$. Let x be in $(0:y^2)$ but not in $(0:y)$. Then xy is a non-zero element of $(0:y)$ and hence $xy = y^2$. Now $x - y$ being in $(0:y)$ it is either 0 or y^2. Thus $(0:y^2)$ is contained in the set $\{0, y, y^2, y + y^2\}$. Since R must have cardinality at least 9 (for $Z(R)$ to have girth 4) and $R/(0:y^2)$ has cardinality 2, this is impossible. Therefore, we conclude that $y^2 = 0$ for each nilpotent y of R.

Let $N(R)$ denote the nil-radical of R. Let a, b be non-zero members of $N(R)$. If $ab \neq 0$, then elements $a, a + ab, ab$ form a triangle in $Z(R)$. This being impossible, $N(R)^2 = 0$. If $N(R) \neq aR$, then for any $c \in N(R) \setminus aR$ elements $a, a + c, b$ form a triangle of $Z(R)$. Hence we must have $N(R) = aR$. But we have already shown that $aR = \{0, a\}$. This establishes our assertion. $\square$

Remark. Observe that if $R := D \times \mathbb{Z}/4\mathbb{Z}$ where D is an integral domain different from $\mathbb{F}_2$, then R has a non-zero nilpotent and $Z(R)$ does have girth 4.

Lemma 2.2. *Assume that the nil-radical of R is zero. Then $Z(R)$ is complete bi-partite if and only if R is a sub-ring of a product of 2 integral domains.*

Proof: The 'if' part is straightforward. Suppose $Z(R)$ is complete bi-partite. Then $Z(R)$ has a partition $\{Z_1, Z_2\}$ where $Z_1 = (0:x) \setminus \{0\}$ for all $x \in Z_2$ and $Z_2 = (0:x) \setminus \{0\}$ for all $x \in Z_1$. Let $P_1 := Z_1 \sqcup \{0\}$. Pick y in Z_2. Now $P_1 = (0:y)$. Assume $a, b \in R$ with $ab \in P_1$. Then $aby = 0$. If $by = 0$, then b is in P_1. Assume $by \neq 0$. Now by is in $Z(R)$ and $(0:y) \subseteq (0:by)$. If by is in Z_1, then $0 = (0:by) \cap (0:y)$ which is absurd since $(0:by) \cap (0:y) = (0:y)$. This forces by to be in Z_2. But then $(0:by) = (0:y)$

and $a \in (0 : by) = (0 : y) = P_1$. In other words, P_1 is a prime ideal of R. Likewise $P_2 := Z_2 \sqcup \{0\}$ is also a prime ideal of R. Since $Z_1 \cap Z_2$ is empty, $P_1 \cap P_2 = 0$. It follows that R is canonically isomorphic to a sub-ring of the product of integral domains R/P_1 and R/P_2. ∎

THEOREM 2.3. *Assume that $Z(R)$ has finite girth. Then the girth of $Z(R)$ is 4 if and only if one of the following holds.*

(i) *R is a sub-ring of a product of two integral domains.*

(ii) *R is isomorphic to $D \times S$ where D is an integral domain and S is either $\mathbb{Z}/4\mathbb{Z}$ or $\mathbb{F}_2[X]/X^2\mathbb{F}_2[X]$.*

PROOF: The proof is divided in two cases, namely the case where the nil-radical $N(R)$ is zero and the case where $N(R)$ is non-zero. Since R must have at least 9 elements for $Z(R)$ to have girth 4, henceforth we tacitly assume that the cardinality of R is at least 9.

First assume that $N(R) \neq 0$. Then, from (2.1) it follows that $N(R) = \{0, a\}$ and the cardinality of $(0 : a)$ is at least 5. Let x be a non-nilpotent in $(0 : a)$. If $(0 : x) \cap (0 : a)$ contains a non-nilpotent y, then x, a, y form a triangle in $Z(R)$. Hence $(0 : x) \cap (0 : a) = N(R)$ for every non-nilpotent x in $(0 : a)$. If every zero-divisor of R is in $(0 : a)$, then $(0 : x) = N(R)$ for all non-nilpotents of $Z(R)$ and hence $Z(R)$ has infinite girth contrary to our assumption. Thus $Z(R) \setminus (0 : a)$ must be non-empty. Pick $y \in Z(R)$ such that y is not in $(0 : a)$. Clearly, $(0 : y) \neq 0$. If $(0 : y) \cap (0 : a)$ is a subset of $N(R)$, then since $ay \neq 0$, we have $(0 : y) \cap (0 : a) = 0$ and hence R is isomorphic to $\mathbb{F}_2 \times R/(0 : y)$ (where the first factor is the field of 2 elements). But the zero-divisor graph of such a product has girth either ∞ or 3. Thus it must be possible to choose a non-nilpotent x in $(0 : y) \cap (0 : a)$. Consider the set $C := \{zy \mid z \in (0 : a)\}$. If C has a member b not in $\{0, x, a\}$, then x, b, a form a triangle of $Z(R)$ contrary to our hypothesis. Therefore, C is contained in $\{0, x, a\}$. Using the fact that $R/(0 : a)$ is the field of two elements, we conclude that $y - 1$ is in $(0 : a)$ and yR is a subset of $\{0, x, a, y, y + x, y + a\}$. Now $(y - 1)a = 0$ implies that a is in yR. So $0, y, a, y + a$ are 4 distinct elements of yR. Clearly, from our choice of x, y it follows that $y + x + a$ can not be in the set $\{0, x, a, y, y + x, y + a\}$. Hence $yR = \{0, y, a, y + a\}$. But then y^2 must belong to $\{y, y + a = y(1 + a)\}$ (observe that $1 + a$ is a unit of R). Consequently, $y^2R = yR$ has exactly 4 elements, $(0 : y) \cap yR = 0$ and $(1 - y)$ is in $(0 : y)$. Thus R is isomorphic to $R/yR \times R/(0 : y)$ where $R/(0 : y)$ is a ring of cardinality 4 containing a non-zero nilpotent (namely, the image of a). In other words (ii) holds.

Finally consider the case where $N(R) = 0$. Let x be in $Z(R)$. Suppose, if possible, that $(0 : x)$ has exactly 2 elements $\{0, y\}$. Then $yR = \{0, y\}$ and hence $(0 : y)$ is a maximal ideal having the residue field of 2 elements. Also, since $y^2 \neq 0$, we must have $y^2 = y$ and $yR \cap (0 : y) = 0$. Hence R is isomorphic to $\mathbb{F}_2 \times R/yR$. But such a ring has girth either ∞ or 3. Thus for each x in $Z(R)$, the cardinality of $(0 : x)$ is at least 3 (in the terminology of [**4**], the graph $Z(R)$ has no 'ends'). Now it follows from [**4**, (2.2)] (see Remark 2. following the assertion [**4**, (2.2)]) that $Z(R)$ is a bi-partite graph. In view of the Lemma (2.2) of this article we see that (i) holds.

Conversely, if either of (i) or (ii) holds (with $Z(R)$ being non-empty of finite girth) then it is easy to verify that the girth of $Z(R)$ is exactly 4. ∎

Lemma 2.4. *Assume that R has at least 10 elements. Let $N(R)$ be the nil-radical of R and assume that $N(R) \neq 0$. Then the following are equivalent.*

(i) *$Z(R)$ has infinite girth.*

(ii) *$N(R) = \{0, y\}$ and $(0 : x) = N(R)$ for all $x \in Z(R)$.*

(iii) *$N(R)$ has cardinality 2 and it is a prime ideal of R.*

Proof: The equivalence of (i) and (ii) follows from [**4**, (2.1)]. Assertion (iii) follows from (ii) in a straightforward manner. Suppose (iii) holds. Then $N(R) = \{0, y\}$ for some non-zero y in R. Let $x \in Z(R)$ be distinct from y. Then x is not in $N(R)$ and $(0 : x) \neq 0$. Consider $0 \neq w \in (0 : x)$. Since $0 = xw \in N(R)$ and $N(R)$ is prime, we must have $w \in N(R)$ and hence $w = y$. Hence (ii) holds. ∎

Definition: Let B be a ring such that its nil-radical $N(B)$ is a prime ideal of cardinality 2 and let $B[X]$ be the polynomial ring in a non-empty set of indeterminates X over B. Let I be an ideal of $B[X]$ such that

1. $I \cap B = 0$,
2. $N(B)B[X] \cdot XB[X] \subseteq I \subseteq 2B[X] + N(B)B[X] + XB[X]$ and
3. $P(B, X, I) := N(B)B[X] + I$ is a prime ideal of $B[X]$.

Then, by $\rho(B, X, I)$ we mean the ring $B[X]/I$.

Theorem 2.5. *Assume that R has at least 10 elements. Let $N(R)$ be the nil-radical of R and assume that $N(R) \neq 0$. Then $Z(R)$ has infinite girth if and only if one of the following holds.*

(i) *$R = \rho(B, X, I)$ where $B = \mathbb{Z}[w]/(w^2, 2w)\mathbb{Z}[w]$ for an indeterminate w over $\mathbb{Z}$.*

(ii) *$R = \rho(B, X, I)$ where $B = \mathbb{F}_2[w]/w^2\mathbb{F}_2[w]$ for an indeterminate w over $\mathbb{F}_2$ and where I is such that $P(B, X, I) \neq N(B)B[X] + XB[X]$.*

(iii) *$R = \rho(B, X, I)$ where $B = \mathbb{Z}/4\mathbb{Z}$ and I is such that $P(B, X, I) \neq N(B)B[X] + XB[X]$.*

Moreover, such a ring is necessarily infinite.

Proof: Our argument will tacitly employ Lemma (2.4). At the outset we show that under our assumptions the characteristic of R is either 0 or 2 or 4. Observe that a sub-ring of R does not contain y if and only if it is an integral domain. On the other hand, if a sub-ring $S \subsetneq R$ contains y, then $N(S) = N(R) \cap S$ is a prime ideal and for any $a, b \in Z(S)$ we have $ab = 0$ if and only if either $a = y$ or $b = y$. Also, it follows that

$(0 : y) \cap S$ is a maximal ideal of S with residue field $\mathbb{F}_2$. In particular, the characteristic of R is an even integer. Let $2n$ denote the characteristic of R. Suppose n is neither 0 nor 1. Then y is in the prime sub-ring π of R. Since the zero-divisors of π have to be contained in the maximal ideal $(0 : y) \cap \pi$, the ring π is a local ring that is, n is a power of 2. But the nil-radical of π has exactly two elements. Hence $n = 2$. It is plain to see that the rings of the form mentioned in (i), (ii), (iii) above have characteristics 0, 2 and 4 respectively.

Suppose R satisfies any one of (i), (ii) and (iii). To simplify the notation set $P := P(B, X, I)$. In the first two cases let t be the canonical image of w in B and in the third case let $t = 2$. Note that $t^2 = 2t = 0$ and $N(B) = \{0, t\}$. We claim that R has to be infinite. This is evident in the case of (i) since $\mathbb{Z}$ is indeed the prime sub-ring of R. In the remaining two cases there is an x in X which is not in $P = tB[X] + I$ Consider the sub-ring A of $B[X]$ obtained by adjoining x to the prime sub-ring. Then A is a polynomial ring in one variable over the prime sub-ring and $I \cap A \subseteq P \cap A$. If A has characteristic 2 then $P \cap A \subseteq xA$ and consequently $P \cap A = 0$. Thus A is (naturally) an infinite sub-ring of R. If A has characteristic 4, then $P \cap A = 2A$ and hence $A/(I \cap A)$ is necessarily an infinite sub-ring of R. Let y denote the canonical image of t in R. Then y is a non-zero nilpotent of R. Let $f \in B[X]$ be in the radical of I. Clearly f has to be in P. Hence $yR = P/I = N(R)$ is a prime ideal of R. It is easy to verify that $N(R) = \{0, y\}$. By Lemma (2.4), $Z(R)$ must have infinite girth.

Conversely, suppose R has at least 10 elements, $N(R) \neq 0$ and $Z(R)$ has infinite girth. In view of Lemma (2.4) if we let $N(R) := \{0, y\}$, then $N(R)$ is a prime ideal and $(0 : y)$ is a maximal ideal with residue field $\mathbb{F}_2$. We have already established that the characteristic of R has to be one of $0, 2, 4$. Our assumption about the cardinality of R ensures that the ideal $(0 : y)$ is distinct from $N(R)$. Let π denote the prime sub-ring of R. Then $B := \pi[y]$ is (isomorphic to) exactly one of the rings appearing in (i), (ii), (iii) above. Clearly, $N(B) = N(R)$ and since for each $r \in R$ either r or $r + 1$ is in $(0 : y)$, the ring R is obtained by adjoining the elements of $(0 : y)$ to B. Observe that $(0 : y) \cap B$ is a maximal ideal of B which equals $J := 2B + N(B)$ and has $\mathbb{F}_2$ as its residue field. Now the B-module $(0 : y)/J$ is in fact a vector-space over $\mathbb{F}_2 = B/J$. Let $T \subset (0 : y)$ be such that T/J is an $\mathbb{F}_2$-basis of $(0 : y)/J$ (we allow T to be the empty set). Then $R = B[T]$. Let X be a set of indeterminates over B equipped with a bijection $s : X \to T \cup \{0\}$. Let $\sigma : B[X] \to R$ be the unique homomorphism of B-algebras which restricts to s on the set X. Then σ is surjective. Let I denote the kernel of σ. Obviously $I \cap B = 0$ and I contains yx for all $x \in X$. If $f := b - g \in I$ with $b \in B$ and $g \in XB[X]$, then $b = \sigma(b) = \sigma(g)$ and $\sigma(g) \in (0 : y)$ imply that b is in J. Consequently, I is contained in $2B[X] + N(B)B[X] + XB[X]$. Finally, since $\sigma(N(B)) = N(R)$ is a prime ideal of R, its inverse image $N(B)B[X] + I$ is also a prime ideal of $B[X]$. □

REMARKS. 1. If R has at least 10 members, $N(R) = 0$ and $Z(R)$ is non-empty, then

$Z(R)$ has infinite girth if and only if R is a product of a domain D and $\mathbb{F}_2$. This assertion follows from the remark at the end of [**4**, (2.1)] (for infinite rings R see [**1**, Theorem 2.5]). In fact, the above Theorem, in conjunction with [**4**, (2.1)] provides a complete (without any cardinality restrictions) characterisation of those non domains R for which $Z(R)$ has infinite girth (that is, $V(R) = \emptyset$ in the notation used in [**4**]).

2. Note that in the above proof it is not essential for us to choose T the way we have chosen it, that is, we did not make any particular use of the fact that T/J is a vector-space basis. On the other hand, it is natural to try to get hold of a 'smallest possible' set T with $R = B[T]$.

Definition: If B is a ring and M is a B-module, then by $B(+)M$ we denote the ring obtained by idealising M (as defined in [**5**]).

Theorem 2.6. *Assume R satisfies the following. As above, $N(R)$ denotes the nil-radical of R.*

(i) *R has at least 5 elements.*

(ii) *There is $y \in Z(R)$ such that $N(R) = \{0, y\}$ and $(0 : x) = N(R)$ for all $x \in Z(R)$.*

Then the characteristic of R is not 4 if and only if R has a sub-domain A such that $A[y] \cong A(+)\mathbb{F}_2$ and for each $r \in R$ there exists a non-zero $a \in A$ (depending on r) with $ar \in A[y]$. Moreover, if r is not in $N(R)$ then $ar \neq 0$. (David F. Anderson (in a private communication) asked whether a non-reduced, non-domain R with $Z(R)$ having infinite girth is of the form $D(+)\mathbb{F}_2$ for some domain D. The above theorem constitutes our response to his question.)

Proof: From the argument at the beginning of the proof of (2.5) it follows that R has characteristic 0 or 2 or 4. If a ring contains a sub-ring of type $D(+)\mathbb{F}_2$ with D a domain, then obviously the characteristic can not equal 4. Henceforth, assume that the characteristic of R is either 0 or 2. We proceed to show that R contains an infinite integral sub-domain. This is evident if R is of characteristic 0. Suppose R has characteristic 2. Condition (i) ensures that $Z(R) \setminus \{y\}$ is non-empty. Let x be an element of $Z(R) \setminus \{y\}$ and $S := \mathbb{F}_2[x]$. Note that $Z(S)$ is contained in the single maximal ideal $P := (0 : y) \cap S$ of S. Since $xS \subseteq P$, we must have $P = xS$. It is easy to see that y is not in xR and hence y is not in S. Thus S is an infinite integral sub-domain of R (in fact a polynomial ring over $\mathbb{F}_2$). Let A denote a maximal sub-ring of R not containing y. Existence of such a sub-ring can be seen in a straightforward manner. As a consequence of the above argument A has to be an infinite integral domain. Since $(0 : y) \cap A$ is a maximal ideal of A with residue field $\mathbb{F}_2$, it follows that A is not a field. For $r \in R$ define

$$I(r, A) := \{a \in A \mid ar \in A[y]\}.$$

Clearly $I(r, A)$ is an ideal of A and $I(r + b, A) = I(r, A)$ for all $b \in A$.

Suppose there is an x in R with $I(x, A) = 0$. Then $I(bx, A) = 0$ for all non-zero $b \in A$. Replacing x by $x + 1$ if needed, we may assume that x is a member of $(0 : y)$. Obviously

x is not in $N(R)$ and hence $xR \cap N(R) = 0$. Consider the A-algebra homomorphism h of the polynomial domain $A[X]$ onto $A[x]$ which maps the indeterminate X to x. Let $J(x)$ denote the kernel of h. Our choice of A ensures that $y \in A[x]$. Thus $J(x)$ is not a radical ideal of $A[X]$. In particular, $J(x) \neq 0$. Let $f \in J(x)$ be a non-zero polynomial of least degree. Let $f := a_0X^d + \cdots + a_d$ where $d \geqslant 2$ is the degree of f and $a_0 \neq 0$. Replacing x by a_0x if needed, we may assume $a_0 = 1$ that is, x is integral over A. The minimality of d allows us to conclude that $J(x) = fA[X]$. More generally, we observe that $J(bx) = b^d f(X/b)A[X]$ for all non-zero $b \in A$. Consequently, for each non-zero $b \in A$, the ring $A[bx]$ contains y and it is a free A- sub-module of $A[x]$ with basis $\{1, bx, \cdots, (bx)^{d-1}\}$. Now, A is an integral domain which is not a field and hence the intersection of all non-zero principal ideals of A is necessarily zero. Thus the intersection of all the sub-modules $A[bx]$ of $A[x]$, as b ranges over non-zero elements of A is exactly A. This is absurd since y is certainly not in A. Therefore we must have $I(r, A) \neq 0$ for all $r \in R$. It is easy to see that $A[y] \cong A(+)\mathbb{F}_2$. ∎

REMARKS. 1. The above theorem allows us to think of R as a "blow-up" of $A(+)\mathbb{F}_2$.

2. A ring R is of the form $S(+)M$ for some non-zero S-module M if and only if R has a non-zero ideal N and a derivation $\delta : R \to N$ such that $N^2 = 0$, $N \cap Ker\ (\delta) = 0$ and the set-theoretic map $R \to R/N \oplus N$ sending $r \in R$ to $\sigma(r) \oplus \delta(r)$ (here σ is the canonical map) is surjective. Now any derivation of R is identically 0 on the prime sub-ring of R. Suppose R satisfies the conditions of the above Theorem and the characteristic of R is 4. Then $N(R) = \{0, 2\}$ and it is the only non-zero ideal of R whose square is zero. Since $N(R)$ is contained in the kernel of every derivation of R, it follows that R is not of the form $S(+)M$ with $M \neq 0$.

3. Consider the 3-variable polynomial ring $B := \mathbb{F}_2[X_1, X_2, X_3]$ with ideal I generated by $\{X_1f, X_2f, X_3f\}$ where $f := X_1X_3 + X_2^2$. Let $R := B/I$ and let u, v, w, y denote the canonical images of X_1, X_2, X_3, f (respectively) in R. It is easy to see that R satisfies the hypotheses of the above Theorem. We leave it to the reader to verify that for A (as in the conclusion of the Theorem) we may take the sub-ring $\mathbb{F}_2[u, v, w^2, vw]$. Further, it is simple to check that $\delta(y) = 0$ for any derivation $\delta : R \to \{0, y\}$. Hence R is not of the form $S(+)M$ with $M \neq 0$. One can construct a similar example in characteristic 0.

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