Canad. Math. Bull. Vol. **37** (4), 1994 pp. 473–481

DISMANTLABILITY REVISITED FOR ORDERED SETS AND GRAPHS AND THE FIXED-CLIQUE PROPERTY

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ABSTRACT. We give a unified treatment of several fixed-point type theorems by using the concept of dismantlability, extended from ordered sets to arbitrary graphs. For a graph G and a vertex x of G we let $N_G(x)$ denote the set of neighbours of x in G. We say that x is a *subdominant* vertex of G if there is a vertex y of G, distinct from x, such that $N_G(x) \cup \{x\} \subseteq N_G(y) \cup \{y\}$. If G has n vertices we say that G is *dismantlable* if the vertices of G can be listed as $x_1, x_2, \ldots, x_i, \ldots, x_n$ such that, for all $i = 1, 2, \ldots, n-1$, x_i is a subdominant vertex of the graph $G_i = G - \{x_j : j < i\}$.

Theorem. *Let G be a graph. Let* Clique(G) *denote the set of all non-empty cliques in G. Then G is dismantlable if, and only if,* (Clique(G), $\subseteq$) *is a dismantlable ordered set.*

Corollaries of this result include known fixed-point type theorems for homomorphisms of trees, and for order reversing functions on dismantlable ordered sets, as well as the following.

Corollary. *Let G be a connected, triangulated graph. Then for any homomorphism $f: G \longrightarrow G$ there is a non-empty clique K in G such that $f[K] = K$.*

1. **Introduction.** The concept of *dismantlability* for ordered sets, introduced by I. Rival in [11], has proven to be a very useful and interesting idea in the study of ordered sets, particularly in connection with the fixed-point property. Since the original proof of the fixed-point property for dismantlable ordered sets in [11], a great deal of significant work in this area has utilized the concept of dismantlability. In particular, we cite the work of K. Baclawski and A. Bjorner in [2], and J. Walker's paper [12]. These papers have provided much of the motivation for the present work.

Our intention in this paper is to use the extended notion of dismantlability for undirected graphs to unify several known fixed-point type theorems, including the result in [2] concerning order-reversing functions on dismantlable ordered sets, and the result in [8] on homomorphisms of trees. The notion of dismantlability for graphs has been employed in [9] to characterize the so-called "cop-win" graphs in connection with a natural vertex to vertex pursuit game, and in the work in [1] on bridged graphs.

We will now set forth the basic definitions and notation to be employed in the sequel. In this paper all graphs and ordered sets are assumed to be finite, and by the term *graph*, we mean a simple, undirected graph without loops.

The author gratefully acknowledges a grant from NSERC in support of this work.

Received by the editors March 11, 1993; revised August 25, 1993.

AMS subject classification: 05C99, 06A10.

Key words and phrases: graph, ordered set, subdominant vertex, dismantlable, graph homomorphism, fixed-point, clique.

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https://doi.org/10.4153/CMB-1994-069-6 Published online by Cambridge University Press

Let $(P, \leq)$ be an ordered set. For elements x, y of P, we say that y *covers* x in P if $x < y$ and there is no element z of P for which $x < z < y$. In this case, x is called a *lower cover* of y in P, and y is called an *upper cover* of x in P. As in [11], an element a of P is said to be an *irreducible* element of P if either a has exactly one upper cover in P or a has exactly one lower cover in P. P is said to be *dismantlable* if the elements of P can be listed as $x_1, x_2, \ldots, x_n$, (where $n = |P|$), such that, for all $i = 1, 2, \ldots, n-1$, x_i is an irreducible element of the ordered set $P - \{x_j : j < i\}$.

Let P and Q be ordered sets and let $f: P \longrightarrow Q$ be a function. Recall that f is said to be *order-preserving* if, for all x, y in P, $x \leq y$ implies $f(x) \leq f(y)$; *order-reversing* if $x \leq y$ implies $f(x) \geq f(y)$, and *comparability-preserving* if, for all x, y in P, $x \leq y$ implies that either $f(x) \leq f(y)$ or $f(x) \geq f(y)$.

Let G be a graph. We may also denote G by writing $G = (V, E)$, where V is the set of vertices of G and E is the set of edges of G. If x is a vertex of G we let $N_G(x) = \{v \in V : v$ is adjacent to x in $G\}$. This set will also be denoted by $N(x)$ when the graph G is clear from context. As usual, a subset K of V is called a *clique* in G if x is adjacent to y for all x, y in K. We let Clique(G) denote the set of all *non-empty* cliques in G. Clique(G) is regarded as an ordered set under set inclusion $\subseteq$. Recall that a vertex a of G is called a *universal vertex* of G if a is adjacent to all other vertices of G, that is if $N_G(a) = V - \{a\}$. And recall that a vertex a of G is called a *simplicial vertex* of G if the set $N_G(a)$ is a clique in G.

As in [9], we now extend the concepts of irreducible element and dismantlability to graphs. Since the definition is closely related to the so-called *vicinal pre-order* on graphs, also called the *domination pre-order* (see [6]), we have chosen to use the term "subdominant" in place of "irreducible". A vertex a in a graph G is called a *subdominant vertex* of G if there is a vertex b in G, distinct from a, such that $\{a\} \cup N_G(a) \subseteq \{b\} \cup N_G(b)$. In this case we will say that b *dominates* a, or that a is dominated by b. This means that a is adjacent to b and every vertex of G which is adjacent to a is also adjacent to b. A graph G is said to *dismantlable* if the vertices of G can be listed as $x_1, x_2, \ldots, x_n$, (where $n = |V|$), such that, for all $i = 1, 2, \ldots, n-1$, x_i is a subdominant vertex of the induced subgraph $V - \{x_j : j < i\}$.

For an ordered set $(P, \leq)$, we will let Comp(P) denote the *comparability graph of P*. This is the graph whose vertex set is P, and in which two vertices x, y of P are adjacent if x and y are comparable in P (that is, if either $x < y$ or $y < x$).

Let G and H be graphs and let $f: G \longrightarrow H$ be a function. f is called a *homomorphism* if, for all vertices x, y of G, if x and y are adjacent in G, then either $f(x) = f(y)$ or $f(x)$ is adjacent to $f(y)$ in H.

Finally, we recall that an ordered set P has the *fixed-point property* if every order-preserving function $f: P \longrightarrow P$ has a fixed-point (that is, there is an element $x \in P$ such that $f(x) = x$.)

The genesis of this paper, and for much of the recent work concerning dismantlability, is I. Rival's result that *if P is a dismantlable ordered set, then P has the fixed point property*. ([11]) While an obvious extension to graphs would be to consider fixed vertices

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of homomorphisms of a graph to itself, a somewhat modified version of this idea will be more useful here, namely *fixed cliques* of homomorphisms of a graph to itself. We will say that a graph G has the *fixed-clique property* if for every homomorphism $f\colon G \longrightarrow G$ there is a non-empty clique K in G such that $f[K] = K$. (Such a clique K will be called a *fixed clique* for f).

The work in this paper will revolve around our generalization of Rival's theorem to graphs, namely, that *every dismantlable graph has the fixed-clique property*. Our proof of this result, (in the next section), and the fact that other fixed-point type theorems can be deduced from it (including some known results established by entirely different methods), come about by relating the graph G and the ordered set $\big(\text{Clique}(G), \subseteq\big)$. These matters will be discussed in the next section. For the moment, let us consider a natural question concerning the two notions of dismantlability for ordered sets and graphs, as applied to an ordered set P and its comparability graph Comp(P). One would expect the two notions to be equivalent (if we have generalized the definition to graphs in a sensible way). This is in fact true, and will be noted explicitly in the following section, but it does not follow immediately from the definitions of dismantlability. It is obvious that, if y is the unique upper cover (or unique lower cover) of x in an ordered set P, then $\{x\} \cup N(x) \subseteq \{y\} \cup N(y)$ in the comparability graph of P, since the latter simply means that any element of P which is comparable to x is also comparable to y. In this case x is a subdominant vertex of Comp(P). These remarks clearly show that, if P is a dismantlable ordered set, then Comp(P) is a dismantlable graph. The converse is not quite as obvious—the point being that it is possible, for elements x, y in an ordered set P, that every element of P which is comparable to x is also comparable to y, even though neither of the two elements covers the other. Now, if x and y are two such elements, we have either $x < y$ or $y < x$. Let us suppose that $x < y$. There is an element x_1 in P such that $x \leq x_1 < y$ and y covers x_1. It is clear that y is the unique upper cover of x_1 in P, and hence x_1 is an irreducible element of P. However this observation is still not sufficient to enable us to deduce that P is dismantlable when the graph Comp(P) is. We will return to this point in the next section, where we will complete the proof that the two notions are in fact equivalent.

2. **The equivalence of the dismantlability of** G **and** Clique(G). Let $G = (V, E)$ be a graph. Let $C_1, C_2, \ldots, C_i, \ldots$ be a listing of all the cliques in G, in order of decreasing cardinality. That is, if C_i and C_j are any two cliques in G for which $|C_i| < |C_j|$, then $i > j$. Such a listing is, of course, not unique. Suppose we have chosen such a listing. It will be useful in describing the order in which we delete the elements of Clique(G) in discussing dismantlability below.

Before we approach the main result of this section, there are two useful observations to make. First, note that, if x and y are vertices of G such that x is dominated by y in G, then for every clique K in G which contains x, $K \cup \{y\}$ is also a clique. And secondly, suppose that G is dismantlable, and that $x_1, x_2, \ldots, x_n$ is a sequence which witnesses this. Then we note that x_1 is a subdominant vertex of G, and that the induced subgraph $V - \{x_1\}$ is also dismantlable.

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Suppose P is an ordered set and that $Q \subseteq P$. We will say that P *can be dismantled to* Q if there is a sequence of elements $x_1, x_2, \ldots, x_k$ in P such that, for all $i = 1, 2, \ldots, k$, x_i is irreducible in $P - \{x_j : j < i\}$, and $Q = P - \{x_i : i = 1, 2, \ldots, k\}$.

LEMMA 2.1. *Let G be a graph and let a be a subdominant vertex of G. Then the ordered set* Clique(G) *can be dismantled to* Clique($G - \{a\}$).

PROOF. Let b be a vertex in G which dominates a. We will first show how to successively delete all the cliques in G which contain a but do not contain b, and then we will successively delete all the cliques in G which contain both a and b. What will remain, of course, is Clique($G - \{a\}$). We let ω denote the largest cardinality of a clique in G.

Let K_1 be the first clique in the given listing which contains a but not b. We claim that K_1 is irreducible in Clique(G). Indeed, $K_1 \cup \{b\}$ is the unique upper cover of K_1 in Clique(G): if L is a clique which properly contains K_1 and does not contain b, then L would precede K_1 in the list, contrary to the choice K_1. We will delete K_1 first in dismantling Clique(G). As this argument shows, all cliques in G which contain a and not b, and having the same cardinality as K_1, are irreducible in Clique(G), and they can all be deleted one after another, in the order they appear in the list, in dismantling Clique(G). We now argue inductively. Suppose m is an integer, and that we have already deleted all cliques in G containing a but not b whose cardinalities are $\geq \omega - m$.

Let $\mathcal{K}_m = \{K \in \text{Clique}(G) : a \in K \text{ and } b \notin K \text{ and } |K| \geq \omega - m\}$. Let K be any clique in G which contains a and not b and such that $|K| = \omega - (m+1)$, and let $\mathcal{L}$ be any set of cliques such that $K \notin \mathcal{L}$ and such that $|L| = \omega - (m+1)$ for all $L \in \mathcal{L}$. We claim that K is irreducible in Clique(G) $- (\mathcal{K}_m \cup \mathcal{L})$. Indeed, the above argument shows that $K \cup \{b\}$ is the unique upper cover of K in Clique(G) $- (\mathcal{K}_m \cup \mathcal{L})$. Therefore, we can further delete, one after another, in the order that they appear in the original listing, all the cliques which contain a and not b and which have size $\omega - (m+1)$. Repeating this procedure for all m, we see that Clique(G) can be dismantled to Clique(G) $- \mathcal{A}$, where $\mathcal{A} = \bigcup_m \mathcal{K}_m$ is the set of all cliques in G which contain a but not b. Now let $\mathcal{B} = \{K \in \text{Clique}(G) : a \in K$ and $b \in K\}$, and for each integer $n \geq 2$, let $\mathcal{B}_n = \{K \in \mathcal{B} : |K| < n\}$. We will now show that Clique(G) $- \mathcal{A}$ can be dismantled to Clique(G) $- (\mathcal{A} \cup \mathcal{B})$ by showing that, for any $n \geq 2$, if K is any clique in G containing both a and b with $|K| = n$, and if $\mathcal{T}$ is any set of cliques such that $K \notin \mathcal{T}$ and such that $|L| = n$ for all $L \in \mathcal{T}$, then K is irreducible in Clique(G) $- (\mathcal{A} \cup \mathcal{B}_n \cup \mathcal{T})$. Indeed, in this case it is clear that $K - \{a\}$ is the unique lower cover of K in Clique(G) $- (\mathcal{A} \cup \mathcal{B}_n \cup \mathcal{T})$: any clique properly contained in K and which contains a has either been deleted in $\mathcal{A}$ (if it doesn't contain b), or in $\mathcal{B}_n$ (if it does). Therefore we can delete, one by one, in order of increasing cardinality, the cliques in $\mathcal{B}$ from Clique(G) $- \mathcal{A}$, thereby dismantling to Clique(G) $- (\mathcal{A} \cup \mathcal{B}) =$ Clique($G - \{a\}$). This completes the proof of the lemma. ∎

We will find it convenient to separate the proofs of the two directions in our main result into two lemmas.

LEMMA 2.2. *Let G be a graph. If G is dismantlable, then* Clique(G) *is a dismantlable ordered set.*

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PROOF. We argue by induction on the number of vertices of G. For $|V| = 1$, the result is trivial. If $x_1, x_2, \ldots, x_n$ is a sequence which witnesses the the dismantlability of G, then, as remarked above, x_1 is a subdominant vertex of G, and $G - \{x_1\}$ is dismantlable. Lemma 2.1 shows that Clique(G) can be dismantled to Clique($G - \{x_1\}$). The induction hypothesis then implies that Clique($G - \{x_1\}$) is dismantlable, which implies that Clique(G) itself is dismantlable. ∎

LEMMA 2.3. *Let G be a graph. If* Clique(G) *is a dismantlable ordered set, then G is dismantlable.*

PROOF. Our proof is by induction on the number of vertices of G, the result being trivial if $|V| = 1$.

Let $K_1, K_2, \ldots, K_i, \ldots, K_r$ be a listing of all the non-empty cliques in G such that, for all $i < r$, K_i is irreducible in Clique(G) $- \{K_j : j < i\}$. We will refer to this ordering of the cliques as "the order in which the cliques are deleted". We say that K_i was "deleted at stage i".

We first show that G contains a subdominant vertex.

Let $\{a\}$ be the first singleton clique which was deleted. Since singletons are minimal elements of Clique(G), at the stage when $\{a\}$ was deleted, $\{a\}$ must have had a unique *upper* cover, say C_a, among the elements of Clique(G) which had not yet been deleted. Thus $\{a\} \subset C_a$. Let b be a vertex in $C_a - \{a\}$. Because C_a is a clique, a and b are adjacent in G. We will show that b dominates a in G. First we establish the following.

CLAIM. If K is any clique in G such that $|K| > 1$, with $a \in K$ and $b \notin K$, then K was deleted before $\{a\}$. If this were not the case, let K be witness. Then, among the cliques which had not yet been deleted when $\{a\}$ was, there is one, call it K_1, for which $\{a\} \subset K_1 \subseteq K$, and such that K_1 covers $\{a\}$ (among the cliques which had not been deleted). But, since $K_1 \neq C_a$, this contradicts the fact that $\{a\}$ had a *unique* upper cover at that stage. This proves the claim.

Now we show that b dominates a in G. For the sake of contradiction, assume not. Then there is a vertex c in G which is adjacent to a but which is not adjacent to b. Now, consider the clique $\{a, c\}$. By the claim, this clique was deleted before $\{a\}$. Let L be the last clique which contains $\{a, c\}$ and which was deleted before $\{a\}$. At the stage when L was deleted, no singleton had yet been deleted, and so there is, for each element $x \in L$, a clique M_x such that $\{x\} \subseteq M_x \subset L$, and such that M_x is a lower cover of L among the cliques which had not been deleted at the stage when L was. If it were the case that $M_x = M_y$ for all x, y in L, then, denoting any of these sets by M, it would follow that M contained *all* the elements of L, contrary to the fact that M is a proper subset of L. Therefore L did not have a *unique* lower cover at the stage when it was deleted—L must have had a unique *upper* cover, call it L_1, at that stage. It follows that $c \in L_1$ and hence that $b \notin L_1$, since b and c are not adjacent. Therefore, by the claim, L_1 must be deleted before $\{a\}$. But since L_1 is deleted after L, this contradicts the fact that L is the *last* clique containing $\{a, c\}$ to be deleted before $\{a\}$. Therefore b dominates a as we wished to show.

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Knowing that G has a subdominant vertex a, the dismantlability of G will clearly follow from the dismantlability of $G - \{a\}$. This follows from the induction hypothesis, once we observe that the dismantlability of the ordered set Clique(G) implies the dismantlability of Clique($G - \{a\}$). This latter implication follows easily from Lemma 5 of [5]: a retract of a dismantlable ordered set is itself dismantlable. Clearly the function r: Clique(G) $\longrightarrow$ Clique($G - \{a\}$) defined by $r(K) = K$ if $a \notin K$ and $r(K) = (K - \{a\}) \cup \{b\}$ if $a \in K$, is a retraction. This completes the proof of Lemma 2.3. ■

We have now established the implications in both directions of the following theorem.

THEOREM 2.4. *Let G be a graph. Then G is dismantlable if, and only if,* Clique(G) *is a dismantlable ordered set.*

We can now also easily establish the equivalence of the dismantlability of an ordered set P and the dismantlability of the comparability graph Comp(P) of P.

COROLLARY 2.5. *Let P be an ordered set. Then P is dismantlable if, and only if,* Comp(P) *is a dismantlable graph.*

PROOF. As discussed in the introduction, only the "if" requires proof. Let $G =$ Comp(P) and suppose G is dismantlable. Then G has a subdominant vertex, and as noted in the introduction, this implies that P has an irreducible element, say a. The dismantlability of P would follow from that of $P - \{a\}$, which we can infer by induction as follows: As the proof of Lemma 2.3 above shows, Clique($G - \{a\}$) is a retract of Clique(G). So, if G is dismantlable, then, by Theorem 2.4, so is Clique(G), and hence so is Clique($G - \{a\}$) by Lemma 5 (on retractions) of [5]. Applying Theorem 2.4 again, we see that $G - \{a\}$ is dismantlable. But clearly $G - \{a\} =$ Comp($P - \{a\}$). ■

3. **The fixed-clique property and some applications.**

Our applications of Theorem 2.4 derive from I. Rival's fixed-point theorem: *If P is a dismantlable ordered set then P has the fixed-point property* (Corollary 2 of [11]). Since any homomorphism $f: G \longrightarrow G$ from a graph G to itself gives rise to the order preserving function $K \longrightarrow f[K]$ from Clique(G) to itself, we obtain the following.

COROLLARY 3.1. *Let G be a dismantlable graph. Then for any homomorphism $f: G \longrightarrow G$ there is a non-empty clique K in G such that $f[K] = K$.*

As we will see, Corollary 3.1 can be used to infer some well-known fixed-point type theorems which were established by entirely different means. First, let us describe some classes of graphs which are dismantlable:

(i) (Comparability graphs of) *dismantlable ordered sets*: These have been extensively studied. Many interesting sufficient conditions for an ordered set to be dismantlable have been found. We cite [2] and [11] in this regard.

(ii) *Graphs which have a universal vertex*: Clearly if a is a universal vertex in G then every other vertex of G is dominated by a and so G is dismantlable.

(iii) *Trees*: If a is a vertex of degree 1 in a graph G then a is a subdominant vertex of G. Hence any tree is dismantlable.

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(iv) *Connected triangulated graphs*: This class of graphs includes the class in (iii). Recall that a graph G is said to be *triangulated* if every cycle in G of length greater than 3 has a chord. A well-known theorem of G. A. Dirac [4] asserts that any triangulated graph which is not a clique contains two non-adjacent simplicial vertices. Now, it is clear that, in any graph G, if a is a non-isolated simplicial vertex of G, then a is a subdominant vertex of G. Indeed, if a is simplicial, we have $\{a\} \cup N(a) \subseteq \{b\} \cup N(b)$ for any vertex b which is adjacent to a. Therefore, if G is a connected, triangulated graph with at least two vertices, Dirac's theorem implies there is a subdominant vertex a in G such that $G - \{a\}$ is a connected, triangulated graph. This easily implies G is dismantlable by induction. This fact has been generalized from triangulated graphs to bridged graphs in [1]. As in [1], a graph G is said to be *bridged* if every cycle C in G of length > 3 contains two vertices whose distance in G is less than their distance on C. Anstee and Farber show in [1] that *every connected bridged graph is dismantlable* (and, conversely, that a dismantlable graph which contains no induced cycles of length 4 or 5 is bridged).

In addition to the graphs included in the above four classes, it is easy to construct examples of dismantlable graphs which are in none of these classes. One such example is in Figure 1. In this graph, for $i > 1$, vertex i is a subdominant vertex in the subgraph consisting of all vertices $\leq i$, and so these vertices can be deleted in order from largest to smallest.

We will now describe several consequences of Corollary 3.1. Of these, Corollary 3.2(ii) was established (among many significant results) in [2] using homological methods applied to ordered sets. The actual result in [2], (see Theorem 2.1 and Corollary 2.8 in [2]) is somewhat stronger: every order-reversing function from a dismantlable ordered set to itself either interchanges a pair of comparable elements or has a unique fixed-point. The uniqueness of the fixed-point does not follow directly from our application of Corollary 3.1. We note that the homological methods in [2] have been extended in [3] to apply to comparability-preserving maps on an ordered set.

The result in Corollary 3.2(iii) is found in 6.19 of [8], and is also established by a completely different argument than the one used here. It is, of course, also a special case of Corollary 3.2(iv).

COROLLARY 3.2. *(i) Let P be a dismantlable ordered set and let $f: P \longrightarrow P$ be any comparability-preserving function. Then there is a non-empty chain C in P such that $f[C] = C$.*

(ii) Let P be a dismantlable ordered set and let $f: P \longrightarrow P$ be an order-reversing function. Then either there is an element $x \in P$ such that $f(x) = x$, or there is a pair of elements x, y in P with $x < y$ such that $f(x) = y$ and $f(y) = x$. ([2])

(iii) Let T be a tree and let $f: T \longrightarrow T$ be any homomorphism. Then either f has a fixed-point or f interchanges two adjacent vertices of T. ([8])

(iv) Let G be any connected, triangulated graph, (or, more generally, any connected bridged graph), and let $f: G \longrightarrow G$ be any homomorphism. Then there is a non-empty clique K in G such that $f[K] = K$. Therefore either f has a fixed point, or there is a vertex $x \in G$ such that $f(x)$ is adjacent to x in G.

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PROOF. Since cliques in the comparability graph of an ordered set are just chains, (i) follows directly from Corollary 3.1. We can deduce (ii) from (i) as follows. If f is an order reversing function as in (ii) then, by (i) there is a non-empty chain C in P which is mapped onto itself by f. If $|C| = 1$ this gives a fixed-point. If $|C| > 1$ then the largest and smallest elements of C must be interchanged since f is order-reversing.

Since, as remarked above, trees are dismantlable, (iii) follows directly from Corollary 3.1 since, in a tree, any clique has at most two elements. Similarly, (iv) follows from the results referred to above, which imply that a graph with the given properties is dismantlable. Note that if $f[K] = K$ and if x is any vertex of K, then x and $f(x)$ both lie in K and hence are either equal or adjacent. ■

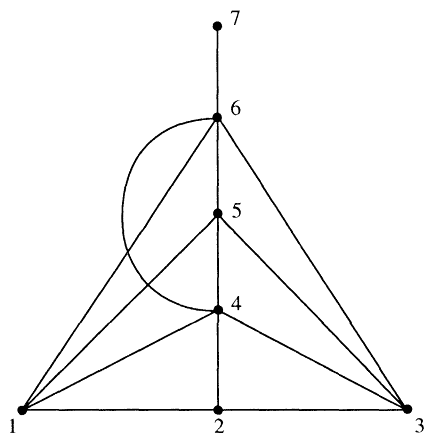

FIGURE 1. A DISMANTLABLE GRAPH

It is interesting to note recent work by A. Quilliot in [10] related to the above results. The condition that, for every homomorphism f on G, there is either a fixed vertex x or a vertex x for which x is adjacent to $f(x)$, follows from the condition that every homomorphism has a non-empty fixed clique, as we noted above in verifying Corollary 3.2(iv). This former condition has been shown to hold in [10] for a class of (finite) connected graphs called *Helly graphs*. (As in [10] a graph G is called a Helly graph if the collection of balls $\{B_r(x) : x \in V, r > 0\}$ has the Helly property, where $B_r(x) = \{v \in V : d(x, v) \leq r\}$.) It is further shown in [10] that every finite Helly graph G contains a clique which is invariant under every automorphism of G. (These results are found in Theorems 1 and 2 in [10].)

We conclude with a remark concerning triangulated graphs. As described in [7], it follows directly from Dirac's theorem that every triangulated graph G has a *perfect elimination scheme*: there is a listing $x_1, x_2, \ldots, x_n$ of all the vertices of G such that, for all

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i, x_i is a simplicial vertex of the induced subgraph $V_i = \{x_j : j \geq i\}$. Note the similarity between this and the definition of dismantlability. The latter requires that the set of neighbours of x_i in V_i has a universal vertex, whereas in a perfect elimination scheme, the set of neighbours of x_i in V_i must be a clique.

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R3B 2E9

https://doi.org/10.4153/CMB-1994-069-6 Published online by Cambridge University Press