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Mazur's incidence structure for projective varieties (I)

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Abstract. Let X be an m dimensional smooth projective variety with a Kähler metric. We construct a metrized line bundle $\mathcal{L}$ with a rational section s over the product $\mathcal{C}_p(X) \times \mathcal{C}_q(X)$ of Chow varieties $\mathcal{C}_p(X)$, $\mathcal{C}_q(X)$ such that

$$\frac{1}{(m-1)!} \log |s(A, B)|^2 = \langle A, B \rangle$$

for disjoint A, B. That gives an answer to a part of Barry Mazur's proposal in a private communication to Bruno Harris about the Archimedean height pairing $\langle A, B \rangle$ on a smooth projective variety X.

Mathematics Subject Classifications (1991): Primary 14C05; Secondary 14G99.

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1. Introduction

In higher dimensional arithmetic geometry, Archimedean height pairing originated from Arakelov's work ([A1], [A2]) was first introduced by Bloch ([B]), Beilinson ([Be]). It is the local pairing at infinity. It was studied by Gillet, Soulé, Bost, and Hain in various contexts. On the other hand, the Chow variety has a longer history that goes back to Cayley, Chow, van der Waerden ([CW]), and Chow coordinates have always been useful in Diophantine geometry ([BGS], [P1], [P2], [P3]). But as for the geometry of the Chow variety, we seem to be so far from understanding it. In 1993, aiming at the geometry of Chow varieties Barry Mazur proposed series of questions ([M]) concerning the incidence relation induced from the Archimedean height pairing.

To view it in a naive way, for a smooth projective variety X of dimension m, we take the incidence set $\mathcal{D} = \{(A, B) \in \mathcal{C}_p(X) \times \mathcal{C}_q(X) : A \cap B \neq \varnothing\}$, where $\mathcal{C}_p(X)$ and $\mathcal{C}_q(X)$ are the Chow varieties of dimension p and q with $p+q = m-1$. The question is: is $\mathcal{D}$ a variety, a divisor, or even more, is it a Cartier divisor? Mazur's proposal is to relate the $\mathcal{D}$ to the Archimedean height pairing. This is based on the fact that Archimedean height pairing is a continuous function outside of $\mathcal{D}$ in $\mathcal{C}_p(X) \times \mathcal{C}_q(X)$, but as the pair of disjoint cycles $A \times B$ approaches

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$\mathcal{D}$, the Archimedean height pairing of A, B goes to infinity ([W3] or [W4]). In the special case where X is a projective space, and $p = 0$, $q = m - 1$, $\mathcal{D}$ is the classical universal hypersurface. In general, to obtain a divisor, we need to associate a 'multiplicity' to each $(A, B) \in \mathcal{C}_p(X) \times \mathcal{C}_q(X)$, which shall reflect the way how A and B intersect. In [M], a Weil divisor D_w supported on the incidence set $\mathcal{D}$ is defined by using intersection theory. Two of the questions asked by Mazur are:

(1) Is $\mathcal{D}_w$ a Cartier divisor?
(2) If it is, let $\mathcal{L}$ be the corresponding line bundle and s the section that defines $\mathcal{D}_w$. Does there exist a metric $||\cdot||$ on $\mathcal{L}$, such that

$$\log ||s||^2(A, B) = \langle A, B\rangle \tag{1.0}$$

for any disjoint cycles $A \in \mathcal{C}_p(X)$, $B \in \mathcal{C}_q(X)$, where $\langle A, B\rangle$ is the Archimedean height pairing?

1.1. THE STATEMENT

The main purpose of this paper is to construct a line bundle and a section s through the calculation of the Archimedean height pairing, which give a formula close to (1.0) (see 1.1.3 below).

DEFINITION 1.1.1. Let X be a smooth, irreducible projective variety of dimension m with a Kähler metric. Let A, B be a linking pair of effective cycles in X of dimensions p and q respectively, i.e. $p + q = m - 1$, and A, B are disjoint. If A and B are disjoint irreducible subvarieties in X, the Archimedean height pairing $\langle A, B\rangle$ of A and B is defined to be the integral $\int_A [G_B]$ (it is well-defined because of the condition (2) of the smoothness below) of a Green's current of B over A. A Green's current ([Bo], 1.2) of B is a current $[G_B]$ of type (p, p) on X satisfying:

(1) As currents $(i/2\pi)\partial\bar{\partial}[G_B] = \delta_B - [\omega_B]$, where δ_B is the current of integration over B and $[\omega_B]$ is the current of a smooth form ω_B on X,
(2) $[G_B]$ is smooth outside of B,
(3) (Normalization) ω_B is harmonic and the harmonic projection of $[G_B]$ is zero.

If, furthermore, $[G_B]$ can be represented by an L^1 form that is smooth on $X\backslash B$, the L^1 form will be denoted by G_B called 'Green's form'.

We linearly extend the definition of Archimedean height pairing, and Green's currents to cycles. Let $\mathcal{C}_r^\alpha(X)$ denote the Chow variety of X which parametrizes the effective cycles of dimension r in X with cohomology class α. By linear extension one can view Archimedean Height Pairing as a function on an open subset $\mathcal{U}$ of the product $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ of the Chow varieties, where $\mathcal{U}$ consists of all the disjoint pairs of cycles.

THEOREM 1.1.2. *There exist a metrized line bundle $\mathcal{L}$ on the closure $\bar{\mathcal{U}}$ of $\mathcal{U}$ in $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$, and a rational section s that is regular, no where zero on $\mathcal{U}$, such that*

$$\lambda \log ||s(A, B)||^2 = \langle A, B\rangle \tag{1.1.3}$$

for $A, B \in \mathcal{U}$, where $\lambda = 1/(m-1)!$. The line bundle is unique, and the metric, the section are unique up to a constant.

The Archimedean height pairing function is not defined on an irreducible component of $\mathcal{C}_p^\alpha(X)\times\mathcal{C}_q^\beta(X)$ that contains only intersecting pairs of cycles A, B. Such a component therefore should not be considered in our theorem. As matter of fact $\bar{\mathcal{U}}$ consists of all the components of $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ that contain at least one pair of disjoint cycles A, B. Our formula is a little different from Mazur's conjecture, because of the coefficient $\lambda = 1/(m-1)!$. We don't know if an $(m-1)!$st root of our section s can be taken to realize the formula (1.0), i.e. $\lambda = 1$. On the contrary we suspect the presence of some nontrivial coefficient might be inevitable. But we are unable to find an example with $\lambda \neq 1$. In [M], the incidence divisor $\mathcal{D}_w$ is defined by using Fulton's intersection theory. Then Mazur's first question (see question 1 above before 1.1) leads us to ask: is $\mathrm{div}(s) = (m-1)!\mathcal{D}_w$?

1.2. SKETCH OF THE PROOF

By the reduction to the diagonal ([Bo]):

$$\langle A, B\rangle_X = \langle A \times B, \Delta\rangle_{X\times X},$$

we only need to consider the height pairing between $m-1$ cycle $C = A \times B$ and the diagonal Δ in $X \times X$ with the cycle Δ fixed. The general principle is that the Archimedean height pairing shall behave well under the correspondence, as the global pairing does ([Be], Lemma 2.1.3). Following that principle, we try to find a suitable correspondence

$$\begin{tikzcd} & \Gamma \subset X \times X \times Y \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \times X & & Y, \end{tikzcd} \tag{1.2.1}$$

such that under Γ and the transpose ${}^t\Gamma$ of Γ, ${}^t\Gamma_* y_\Delta = \Delta$ for a point y_Δ in Y, $\Gamma_* C$ is a divisor in Y, and the Archimedean height pairings satisfy

$$\langle C, {}^t\Gamma_* y_\Delta\rangle_{X\times X} = \langle \Gamma_* C, y_\Delta\rangle_Y + c(y_\Delta, C), \tag{1.2.2}$$

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where $c(y_\Delta, C)$ is a continuous function (therefore bounded) in variables y_Δ, C that comes from the normalization condition of Green's currents, and therefore depends on the Kähler metrics on both X, Y. So, up to a bounded $c(y_\Delta, C)$ ($C = A \times B$), the pairing $\langle A, B\rangle_X$ is equal to the pairing between the divisor $\Gamma_*(A \times B)$ and the point y_Δ with y_Δ fixed. Let L be the line bundle corresponds to the divisor $\Gamma_*(A \times B)$ on Y. The Poincaré–Lelong formula then can be applied to $\langle \Gamma_*(A \times B), y\rangle_Y$, i.e.

$$\langle \Gamma_*(A \times B), y_\Delta\rangle_Y = \log ||\sigma_{A\times B}(y_\Delta)||^2, \tag{1.2.3}$$

where $\sigma_{A\times B}$ is a section of L that defines the divisor $\Gamma_*(A \times B)$ in Y, and $||\cdot||$ is some suitable metric on L. In the meantime, there is a section s' of $\mathrm{O}(1)$ bundle of the projective space $P(H^0(L, Y))$ of divisors on Y (the linear system containing $\Gamma_*(A \times B)$), defined by evaluating the regular section $\sigma \in H^0(L, Y)$ of L at y, i.e. $s'(\mathrm{div}(\sigma)) = \sigma(y) \in \mathbb{C} \backsimeq$ fibre of L at y. Thus (1.2.3) can be written as $\langle \Gamma_*(A \times B), y_\Delta\rangle_Y = \log(||s'(\Gamma_*(A \times B))||_1)^2$ for the metric $||\cdot||_1$ of $\mathrm{O}(1)$ on $P(H^0(L, Y))$ induced from the metric on L. Under the new metric $||\cdot||_2 = ||\cdot||_1 e^{-c(y_\Delta, C)}$, we then have

$$\langle A, B\rangle_X = \log(||s'(\Gamma_*(A \times B))||_2)^2. \tag{1.2.4}$$

Note if $A \times B$ stays in the same cohomology, $\Gamma_*(A \times B)$ will be in the same linear system, provided in Y rational equivalence coincides with homological equivalence for divisors. Therefore the correspondence Γ induces a morphism from product $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ of the Chow varieties to the linear system $P(H^0(L, Y))$ by sending (A, B) to $\Gamma_*(A \times B)$. Using the morphism, we pull back the metrize line bundle $\mathrm{O}(1)$, and the section s' on $P(H^0(L, Y))$ to the product $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ of the Chow varieties to complete the construction.

So the problem boils down to finding such a correspondence (1.2.1) for the fixed cycle Δ. When X is a projective space, one can choose Γ to be the incidence correspondence between points and divisors. Such a detailed construction has been worked out in [W1]. As a corollary of this, for a general smooth projective variety X, if Δ is the intersection cycle of a subvariety and $X \times X$ in a projective space, the construction can still be carried out by using the incidence correspondence for the projective space ([W1], Coro. 2.2.8). But in general Δ is not such an intersection cycle of a subvariety with $X \times X$, not even in the cohomology group. So the idea is to move the diagonal Δ in its rational equivalence class to a new cycle for which above suitable correspondence Γ can be found. That can be done for the cycle $(m-1)!\Delta$ (this is why $(m-1)!$ shows up in (1.1.3)), by the results of Kleiman (see Proposition 4.3) involving a 'twisted imbedding' of $X \times X$ in some Grassmannian $G(k, n)$. Precisely, the new cycle is $\Delta_1 - \Delta_2$ (i.e. $(m-1)!\Delta$ is rationally equivalent to $\Delta_1 - \Delta_2$) here Δ_1 is an intersection cycle of a special Schubert cycle with $X \times X$ in the Grassmannian $G(k, n)$, and Δ_2 is an integer multiple of an intersection cycle of a linear space with $X \times X$ in a projective space.

As we mentioned above, in [W1] the construction for the fixed cycle Δ_2 has been worked out by using the incidence correspondence. In this paper (Section 3 and Proposition 5.3.1) we show that the correspondence (1.2.1) for the fixed cycle Δ_1 can be found, and is the induced correspondence between Grassmannians

$$\begin{tikzcd} & \Gamma \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X \times X \subset G(k,n) & & G(n-k+m,n) \end{tikzcd}$$

and

$$\Gamma = \{(V,W) \in G(k,n) \times G(n-k+m,n) : \mathrm{dim}(V \cap W) \geqslant m+1\}.$$

Then $\Gamma_*(A \times B)$ is a divisor. We call it a 'twisted Chow divisor' (see Section 3). Then ${}^t\Gamma_* y_{\Delta_1}$ is a special Schubert cycle in $G(k,n)$. Finally, we show that moving Δ only changes formula (1.1.3) by adding a logarithm of a rational function (Proposition 5.3.1).

1.3. POSSIBLE FUTURE APPLICATIONS

It is important to point out that the divisor defined by s is supported on the set of intersecting pairs of cycles. Therefore it is appropriate to think of it as a 'universal cycle' (only for $\bar{\mathcal{U}}$), or the collection of intersecting cycles counted with naturally defined 'multiplicity'. As matter of fact it is the generalization of the universal hypersurface.

The study of the incidence line bundle $\mathcal{L}$ and the section s has a wide range of implications in algebraic geometry. We list a couple in the following:

(1.3.1) We define the incidence divisor to be $D = \mathrm{div}(s)$. The noneffectivity of D which sometimes occurs is an obstruction to the existence of a positive Green's current, which is defined by Bost–Gillet–Soulé ([BGS], p. 1015). The ampleness of the incidence divisor was proposed and studied by Griffiths [G2]. We'll discuss these questions in Section 6.

(1.3.2) The line bundle $\mathcal{L}$ induces a duality map $L: CH_p(X) \to \mathrm{Pic}(\mathcal{C}_q(X))$, where $CH_p(X)$ is the Chow group. If two cycles have the same image under L, we say they are incidence equivalent.* Hain's results ([H]) on the the Archimedean height pairing imply that the homomorphism L factors through the intermediate Jacobian via the Abel–Jacobi map. One of the open problems in Hodge theory – understanding the Abel–Jacobi map – therefore can be approached with the aid

* Griffiths defines incidence equivalence in [G2], using smooth parameter spaces and obtains divisors modulo rational equivalence.

of L. In general, rational equivalence implies Abel–Jacobi equivalence, and Abel–Jacobi equivalence implies incidence equivalence. It is a conjecture of Griffiths that Abel–Jacobi equivalence is the same as incidence equivalence ([G1]). We intend to study such a duality map in [W2].

(1.3.3) Here is the outline of the rest of the paper. Our construction is based on Gillet–Soulé's theory of Green's currents. In Section 2, we recall some techniques, namely, the push-forward and the pull-back of cycles, currents, and in particular the Green's currents. In Section 3, we discuss the 'twisted Chow form' of a cycle via special Schubert cycles (instead of planes for the usual Chow form), and then we show a morphism from the usual Chow variety to the 'twisted' one. In Section 4, we discuss 'moving lemmas'. Section 5 is the core of this paper where we present all the calculations of Archimedean height pairing we need for the construction, and prove Theorem 1.1.2. In Section 6 we discuss the positivity and effectivity of the incidence divisor and, furthermore, give some examples.

2. Currents and cycles

We would like to introduce some of the techniques, namely the push forward and the pull back of currents and cycles. Let $Z_p(X)$ be the group of algebraic cycles of dimension p. Let $D_n(X)$ (resp. $D_{p,p}$) be the set of currents of dimension n (resp. (p, p) type), and $A^n(X)$ (resp. $L^{(n)}(X)$) be the set of smooth forms (resp. L^1 forms) of degree n. There are natural maps

$$Z_p(X) \to D_{p,p}; \quad C \to \textstyle\int_C,$$

$$A^n(X) \to D_{m-n}(X); \quad \eta \to \text{ linear functional } \{\alpha \to \textstyle\int_X \alpha \wedge \eta\},$$

$$L^{(n)}(X) \to D_{m-n}(X); \quad \eta \to \text{ linear functional } \{\alpha \to \textstyle\int_X \alpha \wedge \eta\}.$$

Let $C \in Z_p(X)$, and $\phi \in A^n(X)$(resp. $L^{(n)}(X)$), then denote their images in the space of currents by δ_C, $[\phi]$ respectively.

2.1. PUSH-FORWARD

The following are the definitions and properties of the push-forward of cycles and currents. Let X and Y always be smooth algebraic varieties over $\mathbb{C}$, and $R(Z)$ the field of rational functions on algebraic variety Z.

(2.1.1) Let $f\colon X \to Y$ be a proper morphism between smooth projective varieties. Then the push-forward

$$\begin{aligned} f_*\colon D_n(X) &\to D_n(Y) \\ T &\to f_*(T) \end{aligned}$$

of currents is defined by $f_*(T)(\eta) = T(f^*\eta)$, η is a smooth form.

(2.1.2) Suppose f is smooth, and proper. Then the push-forward of a smooth form

$$\begin{aligned} f_*\colon A^{p,q}(X) &\to A^{p-s,q-s}(Y) \\ \eta &\to f_*\eta \end{aligned}$$

is defined by the fibre integration, where s is the complex dimension of the fibres. Using the iterated integral, we have $f_*[\eta] = [f_*\eta]$. f_* commutes with the operator dd^c.

(2.1.3) Let f be proper, Z an irreducible subvariety of X. Denote the space of cycles of dimension k by $Z_k(X)$. We'll denote the cycle represented by Z by the same letter Z. We know $Z' = f(Z)$ is a closed subvariety of Y. Let

$$\deg(Z/Z') = \begin{cases} [R(Z) : R(Z')], & \text{if } \dim(Z) = \dim(Z'), \\ 0, & \text{if } \dim(Z') < \dim(Z). \end{cases}$$

The push-forward f_*Z of V is defined by $f_*Z = \deg(Z/Z')Z'$. f_* can be extended linearly to cycles, and the extension will be denoted by f_* too. The push-forward of a cycle and the push-forward of a current coincide. Precisely, this means

PROPOSITION 2.1.4. *Let $f\colon X \to Y$ be a morphism of smooth projective varieties. Then for any cycle $Z \in Z_p(X)$,*

$$\delta_{f_*(Z)} = f_*\delta_Z. \tag{2.1.5}$$

Proof. It suffices to assume Z is an irreducible subvariety. Let $Z' = f(Z)$. If $\dim(Z') < \dim(Z)$, then both sides of (2.1.5) vanishes. Thus we can assume $\dim(Z) = \dim(Z')$. We only need to prove (2.1.5) locally. Since $i \circ f$ is a morphism, then there is a Zariski open set $U \subset Z'$, such that U lies in the smooth locus of Z', and $f\colon f^{-1}(U) \to U$ is a $\deg(Z'/Z)$ sheets covering. Take any neighborhood (in usual topology) u of an arbitrary point in U, then $f^{-1}(u)$ is the union of $\deg(Z'/Z)$ many copies of u. For any smooth form η on Y,

$$\int_{f^{-1}(u)} f^*\eta = \deg(Z'/Z) \int_u \eta.$$

Thus $f_*\delta_Z = \deg(Z'/Z)\delta_{Z'}$, which is equivalent to what we would like to prove. □

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2.2. PULL-BACK

(2.2.1) Suppose f a smooth morphism. Then the push-forward of differential forms has its dual f^* defined on currents,

$$\begin{aligned} D_n(Y) &\to D_{n+s}(X) \\ T &\to f^*T(\phi) = T(f_*\phi), \end{aligned}$$

where s is the dimension of fibres. f^* commutes with dd^c.

(2.2.2) Suppose the morphism f is flat. Let Z be a subvariety of Y. The pull-back f^*Z of Z is defined by $f^*Z = f^{-1}(Z)$. We then extend it linearly to the cycles to obtain the pull-back on cycles. The pull-back of cycles and the pull-back of currents coincide in the same fashion as in Proposition 2.1.4.

2.3. GREEN'S CURRENTS

This section is adopted from [GS], 2.1.3, 2.1.4 for the convenience of the readers. Let $f\colon X \to Y$ be a morphism between two smooth projective varieties. Let B be a cycle of codimension p on Y. Assume $f^{-1}(|B|)$ has pure codimension p. Then the pull-back f^*B of the cycle B is well-defined (see [GS], [F], Chapter 8). If f is an imbedding, then f^*B is the intersection cycle as defined by J. P. Serre [S].

THEOREM 2.3.1 (Gillet-Soulé)

(1) *Let G_B be an unnormalized Green's form of log-type. Then f^*G_B is a well-defined L^1 form on X. As a current, $[f^*G_B]$ is an unnormalized Green's current for f^*B, i.e. $dd^c[f^*G_B] = \delta_{f^*B} - f^*[\omega_B]$, where ω_B is a smooth form Poincaré dual to B on X.*

(2) *If A, B are two cycles in Y, and they meet properly. Let G_B be an unnormalized Green's form of log-type, then $[G_B] \wedge \delta_A$ is a well-defined current, i.e. for any smooth form ϕ on Y, the integral $\int_A G_B \wedge \phi$ converges. It satisfies $dd^c([G_B] \wedge \delta_A) = \delta_{A\cdot B} - [\omega_B] \wedge \delta_A$.*

If f is an imbedding, then the part (1) says the restriction of Green's form G_B of B to a smooth subvariety X is a Green's form of the intersection cycle $X \cdot B$ in X. See [GS], 2.1.4 for a more general statement.

3. Twisted Chow forms

Let E be a vector space over $\mathbb{C}$ of dimension n. Let $G(k, n)$ be the Grassmannian, i.e. the set of k dimensional linear subspaces in E. We would like to imitate the use of the incidence correspondence in projective space. The correspondence we

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are going to use here is the following that also satisfies Bertini's theorem proved by Kleiman ([K1], Thm. 3.3). Let r be an integer, and $0 < r < k$. Let

$$\Gamma^r = \{(V, W) \in G(k, n) \times G(n-k+r-1, n) : \dim(V \cap W) \geqslant r\}.$$

We have a correspondence

$$\begin{tikzcd} & \Gamma^r \arrow[dl, "\pi_1^{(r)}"'] \arrow[dr, "\pi_2^{(r)}"] & \\ G(k,n) & & G(n-k+r-1,n) \end{tikzcd} \tag{3.0}$$

DEFINITION 3.1. For any $W \in G(n-k+r-1, n)$, $(\pi_1^{(r)})_* \circ (\pi_2^{(r)})^*(W)$ is called an rth special Schubert cycle, denoted by $\sigma_r(W)$.

It is a well known fact that a special Schubert cycle represents the rth Chern class $c_r(Q)$ of the universal quotient bundle Q on $G(k, n)$. The following theorem is Bertini's theorem due to Kleiman ([K1]), Theorem 3.3. We first recall his definition ([K1, 3.1]): an imbedding $A \subset \mathrm{Grass}_n(E) =$ Grassmannian of n-dimensional quotients of E is called twisted if it is the Segre product of a morphism $A \to \mathrm{Grass}_n(E_1)$ and an imbedding $A \to P(E_2)$.

THEOREM 3.2 (Kleiman, [K1], Theorem 3.3). *Let A be a twisted subvariety in $G(k, n)$ of dimension p, then for a general plane $W \in G(n-k+r-1, n)$, the intersection $A \cap \sigma_r(W)$ has pure dimension $p - r$, especially when $p - r < 0$, the intersection is empty, and furthermore it is smooth at the smooth locus of A and $\sigma_r(W)$.*

PROPOSITION 3.3. *Let A be a twisted cycle in $G(k, n)$ of pure dimension p, then $(\pi_2^{(p+1)})_* \circ (\pi_1^{(p+1)})^* A$ is a cycle of codimension 1 in $G(n-k+p, n)$, i.e. a hypersurface.*

Proof. It suffices to assume A is an irreducible variety. Note

$$\Gamma^{(p+1)} \xrightarrow{\pi_1^{(p+1)}} G(k, n)$$

is a flat morphism with each fibre isomorphic to $p+1$st special Schubert cycle in $G(n-k+p, n)$. By Theorem 11.12. in [Ha], $(\pi_1^{(p+1)})^{-1}(A)$ is an irreducible variety of dimension $(n-k+p)(k-p) - 1$.

Let W' be a general $(n-k+p-1)$-plane in E, then by Theorem 3.2. the dimension of the intersection of $\sigma_p(W')$ and A is zero, therefore they intersect properly at finitely many points V_i. Let w be a general vector in E, and $W =$

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$W' + \mathbb{C}w$ a general p-plane, then if $\sigma_{p+1}(W) \cap A$ is not empty, it has to be a subset of $\{V_i\}$, which is finite. That shows among those W that $(\pi_2^{(p+1)})^{-1}(W)$ intersects $(\pi_1^{(p+1)})^{-1}(A)$, such W that $(\pi_2^{(p+1)})^{-1}(W)$ meets $(\pi_1^{(p+1)})^{-1}(A)$ at finitely many points, is generic. Thus $(\pi_2^{(p+1)})_* \circ (\pi_1^{(p+1)})^* A$ is of dimension $(n-k+p)(k-p)-1$. □

(3.4) Let $X \subset G(k,n)$ be a twisted imbedding. Then any subvariety of X will be a twisted subvariety of $G(k,n)$. Proposition 3.3 defines a map ϕ:

$$\begin{aligned} \phi\colon \mathcal{C}_p(X) &\to \mathcal{C}_h(G(n-k+p,n)) \\ A &\to (\pi_2^{(p+1)})_* \circ (\pi_1^{(p+1)})^* A, \end{aligned}$$

where $h = (n-k+p)(k-p)-1$. From now on we denote the hypersurface $(\pi_2^{(p+1)})_* \circ (\pi_1^{(p+1)})^* A$ by D_A. One can view D_A as a 'twisted' Chow form of A. We believe the parameter space of effective cycles, defined via D_A is isomorphic to the Chow variety. Here we are not going to verify our speculation, instead, we only prove

PROPOSITION 3.5. *The map ϕ is a morphism.*

Proof. This is an immediate consequence of Barlet's theorem of intersection ([Ba], Chapter VI, Theorem 10), and theorem of direct image ([Ba], Chapter IV, Theorem 6), which say that the maps between Chow varieties induced by intersection and direct image are analytic. Let $e = n-k+p$, $N_1 = \dim(G(k,n))$ and $N_2 = \dim(G(e,n))$. Let π_1 and π_2 be the projections from $G(k,n) \times G(e,n)$ to the first and the second factors respectively. We view ϕ as a composite map

$$\begin{aligned} \mathcal{C}_p(X) &\to \mathcal{C}_{N_1+p}(G(k,n) \times G(e,n)) \\ &\to \mathcal{C}_{N_2-1}(G(k,n) \times G(e,n)) \to \mathcal{C}_{N_2-1}(G(e,n), \\ A &\to \pi_1^*(A) \to \Gamma^{p+1} \cdot \pi_1^*(A) \to (\pi_2)_*(\Gamma^{p+1} \cdot \pi_1^*(A)), \end{aligned}$$

which is a morphism by Barlet's theorems, where the dot $\cdot$ means the intersecting of cycles.

Clearly $(\pi_2)_*(\Gamma^{p+1}.\pi_1^*(A)) = \phi(A)$. □

Remark 3.6. Note $\mathcal{C}_h(G(e,n))$ is the space of hypersurfaces in $G(e,n)$. Hence it is the union of $\mathcal{C}_h^d(G(e,n)) = P(H^0(G(e,n), O(d)))$–the space of hypersurfaces of degree d in $G(e,n)$, which is a projective space, denoted by P_d.

4. 'Moving lemmas'

Several results concerning 'moving lemmas' are needed for the calculations in Section 5. We list them in this section for the convenience of the readers.

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PROPOSITION 4.1. *Let $X \subset P^n$ be an imbedding of the smooth variety X of dimension m. Let Z be an irreducible subvariety, W a subvariety in X. Let L be a linear space of dimension $n-m-1$, and $C_L(Z)$ the cone with vertex L. Then for generic L,*

(a) *$C_L(Z)$ meets X properly, and the intersection is transversal at Z, i.e.*

$$C_L(Z)\cdot X = Z + \sum_{j=1}^{l} Z_j,$$

where $\sum_j Z_j$ called the residual cycle is a cycle that has dimension $\dim(Z)$, and does not contain Z (There might be repeated Z_j).

(b) *If Z does not meet W properly, then $\dim(Z_i \cap W) < \dim(Z \cap W)$ in particular, if $dim(Z\cap W) = 0$, Z_i is disjoint from W.*

(c) *If Z meets W properly, then the intersection of W and Z_i is generically transversal.*

Parts (a) and (b) were proved by J. Roberts ([R]) and part (c) by Hoyt ([Ho]). The following is a special case of a more general statement proved by Kleiman [K2], 8 Corollary.

PROPOSITION 4.2 (Kleiman, [K2]). *Let Z, W be subvarieties of P^n, then there is a nonempty open set T of $P\mathrm{GL}(n+1)$, such that the intersection of $g(Z)$ and W is generically transversal for all $g \in T$.*

PROPOSITION 4.3 (Kleiman, [K1]). *For any $a \in \mathrm{CH}_p(X)$ (the Chow group), there is a twisted imbedding $i\colon X \to G(k,n)$ such that*

$$(m-p-1)!a = i^*(c_{m-p}(Q)) - l[H]^{m-p},$$

where Q is the universal bundle of $G(k,n)$, [H] a hyperplane section, and l some integer.

PROPOSITION 4.4 (Angéniol, [An], Cor. 8.1.2.2). *Let X be a smooth projective variety. Let U be the subset of the product of the Chow varieties $\mathcal{C}_p(X)\times\mathcal{C}_{p'}(X)$, such that the intersection $A\cdot B$ is well-defined for $(A,B)\in U$. Then U is an open set, and the map*

$$\begin{aligned} U &\to C_q(X)\\ (A,B) &\to A\cdot B \end{aligned}$$

is a morphism, where $q = p + p' - \dim(X)$.

5. Calculation of Archimedean Height Pairing

In this section we always assume X is a smooth projective variety of dimension m equipped with a Kähler metric, and A, B are linking pairs in X, i.e. $\dim(A) = p$, $\dim(B) = q$ with $p + q = m - 1$.

5.1. THE CASE WHERE A IS A HYPERSURFACE

Let's see a special case which serves as a prototype for the formula (1.0). Let X be a smooth projective variety of dimension m, B a fixed point in X ($q = 0$), and L a line bundle on X. Give a Fubini–Study metric on the projective space $P(H^0(X, L)) \neq \varnothing$ i.e. projectivization of the space of holomorphic sections of L, which is a subvariety of the Chow variety of X $\mathcal{C}_{m-1}(X)$ (of dimension $m - 1$, and with the rational equivalence class $[L]$). Take any section f_A, which defines a hypersurface A. Given an admissible Hermitian metric* on L, Poincaré–Lelong formula says $dd^c([\log ||f_A||^2]) = \delta_A - [\omega]$, where ω is the curvature form.

To normalize the Green's function, i.e. $\log ||f_A||^2$, we subtract a suitable number $h(f_A)$, where $h(f_A) = \int_X \log ||f_A||^2 \,\mathrm{dv}$ (dv is the volume form defined by the Kähler form with $\int_X \mathrm{dv} = 1$). By the proposition 1.5.1 in [BGS], $h(f_A)$ is a continuous function on $P(H^0(X, L))$. We then have $\langle A, B\rangle = \log(\mathrm{e}^{-h(f_A)} ||f_A(B)||^2)$. One can view $f_A(B)$ as a section s of the $\mathrm{O}(1)$ bundle on $P(H^0(X, L))$ evaluated at $A \in P(H^0(X, L))$, where s is defined by $\{A : s(A) = 0\} = \{A : f_A(B) = 0\}$. The metric $|| \cdot ||_\nu$ on $\mathrm{O}(1)$, which is only continuous is the induced metric from the admissible metric on L multiplied by $\mathrm{e}^{-h(f_A)}$. With above setting, viewing Archimedean height pairing as a function on $P(H^0(X, L))$, we have $\langle A, B\rangle_X = \log||s(A)||^2_\nu$ with s a section of the line bundle $\mathrm{O}(1)$.

5.2. CALCULATION FOR TWISTED CHOW FORMS

Let X be a twisted smooth subvariety of Grassmannian $G(k, n)$. Assume given a Kähler metric on X. Let $\mathcal{C}^\alpha_p(X)$ be the Chow variety of p cycles in the cohomology class α. Recall in Section 3, we let $P_d = P(H^0(G(e, n), \mathrm{O}(d)))$ be the projective space of the hypersurfaces of degree d in $G(e, n)$, where $e = n - k + p$. For a $W \in G(e, n)$, there is a holomorphic section s' of $\mathrm{O}(1)$ bundle on P_d (which should be in the dual of the vector space $H^0(G(e, n), \mathrm{O}(d))$) defined by evaluating $\sigma \in H^0(G(e, n), \mathrm{O}(d))$ at W, i.e. $s'(\sigma) = \sigma(W) \in \mathbb{C} \backsimeq$ fibre of $\mathrm{O}(d)$ at W.

Recall there is a morphism ϕ via the 'twisted Chow divisor' $\mathcal{C}^\alpha_p(X) \subset \mathcal{C}_p(G(k, n)) \to P_d$, where d is the degree of the 'twisted Chow form' D_A (see 3.4) for $A \in \mathcal{C}^\alpha_p(X)$. Then we define a line bundle $\mathcal{L}$ on $\mathcal{C}^\alpha_p(X)$ to be the pull-back $\phi^*\mathrm{O}(1)$ of $\mathrm{O}(1)$ on P_d, and a holomorphic section s_W to be the pull-back $\phi^* s'$ of s'.

* The curvature form is a harmonic form under the given Kähler metric.

PROPOSITION 5.2.1. *Let B be a fixed q-dimensional intersection cycle of a special Schubert cycle $\sigma_{m-q}(W)$ on X, i.e. $B = \sigma_{m-p}(W)\cdot X$. For above line bundle $\mathcal{L}$ and the holomorphic section s_W on $\mathcal{C}_p^{\alpha}(X)$, there is a continuous metric $||\cdot||$ on $\mathcal{L}$ such that* $\log ||s_W||^2(A) = \langle A, B\rangle_X$ *for all A disjoint from B.*

5.2.2 (*Proof*). Let E be a vector space of dimension n equipped with a Hermitian metric, $G(k,n)$ and $G(n-k+p,n)$ the Grassmannians of subspaces imbedded in projective spaces by the Plücker imbedding. Then $G(k,n)$ and $G(n-k+p,n)$ are equipped with the standard Kähler forms which are the restrictions of the induced Fubini–Study forms μ_1, μ_2 on $P(\wedge^k E)$, $P(\wedge^{n-k+p}E)$ respectively. The $U(n)$ action on E gives actions on $G(k,n)$ and $G(n-k+p,n)$. These actions are transitive and preserve the standard Kähler metrics and they are all denoted by $g\in U(n)$. Let $i\colon X\to G(k,n)$ be the inclusion map. Let G_σ be a normalized Green's form of $\sigma_{m-p}(W)$ in $G(k,n)$ with singularity of log-type. Then by (2.3.1), the restriction $i^*(G_\sigma)$ of G_σ to X defines a current that is an unnormalized Green's current of B in X, i.e. as currents in X $dd^c[i^*(G_\sigma)] = \delta_B - [\omega]$, where ω is some smooth form on X, to be precise, the restriction to X of the harmonic form Poincaré dual to $\sigma_{m-p}(W)$ in $G(k,n)$. Let ω_B be the harmonic form on X that is Poincaré dual to B. Since ω and ω_B are both Poincaré dual to B in X, then there exists a smooth form f, such that $dd^c(f) = \omega - \omega_B$. We choose such f that the harmonic projection of f is zero. Let h be the harmonic projection of $i^*(G_\sigma)$. It is clear that $[i^*(G_\sigma) - h + f]$ is a normalized Green's current of B in X. Let A be a p-cycle in Z disjoint from B. We obtain

$$\langle A, B\rangle_X = \langle A, \sigma_{m-p}(W)\rangle_{G(k,n)} + c(A) \tag{5.2.3}$$

where $c(A) = \int_A (f-h)$, is a continuous function of A in $\mathcal{C}_p(X)$, that depends on the Kähler metrics on X and $G(k,n)$, and cohomology class of B.

Next we calculate $\langle A, \sigma_{m-p}(W)\rangle_{G(k,n)}$. Let π_1, π_2 be the projections of

$$G(k,n)G(n-k+p,n)$$

to the two factors, and Γ^{p+1} be the correspondence defined in (3.0). Let $Z = \pi_1^{-1}(A)$ and G_A Green's form of log-type. Note that $\pi_1^*G_A$ defines Green's form of Z in $G(k,n)\times G(n-k+p,n)$. Then $\pi_1^*[G_A]\wedge\delta_{\Gamma^{p+1}}$ is a well defined current (see [GS],(2.1.3.2)) on the product these Grassmannians. Thus $(\pi_2)_*(\pi_1^*[G_A]\wedge\delta_{\Gamma^{p+1}})$ is also well-defined. By (2.3.1),

$$dd^c(\pi_1^*[G_A]\wedge\delta_{\Gamma^{p+1}}) = \delta_{Z\cdot\Gamma^{p+1}} - \pi_1^*[\omega_A]\wedge\delta_{\Gamma^{p+1}},$$

where ω_A is the harmonic form Poincaré dual to A in $G(k,n)$. Thus

$$dd^c((\pi_2)_*(\pi_1^*[G_A]\wedge\delta_{\Gamma^{p+1}})) = (\pi_2)_*(\delta_{Z\cdot\Gamma^{p+1}}) - (\pi_2)_*(\pi_1^*[\omega_A]\wedge\delta_{\Gamma^{p+1}}), \tag{5.2.4}$$

where $(\pi_2)_*(\delta_{Z\cdot\Gamma^{p+1}}) = \delta_{D_A}$ by (3.4).

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Let ω_α be the harmonic $(1,1)$ form Poincaré dual to D_A. Note ω_α only depends on the cohomology class α and the Fubini–Study metric on $G(n-k+p,n)^\star$

It follows from (5.2.4) that, in order to show

$$(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}}) \tag{5.2.5}$$

is a normalized Green's function of D_A, it suffices to prove that

(1) the harmonic projection of $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})$ is zero, i.e.

$$\int_{G(n-k+p,n)} (\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}}) \wedge \mu_2^{N_2} = 0,$$

where $N_2 = \dim(G(n-k+p,n))$, and μ_2 is the Kähler form.

(2) $\omega_\alpha = (\pi_2)_*(\pi_1^*[\omega_A] \wedge \delta_{\Gamma^{p+1}})$, i.e. $(\pi_2)_*(\pi_1^*[\omega_A] \wedge \delta_{\Gamma^{p+1}})$ is harmonic.

(3) $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})$ is smooth outside of $|D_A|$.

To prove (1), it suffices to show

$$\int_{G(n-k+p,n)} (\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}}) \wedge \mu_2^{N_2}$$
$$= \int_{G(k,n)} G_A \wedge ((\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}]) = 0.$$

Let $g \in U(n)$ act on currents, we have $g(\Gamma^{p+1}) = \Gamma^{p+1}$. So

$$g((\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}])$$
$$= (\pi_1)_*(g\delta_{\Gamma^{p+1}} \wedge g(\pi_2)^*[\mu_2^{N_2}]) = (\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}]).$$

Hence, $(\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}])$ is $U(n)$ invariant on a symmetric space and, therefore, a harmonic form. By the normalization condition on G_A,

$$\int_{G(k,n)} G_A \wedge ((\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}])) = 0.$$

To prove part (2), we reverse the order of π_1 and π_2. Then we see that part (2) is equivalent to the above assertion: $(\pi_1)_*(\delta_{\Gamma^{p+1}} \wedge [\pi_2^*\mu_2^{N_2}])$ is harmonic. By the virtue of above proof, we prove part (2).

To prove part (3), we let G_{D_A} be Green's function of D_A in $G(n-k+p,n)$. Then according to (5.2.4),

$$dd^c((\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}}) - G_{D_A}) = 0.$$

* Actually $\omega_\alpha = \mathrm{d}\,\mu_2$ for some integer d depending on α, and clearly d equals to the topological intersection number of α with special Schubert cycle of complimentary dimension in $G(n-k+p,n)$.

Note that we are dealing with currents of degree 0. Therefore $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}}) - G_{D_A}$ is a harmonic current of degree 0 and, in particular, a smooth function. The smoothness of G_{D_A} outside of $|D_A|$, implies the smoothness of $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})$ outside of $|D_A|$. Thus $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})$ is a normalized Green's current of D_A.

By the proposition (1,5.2) of [GBS], the fibre integral $\int_{\pi_2^{-1}(W)\cdot\Gamma^{p+1}} G_A$ coincides with the continuous function $(\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})$ outside of $|D_A|$. Hence

$$\begin{aligned}
&\langle D_A, W\rangle_{G(n-k+p,n)}\\
&\quad = (\pi_2)_*(\pi_1^*[G_A] \wedge \delta_{\Gamma^{p+1}})(W) = \int_{(\pi_2^{-1}(W))\cdot\Gamma^{p+1}} G_A\\
&\quad = \langle A, \sigma_{m-p}(W)\rangle = \langle A, B\rangle_X - c(A).
\end{aligned}$$

Applying 5.1 to the pairing $\langle D_A, W\rangle_{G(n-k+p,n)}$, we know there exist a section s' of the O(1) bundle on $P(H^0(G(n-k+p,n), \mathrm{O}(d))$, and a continuous metric $||\cdot||_1$ on $\mathrm{O}(1)$, such that $\langle D_A, W\rangle = \log||s'||_1^2$. Note $\langle A, B\rangle = \langle D_A, W\rangle - c(A)$. Then pulling back the section s', and the line bundle $\mathrm{O}(1)$ to $\mathcal{C}_p^\alpha(X)$ via the map ϕ (from (3.5)), and multiplying the pull-back metric by $\mathrm{e}^{\frac{-c(A)}{2}}$, we obtain the section $s = \phi^* s'$, the line bundle $\mathcal{L} = \phi^*(\mathrm{O}(1))$ and the metric $||\cdot||$, such that $\langle A, B\rangle = \log||s||^2(A)$. This completes the proof of 5.2.1. □

5.3. THE PROOF OF THEOREM 1.1.2

We now go back to the general X. Recall $\mathcal{C}_p^\alpha(X)$ is the Chow variety of the p-cycles in the cohomology class α. Let $\mathcal{U}_B$ be the open set of $\mathcal{C}_p^\alpha(X)$ that consists of the p-cycles A disjoint from B (i.e. their supports are disjoint) for a q-cycle B.

PROPOSITION 5.3.1. *Let B be a q-cycle rationally equivalent to zero. Then there exists a rational function r on the closure $\bar{\mathcal{U}}_B$ of $\mathcal{U}_B$, such that $\log|r|^2(A) = \langle A, B\rangle$ for $A \in \mathcal{U}_B$.*

One can prove the proposition by using Hain's result in [H]. Such a proof will appear in [W2], where the relation with Intermediate Jacobians will be emphasized. But here we present another proof by using Gillet–Soulé's analytic tool – 'Green's Currents' – for on the one hand, we would like to maintain the consistent approach of 'Green's currents' throughout the paper, and on the other hand we hope this method alone might be of interest in the study of the incidence relation.

Proof. Note first that since the height pairing is additive on X, it suffices to assume there is only one irreducible $q+1$ variety Z in X with a rational function r_1 on Z, such that $\mathrm{div}(r_1) = B$ and, secondly, for the technical reason we need to work on an irreducible component of $\mathcal{C}_p^\alpha(X)$, where there is at least one cycle disjoint

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from B. But we'll see the construction of the rational function r is independent of such an irreducible component. Therefore, r will be defined on $\mathcal{U}_B$, which is the union of all these irreducible components. We let $\mathcal{C}$ be an irreducible component of $\mathcal{C}_p^\alpha(X)$, in which there is at least one cycle A disjoint from B (i.e the intersection of $\mathcal{U}_B$ with $\mathcal{C}$ is nonempty). We will work on $\mathcal{C}$.

Step 1. In this step we assume

$(*)$ There is an $A \in \mathcal{C}$ such that A intersects Z properly at the smooth locus of Z, i.e $|A| \cap |Z|$ consists of finitely many smooth points of Z.

Let $\pi\colon \tilde{Z} \to Z$ be a smooth resolution of Z, $i\colon Z \to X$ the inclusion map, and $f = i \circ \pi$. Give a Kähler metric to $\tilde{Z}$, such that the induced volume form μ is normalized, i.e. $\int_{\tilde{Z}} \mu = 1$. Let a be the intersection number of Z with α. Let G_A be a normalized Green's form of A of logarithmic type. Then by (2.3.1), $[f^*(G_A)]$ is a current on $\tilde{Z}$, and $dd^c[f^*(G_A)] = \delta_{f^*A} - [f^*\omega_\alpha]$, where ω_α is the harmonic form representing α on X. Note that since we assume A intersects Z properly at smooth locus of Z, f^*A is supported on a points (with multiplicity). That shows $[f^*G_A]$ is an unnormalized Greens' current of f^*A. As before, in order to obtain a normalized Green's current, one needs to subtract the harmonic projection ψ of $[f^*(G_A)]$ and add a smooth form ϕ_α orthogonal to closed forms on $\tilde{Z}$, which satisfies $dd^c\phi_\alpha = f^*\omega_\alpha - a\mu$. Then

$$f^*G_A + \phi_\alpha - \psi \tag{5.3.2}$$

is a normalized Green's form of f^*A.

We shall prove

$$f_*(\delta_{f^*B}) = \delta_B. \tag{5.3.3}$$

By Proposition 2.1.4, it suffices to prove $f_*f^*B = B = \operatorname{div}(r_1)$ in $Z_{q+1}(Z)$. A formula in [F], p. 34, says $f_*(f^*B) = \deg(\tilde{Z}/Z)B$. Here $\deg(\tilde{Z}/Z) = 1$, since $\tilde{Z}$ and Z are birational. Therefore $f_*(f^*B) = B$. This proves (5.3.3).

Next we calculate $\langle A, B\rangle_X$. Because G_A is smooth in a neighborhood of $|B|$, then

$$\langle A, B\rangle_X = \int_B G_A$$

(by 5.3.3)

$$= \int_{f_*(f^*B)} G_A$$

(by the definition of the push forward of currents. Note $\operatorname{div}(r_1 \circ \pi) = f^*B$)

$$= \int_{f^*B} f^*G_A = \int_{\operatorname{div}(r_1\circ\pi)} f^*(G_A)$$

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(by 5.3.2)

$$= \langle \mathrm{div}(r_1 \circ \pi), f^*A \rangle - \int_{f^*B} \phi_\alpha + \int_{f^*B} \psi.$$

Note $\int_{f^*B} \psi = 0$ (ψ is harmonic, and f^*B is homologous to zero), and

$$\langle\, \mathrm{div}(r_1 \circ \pi), f^*A \rangle = \left(\int_{f^*A} \log |r_1 \circ \pi|^2 \right) - ka,$$

where $k = \int_{\tilde{Z}} \log |\pi \circ r_1|^2 \mu$ is the harmonic projection of $\log |r_1 \circ \pi|^2$ (since the Green's form of $\mathrm{div}(r_1 \circ \pi)$ is $\log |r_1 \circ \pi|^2 - k$ by Poincaré Lelong formula, i.e

$$dd^c(\log |r_1 \circ \pi|^2) = \delta_{f^*A} \quad \text{on } \tilde{Z}).$$

Therefore

$$\langle A, B \rangle_X = \left(\int_{f^*A} \log |r_1 \circ \pi|^2 \right) - \left(\int_{f^*B} \phi_\alpha \right) - ka \tag{5.3.4}$$

To view (5.3.4) as the Chow variety, we let $\mathrm{Sym}^a(\tilde{Z})$ be the symmetric product of $\tilde{Z}$, which is the 0-Chow variety of $\tilde{Z}$ with degree a. Then the rational function $r_1 \circ \pi$ induces a rational function r_2 on $\mathrm{Sym}^a(\tilde{Z})$

$$r_2(z_1, \cdots, z_a) = \prod_i r_1 \circ \pi(z_i).$$

Let $z_1(A) + \cdots + z_a(A)$ be the zero cycle f^*A, then the formula (5.3.4) yields

$$\log |r_2(z_1(A), \ldots, z_a(A))|^2 - \langle A, B \rangle_X = ka + \int_{f^*B} \phi_\alpha. \tag{5.3.5}$$

Finally, we interpret f^*A. By Proposition 4.4, there is an open set U in $\mathcal{C}$, which is nonempty by our assumption (consisting of the cycles A which have proper intersection with Z), such that intersection map Z

$$U \to \mathcal{C}_0(Z) = \mathrm{Sym}^a(Z)$$

$$A \to Z \cdot A$$

is a morphism. Thus we have a composite rational map j

$$\mathcal{C}_p^\alpha \to \mathcal{C}_0(Z) = \mathrm{Sym}^a(Z) \xrightarrow{\pi_a^{-1}} \mathrm{Sym}^a(\tilde{Z}),$$

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where π_a^{-1} is the obvious rational map induced from π^{-1}. Formula (5.3.5) becomes

$$\log|r_2\circ j(A)|^2 - \langle A, B\rangle_X = ka + \int_{f^*B}\phi_\alpha, \tag{5.3.6}$$

where the right-hand side is independent of A and $r_2\circ j$ is a rational function on $\mathcal{C}$. Note that (5.3.6) only holds for those A that intersect Z properly at smooth points. But the set of those A is an open set, which by our assumption is nonempty, therefore dense in the irreducible component $\mathcal{C}$ of the Chow variety. The continuity of the Archimedean height pairing allows us to extend (5.3.6) to the entire $\mathcal{C}\cap\mathcal{U}_B$.

The constant term (with respect to A) will now be calculated and shown to be independent of the resolution $\tilde{Z}$ of Z and so dependent only on Z, B and r_1.

$$ka + \int_{f^*B}\phi_\alpha = ka + \delta_{f^*B}(\phi_\alpha)$$

(use the formula just before 5.3.4)

$$= ka + \int_{\tilde{Z}}\log|\pi\circ r_1|dd^c\phi_\alpha$$

$$= ka + \int_{\tilde{Z}}\log|\pi\circ r_1|^2(f^*\omega_\alpha - a\mu)$$

(use the definition of k)

$$= \int_Z \log|r_1|^2\omega_\alpha.$$

At this point we have proved Proposition 5.3.1 under the assumption $(*)$ except for a constant which can be absorbed into the rational function $r_2\circ j$.

Step 2. We now drop the assumption $*$ in step 1. We first fix an imbedding $X\subseteq P^n$, and a rational function r_2 on P^n inducing the rational function r_1 on Z whose divisor is B. We proceed by the induction on $\dim(A\cap Z)$. Suppose for A, $e=\dim(A\cap Z)>0$. By Propositions 4.1, 4.2 there is a linear space L of dimension $n-m-1$ in P^n, such that the cone $C_L(Z)$ meets X transversally at Z and

(a) $$C_L(Z)\cdot X = Z + \sum_j Z_j,$$

(b) for each irreducible variety Z_j, $\dim(Z_j\cap A) < e$, Z_j is not contained in $\mathrm{div}(r_2)$ and $\mathrm{div}(r_2|_{Z_j})$ is disjoint from A,

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(c) also there is a Zariski open set T in $PGL(n+1)$, such that for $g \in T$, A meets $g(C_L(Z))$ transversally at finitely many points, the intersection

$$g(C_L(Z)) \cdot X = \sum_i Z'_{g,i}$$

is proper, and $\operatorname{div}(r_2|_{Z'_{g,i}})$ is well-defined and disjoint from A.

Note the support of A is contained in X, and $g(C_L(Z)) \cdot X = \sum_i Z'_{g,i}$. We have

$$A \cdot \sum_i Z'_{g,i} = A \cdot g(C_L(Z)).$$

Since the intersection of A and $g(C_L(Z))$ is transversal, then the intersection of A and $\sum_i Z'_{g,i}$ is also transversal. The assumption $(*)$ in the first step then is satisfied for the cycles A, $\sum_i Z'_{g,i}$. We can apply the step 1 to the cycle $B = \sum_i \operatorname{div}(r_2|_{Z'_{g,i}})$, i.e. by Equation (5.3.6) in step 1, there is a constant c_g, and rational function r_g on $\mathcal{C}$ such that

$$\sum_i \langle \operatorname{div}(r_2|_{Z'_{g,i}}), A\rangle = \log|r_g|^2(A) - c_g. \tag{5.3.7}$$

We will discuss the behavior of each term in (5.3.7) as g approaches to the identity I. Recall $c_g = \sum_i \int_{Z'_{g,i}} \log|r_2|^2\omega_\alpha$. By the continuity of fibre integrals in [BGS], Proposition 1.5.1, the limit c_I of c_g is $\int_Z \log|r_2|^2\omega_\alpha + \sum_j \int_{Z_j} \log|r_2|^2\omega_\alpha$ which is finite. Also due to the continuity of fibre integrals, the limit of

$$\sum_i \langle \operatorname{div}(r_2|_{Z'_{g,i}}), A\rangle = \int_{\sum_i \operatorname{div}(r_2|_{Z'_{g,i}})} G_A$$

is (here the cycles being paired do not intersect and so we are integrating smooth form G_A) $\langle B, A\rangle + \sum_j \langle \operatorname{div}(r_2|_{Z_j}), A\rangle$, which is finite. Lastly we examine r_g. By (4.4), there is a rational map (using intersection),

$$T \times \mathcal{C} \xrightarrow{\phi} \operatorname{Sym}^a(X)$$

$$(g, A) \longrightarrow g(C_L(Z)) \cdot A.$$

By the construction of r_g in the step 1, r_g is the composition with ϕ of a rational function on $\operatorname{Sym}^a(X)$ which is the product of a copies of the rational function r_2. Therefore r_g is a rational function $r(g, A)$ on $T \times \mathcal{C}$ or $PGL(n+1) \times \mathcal{C}$. The existence of the limits of other two terms in (5.3.7) as $g \to I$ implies $r(g, A)$ restricts to a well-defined rational function $r(I, A) = r_I$ on $\mathcal{C}$. Taking the limits of (5.3.7), we have

$$\langle B, A\rangle + \sum_j \langle \operatorname{div}(r_2|_{Z_j}), A\rangle = \log|r_I|(A) - c_I. \tag{5.3.8}$$

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Thus

$$\langle B, A\rangle = \log|r_I|(A) - c_I - \sum_j \langle \mathrm{div}(r_2|_{Z_j}), A\rangle,$$

where $\dim(Z_j \cap A) < e$. The induction can be performed on $\dim(Z_j \cap A)$ until the residual cycle $\sum_j Z_j$ intersects A transversally. Then apply step 1 to complete the proof. $\square$

(5.3.9) *Proof of Theorem 1.1.2*. By the reduction to the diagonal ([Bo]), $\langle A, B\rangle_X = \langle A \times B, \Delta\rangle_{X\times X}$, where Δ is the diagonal. We then consider the height pairing between $A \times B$ and Δ in $X \times X$ with the cycle Δ fixed. By Proposition 4.3, there exist a twisted imbedding i: $X \times X \to G(k, n)$, a special Schubert cycle σ_m that represents the mth Chern class of the quotient bundle, and a m-codimensional plane section H^m such that

$$(m-1)!\Delta - (\sigma_m \cdot i(X \times X) - lH^m)$$

is rationally equivalent to zero on $X \times X$, where l is some integer. Let $\mathcal{C}^{\gamma}_{m-1}(X \times X)$ be the Chow variety with cohomology class γ of $A \times B$ in $X \times X$. Let $\mathcal{U}_\Delta$ be the open set of $\mathcal{C}^{\gamma}_{m-1}(X \times X)$ that consists of the $(m-1)$-cycles in $X \times X$ disjoint from Δ. Let $\mathcal{U}_1$, $\mathcal{U}_2$ be the open sets in $\mathcal{C}^{\gamma}_{m-1}(X \times X)$ that consist of $(m-1)$ cycles disjoint from $\sigma_m \cdot i(X \times X)$ and H^m respectively. We may also assume in each irreducible component of $\mathcal{U}_\Delta$ there is at least one cycle C disjoint from σ_m and H^m (it can be achieved by choosing a general position of the special Schubert cycle σ_m, and the plane section H^m). Therefore the closure $\overline{\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta}$ is equal to the closure $\bar{\mathcal{U}}_\Delta$. Note for a cycle $C \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta$, we have

$$\begin{aligned}\langle (m-1)!\Delta, C\rangle &= \langle (m-1)!\Delta - (\sigma_m \cdot i(X \times X) - lH^m), C\rangle + \\ &\quad + \langle \sigma_m \cdot i(X \times X), C\rangle - \langle lH^m, C\rangle.\end{aligned}$$

By (5.3.1), there is a rational function r on $\bar{\mathcal{U}}_\Delta = \overline{\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta}$, such that

$$\langle (m-1)!\Delta - (\sigma_m \cdot i(X \times X) - lH^m), C\rangle = \log|r|^2 \tag{5.3.10}$$

for $C \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta$. By (5.2.1), and Corollary 2.2.8 in [W1], there are metrized line bundles L_1, L_2, with rational sections s_1 and s_2 on $\mathcal{C}^{\gamma}_{m-1}(X \times X)$ such that

$$\langle \sigma_m \cdot i(X \times X), C\rangle = \log||s_1||^2, \tag{5.3.11}$$

$$\langle lH^m, C\rangle = \log||s_2||^2 \tag{5.3.12}$$

for $C \in \mathcal{U}_1 \cap \mathcal{U}_2$. Let $L = L_1 \times L_2^{-1}$, and $s' = r \cdot s_1 \cdot s_2^{-1}$. Metrize L by multiplying two continuous metrics of L_1 and L_2. We obtain the metrized line bundle L and a section s' on $\bar{\mathcal{U}}_\Delta$. Combining formulas (5.3.10), (5.3.11) and (5.3.12), we obtain

$$\langle \Delta, C\rangle_{X\times X} = \frac{1}{(m-1)!} \log||s'||^2 \tag{5.3.13}$$

for $C \in \mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta$. By the continuity of the Archimedean height pairing function, and the fact the closure of $\mathcal{U}_1 \cap \mathcal{U}_2 \cap \mathcal{U}_\Delta$ is equal to the closure of $\mathcal{U}_\Delta$, we can extend (5.3.13) to $\mathcal{U}_\Delta$.

Finally we need to have everything on $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$. By a proposition of Barlet ([Ba], Chapter VI Prop. 1),

$$\begin{aligned} &\times\colon \mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X) \to \mathcal{C}_{m-1}^\gamma(X \times X) \\ &(A, B) \to A \times B \end{aligned}$$

is a morphism. Especially the restriction to $\times|_{\bar{\mathcal{U}}}\colon \bar{\mathcal{U}} \to \bar{\mathcal{U}}_\Delta$ is a morphism (recall $\mathcal{U}$ is an open set of $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ consists of all the disjoint pairs of cycles A, B). Using the map $\times|_{\bar{\mathcal{U}}}$ to pull back L and s', we obtain a line bundle $\mathcal{L}$ and a section s on $\bar{\mathcal{U}}$ as required.

To see the uniqueness, we let $\mathcal{L}_1$ and s_1 be another metrized line bundle and section satisfying 1.1.3. Then

$$\log \frac{||s||}{||s_1||_1} = 1.$$

This implies that the line bundle $L \times L_1^{-1}$ is a trivial bundle, and $s \cdot s_1^{-1}$ is a constant. This completes the proof. □

6. Positivity and effectivity of the incidence divisor

6.1. EFFECTIVITY

The incidence divisor is defined to be $D = \mathrm{div}(s)$. We can now define the duality map induced by the intersection with D. That is

$$\begin{aligned} \mathcal{C}_p^\alpha(X) \quad &\xrightarrow{\mathcal{D}} \quad Ca(X) \\ A \quad &\to \quad (P_2)_*(D \cdot (\{A\} \times \mathcal{C}_q^\beta(X))), \end{aligned}$$

where P_2 is the second projection from $\mathcal{C}_p^\alpha(X) \times \mathcal{C}_q^\beta(X)$ to $\mathcal{C}_q^\beta(X)$, and $Ca(X)$ is the group of Cartier divisors. Note $\mathcal{D}(A)$ is not always well-defined on all the components of $\mathcal{C}_p^\alpha(X)$ (see Example 6.3.3). Since the section s may be identically 0, or has a pole everywhere on some components. But this will not happen on the component of $\mathcal{C}_q^\beta(X)$ in which there is at least one cycle B disjoint from A.

(6.1.1) We would like to make some comments on our definition of the incidence divisor $\mathcal{D}(A)$. We believe the definition above is equivalent to those of Griffiths in [G1], [G2] (up to torsion). In [G1], Griffiths defined the incidence divisor $\mathcal{D}(A)$ on a suitable irreducible component of Chow variety, later in [G2] he, in a precise way, defined $\mathcal{D}(A)$ modulo rational equivalence for any smooth parameter space S. His approach put the emphasis on algebraic cycles and their equivalences, not

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the parameter space. Thus the study of the Chow variety was not issue there. Our construction with the approach of Archimedean height pairing allows us to define $\mathcal{D}(A)$ in a universal sense as a Cartier divisor, therefore also contains the information of Chow varieties. More on this will appear in [W2].

In [BGS], Bost, Gillet, and Soulé defined an effective Arithmetic cycle (A, G_A) by imposing a positivity condition on G_A, where A is an effective cycle, and G_A is an unnormalized, positive Green's form. They further studied the existence of such positive Green's currents. From our point of view, the effectivity of incidence divisor $D = \mathrm{div}(s)$ is related to the Bost–Gillet–Soulé positivity of Green's currents.

PROPOSITION 6.1.2. *If there exists a positive Green's current G_A for the effective cycle A on X, then the divisor $\mathcal{D}(A)$ is effective on $\mathcal{C}_q^\beta(X)$. Furthermore, if there exist positive Green's currents for the diagonal in $X \times X$, then the incidence divisor D is effective on $\mathcal{C}_p(X) \times \mathcal{C}_q(X)$.*

Proof. If $\mathcal{D}(A)$ is not effective, let V_- be an irreducible subvariety with a negative coefficient in $\mathcal{D}(A)$. Choose a point B_0 on V_-, but not on any other varieties appearing in $\mathcal{D}(A)$. Take any holomorphic path B_t on $\mathcal{C}_q(X)$ passing through B_0 at $t = 0$. Then the section $s(A, B_t)$ has a pole at $t = 0$. Due to the positivity of Green's current, as $t \to 0$, the left-hand side of the formula $\langle A, B_t\rangle = \lambda \log ||s(A, B_t)||^2$ is bounded above, while the right-hand side clearly goes to positive infinity. That is a contradiction. The same proof is applied to the second assertion.

Remark 6.1.3. We don't see any convincing evidence that the converse statement of Proposition 6.1.2 should be true. In [BGS], it is proved that on a compact homogeneous space, the diagonal has a positive Green's current, therefore the incidence divisor for such a space is effective, in other words the incidence section s in Theorem 1.1.2 is holomorphic.

6.2. POSITIVITY

Following the definition of Griffiths ([G2]), we define:

DEFINITION 6.2.1. An algebraic cycle A is positive, if $\mathcal{D}(A)$ is ample on any component $\mathcal{C}_q^\beta(X)$.

PROPOSITION 6.2.2. *Let A be an algebraic cycle, and its rational equivalence class is a positive linear combination of Chern classes of very ample vector bundles, then A is positive.*

Proof. It suffices to assume the rational equivalence class of A is the Chern class of a very ample vector bundle. By Proposition 5.3.1, changing the cycle A in its rational equivalence class does not change the line bundle determined by $\mathcal{D}(A)$. We may then assume that X is imbedded in a Grassmannian, and A is the cut-out of a special Schubert cycle. The situation is the same as in Proposition 5.2.1, in which the incidence line bundle of A we constructed is ample.

6.3. EXAMPLES

We discuss the coefficient $1/(m-1)!$, which we don't know if is the best, i.e. closest to 1 for all the X. For a specific X, the best λ is a numerical invariant of X, denoted by λ_X. For a specific irreducible component $\mathcal{C}$, the best λ is an invariant of $\mathcal{C}$, denoted by $\lambda_{X,\mathcal{C}}$. In [W1] we see for a projective space $\mathbb{P}^n$, $\lambda_{\mathbb{P}^n} = 1$, thus $\lambda_{\mathbb{P}^n,\mathcal{C}} = 1$ for all the components $\mathcal{C}$. It seems that, in general, for most of the components, the coefficient should be 1. But we can't prove it is always the case, nor can we give any example with $\lambda \neq 1$.

Intersection with the incidence divisor provides us with a microscope to look into the singular nature of Chow varieties. The full account of intersection theory involving D will not be clear until the coefficients $\lambda_{X,\mathcal{C}}$ are understood. In the following discussion, for the sake of convenience, we'll fixed cycle A.

EXAMPLE 6.3.1. Let $X \subset P^n$. Take a subvariety Z of dimension $q+1$ in X. Assume A intersects Z transversally at d many points $\{x_i\}$. Let H_t be a family of hyperplanes with $t \in P^1$, and $\cup_t H_t = P^n$. Assume each H_t contains at most one x_i. Corresponding t is denoted by t_i. Then $Z_t = Z \cdot H_t$ gives a morphism f, $P^1 \to \mathcal{C}_q(X)$ whose image meets $\mathcal{D}(A)$ at the points Z_{t_i}. Let $\mathcal{C}^Z$ be the irreducible component of $\mathcal{C}_q(X)$, that contains the image. By the asymptotic calculation in [W3] or [W4], as $t \to t_i$

$$\langle A, Z_t\rangle = \log|t-t_i|^2 + \mathrm{O}(1).$$

By the definition in [F], Chapter 7, as $t \to t_i$,

$$s(A, Z_t) = (t-t_i)^{\mu} + \mathrm{O}(1),$$

where μ is the intersection multiplicity $i(t_i, \mathcal{D}(A)\cdot P^1, \mathcal{C}^Z)$. Comparing both sides of

$$\lambda_{X,\mathcal{C}^Z} \log|s(A, Z_t)|^2 = \langle A, Z_t\rangle,$$

we then have

$$\lambda_{X,\mathcal{C}^Z} = \frac{1}{i(t_i, \mathcal{D}(A)\cdot P^1, \mathcal{C}^Z)}. \tag{6.3.2}$$

EXAMPLE 6.3.3. Let $\tilde{P}$ be the blow-up of projective space P^2 of dimension 2 at one point, $E \subset \tilde{P}$ the exceptional divisor. Use the complex variable $t \in \mathbb{C} \cup \{\infty\}$ to express the points in projective space P^1 of dimension 1. Let $X = \tilde{P} \times P^1$.

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(1) Let $A = E \times \{0\}$ and $B_t = E \times \{t\}$. P^1 is the parameter space that parameterizes cycles B_t on X. In this case, the incidence divisor $\mathcal{D}(A)$ is $-\{0\}$ which is not effective.
(2) Let $A = \{e\} \times P^1$ for $e \in E$ and $B_t = E \times \{t\}$. Then the incidence divisor is not defined on P^1, since A meets all the cycle B_t.
(3) Let C be a curve in $\tilde{P}$ that does not meet E. Let $A = C \times \{0\}$. Then $\mathcal{D}(A) = 0$ on P^1.

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