The distribution of prime ideals
of a Dedekind domain

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Let G be an abelian group, and let S be a subset of G.
Necessary and sufficient conditions on G and S are given in
order that there should exist a Dedekind domain D with class
group G with the property that S is the set of classes that
contain maximal ideals of D. If G is a torsion group, then
S is the set of classes containing the maximal ideals of D if
and only if S generates G. These results are used to
determine necessary and sufficient conditions on a family $\{H_\lambda\}$
of subgroups of G in order that there should exist a Dedekind
domain D with class group G such that $\{G/H_\lambda\}$ is the family
of class groups of the set of overrings of D. Several
applications are given.

0. Introduction

In [3], Claborn proved that each abelian group G is the class group
of a Dedekind domain D. Claborn extended his results in [4] and [5],
Chapter III, to prove that some restrictions can be imposed on the prime
ideal structure of D. Leedham-Green established similar results in [8].
The results of [4] enable Claborn to obtain a number of interesting
examples - for instance, a Dedekind domain D_1, not a principal ideal
domain, such that each proper overring of D_1 is a principal ideal domain,

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and a Dedekind domain D_2 such that D_2 does not have the QR-property ($cf.$ [7]), but each proper overring of D_2 has the QR-property. In [9], Chapter 3, Levitz uses Claborn's results to give several other interesting examples.

In this paper, we investigate two questions concerning Dedekind domains and their class groups that are much in the same vein as those considered by Claborn and Leedham-Green. Specifically, we ask:

(1)   Let G be an abelian group, and let S be a subset of G. Determine necessary and sufficient conditions on G and S in order that there should exist a Dedekind domain D with class group isomorphic to G under an isomorphism Φ, with the property that $\Phi(S)$ is the set of elements of the class group of D determined by a maximal ideal of D.[1]

(2)   Let G be an abelian group, and let $\{H_\lambda\}$ be a family of sub-groups of G. Determine necessary and sufficient conditions on G and $\{H_\lambda\}$ in order that there should exist a Dedekind domain D with class group G such that $\{G/H_\lambda\}$ is the family of class groups of the set of overrings of D.

Before proceeding further, some comment on (2) is appropriate. If D is an integral domain with identity with quotient field K, then an *overring* of D is a subring of K that contains D. If D is a Dedekind domain and if J is an overring of D, then there is a canonical homomorphism of the class group $C(D)$ of D onto the class group $C(J)$ of J induced by the mapping $A \rightarrow AJ$ of the group of nonzero fractional ideals of D onto the group of nonzero fractional ideals of J. The kernel H_J of this homomorphism is generated by the set of classes $\{\overline{M} \mid M$ is a maximal ideal of D and $MJ = J\}$ [2, p. 800]. Thus $C(J) \simeq C(D)/H_J$. For convenience, we use the phrase "G/H is the class group of J" to mean that the subgroup H of G is the kernel of the canonical homomorphism of $G = C(D)$ onto $C(J)$. It is in this sense that we intend question (2).

We answer questions (1) and (2) in §1 and §2. In §3 we give several applications of our results. One of the examples produced relates to the

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[1]   This question was suggested to Professor Gilmer by Joe L. Mott.

question of whether each nonzero element of a domain of finite character
has at most a finite number of non-associate irreducible divisors (*cf*. [*1*,
p. 87]).

1.   Partitions of the class group

Let D be a Dedekind domain with class group $C(D)$. We partition
$C(D)$ into two subsets S and T , where S consists of the classes that
contain a maximal ideal of D and T consists of the classes that do not
contain a maximal ideal. In our first proposition, two properties of S
are established that will play a key role in our answer to question (1).

PROPOSITION 1.1. *The set S has the following properties:*

(1) *S generates G ;*

(2) *if s_1, ..., s_k are elements of S and n_1, ..., n_k are*
nonnegative integers, then there exist elements r_1, ..., r_t
of S distinct from the s_i and nonnegative integers

m_1, ..., m_t such that $\sum_{i=1}^{k} n_i s_i + \sum_{i=1}^{t} m_i r_i = 0$.

Proof. (1) If A is a nonzero fractional ideal of D , then there
are maximal ideals M_1, ..., M_k of D and integers n_1, ..., n_k such

that $A = M_1^{n_1} \ldots M_k^{n_k}$. Hence $\overline{A} = n_1 \overline{M_1} + \ldots + n_k \overline{M_k}$, and S generates
$C(D)$.

(2) This is a form of the approximation theorem for Dedekind
domains.   //

In order to provide a converse to Proposition 1.1 and an answer to
question (1), we need a theorem of Claborn. For completeness, we give his
terminology and theorem here.

Let X be a set, and let $Z^{(X)}$ denote the free abelian group based
on X . We let $Z_+^{(X)}$ be the set of elements $\sum n_\lambda x_\lambda$ in $Z^{(X)}$ such that
each n_λ is nonnegative. If X' is a subset of X , then the restriction
map $\rho : Z^{(X)} \to Z^{(X')}$ defines a group homomorphism which is a surjection.

If P is a subset of $Z_+^{(X)}$, we say that P is *finitely dense* in $Z^{(X)}$ if $\rho(P) = Z_+^{(X')}$ for each finite subset X' of X; P is *dense* if $\rho(P) = Z_+^{(X')}$ for each subset X' of X with $\operatorname{card}(X') < \operatorname{card}(X)$. (If X is countable, then the conditions finitely dense and dense are equivalent; finitely dense is equivalent to Claborn's condition (α) in [4].)

THEOREM 1.2 (Claborn [5, p. 78]). *Let X be a set and P, contained in $Z_+^{(X)}$, a dense subset. Let H be the subgroup of $Z^{(X)}$ generated by P. Then there exists a Dedekind domain D with set $\{M_x\}_{x \in X}$ of maximal ideals such that the class group $C(D)$ is isomorphic to $Z^{(X)}/H$ under an isomorphism which sends $x + H$ to $\overline{M}_x$ for each $x \in X$.*

We observe that if H is a subgroup of $Z^{(X)}$, if $P = H \cap Z_+^{(X)}$, and if P is finitely dense in $Z^{(X)}$, then P generates H. For if $h = n_1 x_1 + \dots + n_k x_k \in H$, let $n_{i(1)}, \dots, n_{i(s)}$ be the negative integers occurring among the integers $n_1, \dots, n_k$. Then there exists an element $f \in P$ whose coefficient on $x_{i(j)}$ is $-n_{i(j)}$, $1 \le j \le s$. Hence $h + p \in H \cap Z_+^{(X)} = P$, so that P generates H. This proves the following lemma.

LEMMA 1.3. *Let $Z^{(X)}$ be the free abelian group based on the set X. Let H be a subgroup of $Z^{(X)}$, and let P be the subset of H consisting of the elements all of whose coefficients are nonnegative. If P is finitely dense, then P generates H.*

We now provide an answer to question (1).

THEOREM 1.4. *Let G be an abelian group, and let S be a subset of G satisfying the following two conditions:*

(1) *S generates G;*

(2) *if* $\{s_1, \ldots, s_k\}$ *is a finite subset of* S , *then there*
     exist positive integers $n_1, \ldots, n_k$, *elements* $r_1, \ldots, r_t$
     of S , *and nonnegative integers* $m_1, \ldots, m_t$ *such that*
     $$n_1 s_1 + \ldots + n_k s_k + m_1 r_1 + \ldots + m_t r_t = 0 .$$

Then there exists a Dedekind domain D *whose class group* $C(D)$ *is*
isomorphic to G *under an isomorphism* Φ *such that for each maximal ideal*
M *of* D , $\Phi(\overline{M}) \in S$, *and conversely, for each* $s \in S$, *there is a maximal*
ideal M *of* D *such that* $\Phi(\overline{M}) = s$. *Moreover, if* λ *is any infinite*
cardinal greater than or equal to the cardinality of G , *then* D *can be*
chosen to have λ *maximal ideals and such that for each* $s \in S$, *there are*
λ *maximal ideals* M *of* D *such that* $\Phi(\overline{M}) = s$.

     Proof. Let Λ be an indexing set for S , and let Δ be an infinite
set of cardinality greater than or equal to that of G . Let $T = \{T_\lambda\}_{\lambda \in \Lambda}$
be a partition of Δ into $|\Lambda|$ disjoint sets, each containing $|\Delta|$
elements. Let $J = \sum_{\delta \in \Delta} \mathbb{Z} x_\delta$ be the free abelian group based on the set
$\{x_\delta\}_{\delta \in \Delta}$. Define a homomorphism $\psi : J \to G$ as follows:

$$\psi(x_\delta) = s_\lambda \quad \text{if} \quad \delta \in T_\lambda ,$$

and, in general,

$$\psi\left(\sum m_i x_{\delta_i}\right) = \sum m_i \psi\left(x_{\delta_i}\right) .$$

     Since S generates G , ψ maps J onto G . Let H be the kernel
of ψ , and let P be the subset of H consisting of the elements all of
whose coefficients are nonnegative. We shall verify that P is dense. It
will then follow from Lemma 1.3 that P generates H . Hence by Theorem
1.2, there exists a Dedekind domain D with maximal ideals $\{M_\delta\}_{\delta \in \Delta}$ and
class group $C(D)$ such that $C(D)$ is isomorphic to J/H under an
isomorphism τ determined by the relations $\tau(x_\delta + H) = \overline{M}_\delta$. We claim that
D is the desired Dedekind domain. To see that this true, let $\overline{\psi} : J/H \to G$
be the induced isomorphism from ψ , and let $\Phi = \overline{\psi}\tau^{-1}$; Φ is an
isomorphism from $C(D)$ onto G . If M_δ is a maximal ideal of D , then

$\Phi(\overline{M}_\delta) = \overline{\psi}\left(\tau^{-1}(\overline{M}_\delta)\right) = \overline{\psi}(x_\delta + H) = \psi(x_\delta) \in S$. Conversely, if $s_\lambda \in S$, pick $\delta \in T_\lambda$. Then $\Phi(\overline{M}_\delta) = \overline{\psi}(x_\delta + H) = \psi(x_\delta) = s_\lambda$.

It remains to show that P is dense. Let X' be a subset of $\{x_\delta\}$ such that $|X'| < |\Delta|$, and let $f \in \sum_{x' \in X'} \mathbb{Z}x'$. Label the subscripts of the nonzero coefficients occurring in f as $T_f = \{n_1, \ldots, n_r\}$. These integers (viewed as elements of Δ) belong to at most r elements of T - say $T_{\lambda_1}, \ldots, T_{\lambda_v}$. To simplify the notation, set $U_i = T_{\lambda_i} \cap T_f$. By hypothesis, there exist positive integers $k_1, \ldots, k_v$, elements $s_{\gamma_1}, \ldots, s_{\gamma_t}$ of S and nonnegative integers $m_1, \ldots, m_t$ such that

$$\sum_{i=1}^{v} k_i s_{\lambda_i} + \sum_{i=1}^{t} m_i s_{\gamma_i} = 0 .$$

By multiplying by a suitably large positive integer, we may assume that

$$k_i \geq \sum_{n_j \in U_i} n_j .$$

Since $|X'| < |\Delta| = |T_\lambda|$, we can select $d_i \in T_{\lambda_i} - X'$, $1 \leq i \leq v$.

Similarly, for $1 \leq j \leq t$, we can select an element $c_j \in T_{\gamma_j} - (X' \cup \{d_1, \ldots, d_v\})$. Then if

$$g = f + \left(k_1 - \sum_{n_j \in U_1} n_j\right)x_{d_1} + \ldots + \left(k_v - \sum_{n_j \in U_v} n_j\right)x_{d_v} + m_1 x_{c_1} + \ldots + m_t x_{c_t} ,$$

then g projects onto f under the restriction map induced by X' . Moreover g belongs to P since all its coefficients are nonnegative and since

$$\psi(g) = \left[\sum_{n_j \in U_1} n_j\right] s_{\lambda_1} + \ldots + \left[\sum_{n_j \in U_\nu} n_j\right] s_{\lambda_\nu} + \left[k_1 - \sum_{n_j \in U_1} n_j\right] s_{\lambda_1} + \ldots +$$

$$+ \left[k_\nu - \sum_{n_j \in U_\nu} n_j\right] s_{\lambda_\nu} + m_1 s_{\gamma_1} + \ldots + m_t s_{\gamma_t} = 0 \; .$$

Thus P is dense, and this completes the proof of the theorem.   //

It is an interesting consequence of Theorem 1.4 that if S is a generating set for G satisfying condition (2) of the theorem, then S also satisfies the seemingly stronger condition (2) of Proposition 1.1.

We observe that if each element of S has finite order or if $-S = \{-s \mid s \in S\} \subseteq S$, then S satisfies condition (2) of Theorem 1.4. Hence the following two corollaries follow easily from the theorem.

COROLLARY 1.5. *Let G be an abelian torsion group, and let S be a subset of G . Then there exists a Dedekind domain with class group (isomorphic to) G such that the classes which contain maximal ideals are precisely the elements of S if and only if S generates G .*

COROLLARY 1.6. *Let G be an abelian group, let S be a subset of G , and let $S' = S \cup -S$. Then there exists a Dedekind domain with class group G such that the classes that contain maximal ideals are precisely the elements of S' if and only if S generates G .*

2.   Class groups of overrings

If D is a Dedekind domain with class group $C(D)$ and if J is an overring of D , then the class group of J is isomorphic to a factor group of $C(D)$. It need not be true, however, that if H is a subgroup of $C(D)$, then there is an overring of D with class group $C(D)/H$. For example, Claborn [4, p. 253] and Levitz [9] construct Dedekind domains with cyclic class groups (both finite and infinite) such that all overrings are principal ideal domains, and hence have trivial class group. Question (2) is concerned with this type of problem. In order to provide an answer, we need the following result concerning the structure of the set of overrings of a Dedekind domain.

THEOREM 2.1 [6, p. 817]. *Let D be a Dedekind domain, and let J be an overring of D . Then there exists a unique collection $\{M_\lambda\}$ of*

maximal ideals of D such that $J = \cap D_{M_\lambda}$. Precisely,

$\{M_\lambda\} = \{M \mid M$ is a maximal ideal of D and $MJ \subset J\}$.[2]

Let D be a Dedekind domain with class group G , and let $\{G/H_\lambda\}$ be the family of factor groups of G that are class groups of overrings of D . By our convention, $\{H_\lambda\}$ is the collection of kernels of the canonical map from G onto $C(J)$, where J ranges over the set of all overrings of D . Let S be the subset of G that consists of all classes containing a maximal ideal. With this notation, we have the following result. The proof is straightforward and will be omitted.

PROPOSITION 2.2. *$\{H_\lambda\}$ is the family of subgroups of G generated by the family of subsets of S .*

The next result provides a converse to Proposition 2.2.

THEOREM 2.3. *Let G be an abelian group, and let S be a subset of G that generates G . Then there exists a Dedekind domain D with class group G such that the set of class groups of overrings of D is $\{G/H_\lambda\}$, where $\{H_\lambda\}$ is the family of subgroups generated by the subsets of S .*

Proof. By Corollary 1.6, there exists a Dedekind domain D with class group G such that

$$S' = S \cup -S = \{\overline{M} \mid M \text{ is a maximal ideal of } D\} .$$

We show that D is the desired domain.

Let $H \in \{H_\lambda\}$. If $H = G$, let M be a maximal ideal of D , and let $J = D_M$. Then $C(J) = \{0\} = G/H$. If $H \subset G$, let $J = \cap\{D_M \mid \overline{M} \notin H\}$. Then $PJ = J$ if and only if $P \notin \{M \mid \overline{M} \notin H\}$ — that is, if and only if $P \in \{M \mid \overline{M} \in H\}$. Hence $C(J) = G/H$.

Conversely, let J be an overring of D . The class group of J is

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[2] Dedekind domains are not characterized by this property – that is, each overring is a unique intersection of localizations. In fact, if D is a Prüfer domain, then distinct sets of maximal ideals of D determine distinct overrings of D if and only if there exists, for each maximal ideal M of D , a finitely generated ideal A contained in M such that M is the only maximal ideal of D that contains A [7, p. 283].

G/H , where H is generated by $\{\overline{M} \mid MJ = J\} = H \cap S'$. Since $S' = S \cup -S$, H is also generated by a subset of S .   //

The following theorem answers question (2). It is a direct consequence of Theorem 2.3.

THEOREM 2.4. *Let G be an abelian group, let $H = \{H_\lambda\}$ be a collection of subgroups of G , and let $H' = \{G/H_\lambda\}$ be the corresponding family of factor groups of G . Suppose that there exists a set S of generators for G such that H is the family of subgroups generated by the subsets of S . Then there exists a Dedekind domain D with class group G such that H' is the family of class groups of the set of overrings of D .*

3. Examples

EXAMPLE 3.1. A family $\{D_\lambda\}$ of domains with common quotient field K is said to be of *finite character* if each element of K is a unit in all but a finite number of the domains D_λ . Exercise 25 (a) in Bourbaki [1, p. 87] asserts that if D is an intersection of a family $\{V_\lambda\}$ of valuation rings of finite character, then each nonzero element of D has at most a finite number of nonassociate irreducible divisors. The hint given is to show that if $x \in D - \{0\}$ is a nonunit in precisely n of the valuation rings and if $y_1, \ldots, y_{n+1}$ are $n + 1$ nonassociate irreducible divisors of x , then $\left(y_i, y_1 \cdots \hat{y}_i \cdots y_{n+1}\right)_v = D$, where the caret $\hat{\ }$ signifies that y_i is omitted from the product and where the ideal on the left is the v-ideal generated by y_i and $y_1 \cdots \hat{y}_i \cdots y_{n+1}$. If the statement in the hint were true, then it would follow easily that x has at most n nonassociate irreducible divisors. We shall use the results of §2 to give an example where this is not true. This will show that the "fact" given in the hint is false. Moreover, the example produced leads us to believe that the Bourbaki exercise is false, although we know of no counterexample.

For each positive integer $n > 1$, let C_n denote the cyclic group of order n , and let G be the direct sum of the groups $\{C_n\}_{n>1}$. Let

$S = \{(\overline{1}, 0, \ldots), (0, \overline{1}, 0, \ldots), \ldots\}$ be the standard generating set for G . (We use the same symbol $\overline{1}$ for the generator of each C_n .) By Corollary 1.5, there exists a Dedekind domain D with class group G such that the classes which contain maximal ideals are precisely the elements of S . Moreover, we can assume that for each $s \in S$, there are infinitely many maximal ideals M of D such that $\overline{M} = s$. We claim that D has the following property.

Let n and m be positive integers with $1 < n \leq m$. Then there is a nonzero element x of D such that x is a nonunit in precisely n valuation overrings of D , but x has at least m nonassociate irreducible divisors $y_1, \ldots, y_m$. Moreover, $(y_1, y_2 \ldots y_m)_v \neq D$.

To see that this is true, let $\varepsilon_m = (0, \ldots, \overline{1}, 0, \ldots)$ be the element of S with the $\overline{1}$ in the m-th slot. We can choose n distinct maximal ideals $M_1, \ldots, M_n$ of D such that $\overline{M}_i = \varepsilon_m$, $1 \leq i \leq n$. Then the following ideals of D are principal:

$$(y_1) = M_1^m$$

$$(y_2) = M_1^{m-1}M_2$$

$$\vdots$$

$$(y_{m-1}) = M_1^2 M_2^{m-2}$$

$$(y_m) = M_2^{m-n+2}M_3 \ldots M_n \ .$$

Each y_i is irreducible, for if any maximal ideals are omitted from the factorization of (y_i) , the resulting ideal is not principal. For example, $M_1^{m-2}M_2$ is not principal since

$$\overline{M_1^{m-2}M_2} = (m-2)\varepsilon_m + \varepsilon_m = (m-1)\varepsilon_m \neq 0$$

in $C(D)$.

Let $x = y_1 \ldots y_m$. Then x is a nonunit in precisely the valuation rings $D_{M_1}, \ldots, D_{M_n}$. Moreover, x has at least m nonassociate

irreducible divisors $y_1, \ldots, y_m$, and $(y_1, y_2 \ldots y_m)_v \subseteq (M_1)_v \neq D$.

EXAMPLE 3.2. If D is an integral domain with identity, then D is said to have the *QR-property* if each overring of D is a quotient ring of D. A Dedekind domain has the QR-property if and only if its class group is a torsion group. This is an example of a Dedekind domain D with mixed class group such that each proper overring of D has the QR-property; moreover, the torsion part of $C(D)$ is not of bounded order. Let $\{n_i\}_{i=1}^{\infty}$ be a strictly increasing sequence of positive integers with $n_1 > 1$. Let G be the direct sum of the family $\{Z\} \cup \left\{C_{n_i}\right\}$ of groups.

Set $s = (1, 0, 0, \ldots)$, $s_1 = (-1, \overline{1}, 0, \ldots)$, $s_2 = (-1, 0, \overline{1}, 0, \ldots)$, and, in general, $s_i = (-1, 0, \ldots, \overline{1}, 0, \ldots)$. By Theorem 1.4, there exists a Dedekind domain D with class group G such that $S = \{s, s_1, s_2, \ldots\}$ is the set of classes determined by maximal ideals of D. It is easy to see that if S' is a nonempty subset of S and if H is the subgroup of G generated by S', then G/H is torsion. Hence each proper overring of D has the QR-property.

EXAMPLE 3.3. We construct a Dedekind domain such that the set of class groups of overrings is a prescribed set of cyclic groups. Let $S = \{n_i\}_{i \in I}$ be a set of integers with the property that $1 \in S$ and that for each $n, m \in S$, their greatest common divisor also belongs to S. Then by Theorem 2.4, there exists a Dedekind domain D with class group Z such that $\{C(J) \mid J$ is an overring of $D\} = \left\{C_{n_i}\right\}_{i \in I} \cup \{(0), Z\}$.

EXAMPLE 3.4. In [2], Claborn investigated conditions under which a Dedekind domain contains a semi-prime element – that is, an element that does not belong to the square of any maximal ideal. If D is a Dedekind domain constructed as in Theorem 1.4, then D contains semi-prime elements. In fact, each nonzero element of D gives rise to a semi-prime element in the following way. Suppose that $d \in D - \{0\}$ and that

$(d) = M_1^{n_1} \ldots M_r^{n_r}$ is the prime factorization of (d). Then in $C(D)$,

the relation $n_1\overline{M}_1 + \ldots + n_r\overline{M}_r = 0$ holds. For each maximal ideal M of D, there are infinitely many maximal ideals P_i of D such that $\overline{M} = \overline{P}_i$. Hence for $1 \leq i \leq r$, we can select n_i maximal ideals $P_{i(1)}, \ldots, P_{i(n_i)}$ of D such that $\overline{M}_i = \overline{P}_{i(j)}$, $1 \leq j \leq n_i$, where $P_{i(j)} = P_{k(s)}$ if and only if $i = k$ and $j = s$. Since

$$\sum_{j=1}^{n_1} \overline{P}_{1(j)} + \ldots + \sum_{j=1}^{n_r} \overline{P}_{r(j)} = 0 \text{ , the ideal}$$

$$P_{1(1)} \cdots P_{1(n_1)} \cdots P_{r(1)} \cdots P_{r(n_r)} = (d')$$

is principal, and d' is a semi-prime element.

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