BULL. AUSTRAL. MATH. SOC. 26A15, 41A35, 41A63, 26D15, 41A44
VOL. 61 (2000) [489–506]

GLOBAL SMOOTHNESS PRESERVATION BY MULTIVARIATE SINGULAR INTEGRALS

GEORGE A. ANASTASSIOU AND SORIN G. GAL

By using various kinds of moduli of smoothness, it is established that the multivariate variants of the well-known singular integrals of Picard, Poisson-Cauchy, Gauss-Weierstrass and their Jackson-type generalisations satisfy the "global smoothness preservation" property. The results are extensions of those proved by the authors for the univariate case.

1. INTRODUCTION

Let f be a function defined on $\mathbf{R}^m$ with values in $\mathbf{R}$. Throughout the article, we use δ, x, h consistently to represent m-tuples $\delta = (\delta_1, \dots, \delta_m)$, $x = (x_1, \dots, x_m)$, $h = (h_1, \dots, h_m)$ of real numbers. We adopt also the notation

$$\Delta_h^r f(x) := \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} f(x + ih), \; r \in \mathbf{N}.$$

We define the rth-L^p-modulus of smoothness over $\mathbf{R}^m$, $1 \leqslant p \leqslant \infty$, by

$$\omega_r(f;\delta)_p := \sup_{0 \leqslant h \leqslant \delta} \left\| \Delta_h^r f(\cdot) \right\|_{L^p(\mathbf{R}^m)}, \tag{1}$$

(see, for example [3, p.126]), where

$$\|f\|_{L^p(\mathbf{R}^m)} := \left\{ \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \left| f(x_1, \dots, x_m) \right|^p dx_1 \dots dx_m \right\}^{1/p}, \quad \text{if } 1 \leqslant p < +\infty,$$

$$\|f\|_{L^\infty(\mathbf{R}^m)} := \sup\left\{ \left| f(x_1, \dots, x_m) \right| ;\; x_i \in \mathbf{R},\; i = \overline{1,m} \right\}, \quad \text{if } p = +\infty.$$

Here as subsequently $0 \leqslant h \leqslant \delta$ means $0 \leqslant h_i \leqslant \delta_i$, $i = \overline{1,m}$.

We define also the rth-L^p-modulus of smoothness over $I = [a,b]^m$, $a, b \in \mathbf{R}$, $a < b$, $1 \leqslant p \leqslant \infty$, by

$$\omega_r(f;\delta)_p := \omega_r(f;\delta)_{L^p(I)} := \sup_{0 \leqslant h \leqslant \delta} \left\| \Delta_h^r f(\cdot) \right\|_{L^p(I_{r,h})}, \tag{2}$$

Received by the Bulletin 20th January, 2000. Submitted to the Journal of the Australian Mathematical Society Series B, 9th October 1998, and subsequently transferred to the Bulletin.

where $I_{r,h} = [a, b - rh_1] \times \ldots \times [a, b - rh_m]$ and $0 \leqslant h \leqslant \delta$.

When $f \in L^p_{2\pi}(\mathbf{R}^m) = \Big\{f : \mathbf{R}^m \to \mathbf{R};\ f$ is 2π-periodic in each variable and $\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} < +\infty\Big\}$, we define the rth-L^p-modulus of smoothness by

(3)
$$\omega^*_r(f;\delta)_p := \sup_{0 \leqslant h \leqslant \delta} \Big\|\Delta^r_h f(\cdot)\Big\|_{L^p_{2\pi}(\mathbf{R}^m)},$$

where $0 \leqslant h \leqslant \delta$ and

$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} := \left\{\int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \Big|f(x_1, \ldots, x_m)\Big|^p dx_1 \ldots dx_m\right\}^{1/p}, \qquad \text{if } 1 \leqslant p < \infty,$$
$$\|f\|_{L^p_{2\pi}(\mathbf{R}^m)} := \sup\Big\{\Big|f(x_1, \ldots, x_m)\Big|;\ x_i \in [-\pi, \pi],\ i = \overline{1, m}\Big\}, \qquad \text{if } p = +\infty.$$

Next we define the multivariate Ditzian-Totik modulus of smoothness over $[a, b]^m$ (see [2]). First we define the rth symmetric difference

$$\widetilde{\Delta}^r_h f(x) := \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \frac{r}{2}h - kh\right), & \text{if } x \pm \dfrac{rh}{2} \in [a, b]^m \\ 0, & \text{otherwise.} \end{cases}$$

For $r \in \mathbf{N}$ and $f \in C\Big([a, b]^m\Big)$, the space of all real functions continuous on $[a, b]^m$, the rth uniform Ditzian-Totik modulus is

(4)
$$\omega^r_\phi(f;\delta)_\infty := \sup_{0 \leqslant h \leqslant \delta} \Big\|\widetilde{\Delta}^r_{h\phi(x)} f(x)\Big\|_{C([a,b]^m)},$$

where $0 \leqslant h \leqslant \delta$, $\phi(x) = \Big(\varphi(x_1), \ldots, \varphi(x_m)\Big)$, $h\phi(x) := \Big(h_1\varphi(x_1), \ldots, h_m\varphi(x_m)\Big)$ and

$$\|f\|_{C([a,b]^m)} := \sup\Big\{\Big|f(x_1, \ldots, x_m)\Big|;\ x_i \in [a, b],\ i = \overline{1, m}\Big\}.$$

In the above definitions and in what follows, we consider only functions with finite modulus of smoothness.

Put

(5)
$$\operatorname{Erf}(x_i) := \frac{2}{\sqrt{\pi}} \int_0^{x_i} e^{-t_i^2}\, dt_i,\ x_i \in \mathbf{R},$$

and note that

(6)
$$\frac{1}{2\xi_i} \int_{-\infty}^{+\infty} e^{-|t_i|/\xi_i}\, dt_i = 1,\ \xi_i \in \mathbf{R},\ \xi_i > 0$$

and

(7)
$$\int_{-\pi}^{\pi} \frac{dt_i}{t_i^2 + \xi_i^2} = \frac{2}{\xi_i} \tan^{-1}\left(\frac{\pi}{\xi_i}\right),\ \xi_i \in \mathbf{R},\ x_i > 0.$$

Also, $(2/\xi_i)\tan^{-1}(\pi/\xi_i)$, $\operatorname{Erf}\Big(\pi/\sqrt{\xi_i}\Big)$ both tend to 1 as $\xi_i \to 0$.

Next, for $\xi > 0$ we define the multivariate Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals

$$P_\xi(f)(x) := \left[\prod_{i=1}^m (2\xi_i)\right]^{-1} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1+t_1,\dots,x_m+t_m) \cdot \left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right) dt_1\dots dt_m, \tag{8}$$

$$Q_\xi(f)(x) := \left[\prod_{i=1}^m \left(\frac{\xi_i}{\pi}\right)\right] \cdot \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \frac{f(x_1+t_1,\dots,x_m+t_m)}{\prod_{i=1}^m \left(t_i^2+\xi_i^2\right)}\, dt_1\dots dt_m \tag{9}$$

and

$$W_\xi(f)(x) := \left[\prod_{i=1}^n \sqrt{\pi\xi_i}\right]^{-1} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} f(x_1+t_1,\dots,x_m+t_m) \cdot \left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right) dt_1\dots dt_m. \tag{10}$$

We study also the generalised multivariate singular integrals

$$P_{n,\xi}(f)(x) = -\left[\prod_{i=1}^m (2\xi_i)\right]^{-1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(x_1+kt_1,\dots,x_m+kt_m) \left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right) dt_1\dots dt_m, \tag{11}$$

$$Q_{n,\xi}(f)(x) = -\left\{\prod_{i=1}^m \left[\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\right\}^{-1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \int_{-\pi}^{\pi} \cdots \int_{\pi}^{\pi} \frac{f(x_1+kt_1,\dots,x_m+kt_m)}{\prod_{i=1}^m \left(t_i^2+\xi_i^2\right)}\, dt_1\dots dt_m \tag{12}$$

and

$$W_{n,\xi}(f)(x) = -\left[\prod_{i=1}^m C(\xi_i)\right]^{-1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \int_{-\pi}^{\pi} \cdots \int_{\pi}^{\pi} f(x_1+kt_1,\dots,x_m+kt_m) \left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right) dt_1\dots dt_m \tag{13}$$

of Jackson type for $\xi > 0$, where $C(\xi_i) = \int_{-\pi}^{\pi} e^{-t_i^2/\xi_i}\, dt_i$, $i = \overline{1,m}$.

Finally, when $f \in C\big([0,1]^m\big)$ or $f \in L^p\big([0,1]^m\big)$, $1 \leqslant p < \infty$, we study the multivariate Picard-type singular integral

$$L_\xi(f)(x) = \left[\prod_{i=1}^m \xi_i\right]^{-1} \int_0^{\infty} \cdots \int_0^{\infty} f\left(\frac{x_1}{e^{t_1}},\dots,\frac{x_m}{e^{t_m}}\right) \left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1\dots dt_m, \tag{14}$$

for $\xi > 0$. Obviously $L_\xi(f)(x) \in \mathbf{R}$, for all $x \in [0,1]^m$ and $f \in C\big([0,1]^m\big)$. Otherwise we assume $f \in L^p\big([0,1]^m\big)$. Also

$$\frac{1}{\xi_i}\int_0^\infty e^{-t_i/\xi_i}\,dt_i = 1,\ \xi_i \in \mathbf{R},\ \xi_i > 0,\ i = \overline{1,m}. \tag{15}$$

In [**1**] the authors obtained results regarding global smoothness preservation by the univariate cases of the operators defined by (8)-(14). The purpose of the present paper is to extend these results to the above multivariate singular integrals given by (8)-(14). Our global smoothness inequalities involve all kinds of moduli of smoothness introduced by (1)-(4).

2. Main Results

The first main result is as follows.

Theorem 1. *Let $f : \mathbf{R}^m \to \mathbf{R}$ have $\omega_r(f;\delta)_\infty < +\infty$, $r \in \mathbf{N}$, for any $\delta > 0$, and be such that $P_\xi(f)(x)$, $Q_\xi(f)(x)$, $W_\xi(f)(x) \in \mathbf{R}$, for all $x \in \mathbf{R}^m$, where $\xi > 0$. Then for any $\delta > 0$*

$$\omega_r\big(P_\xi(f);\delta\big)_\infty \leqslant \omega_r(f;\delta)_\infty, \tag{16}$$

$$\omega_r\big(Q_\xi(f);\delta\big)_\infty \leqslant \left[\prod_{i=1}^m \left(\frac{2}{\pi}\cdot\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right)\right]\omega_r(f;\delta)_\infty \tag{17}$$

and

$$\omega_r(W_\xi(f);\delta)_\infty \leqslant \left[\prod_{i=1}^m \left(\operatorname{Erf}\left(\frac{\pi}{\sqrt{\xi_i}}\right)\right)\right]\omega_r(f;\delta)_\infty. \tag{18}$$

The inequalities are sharp, being attained by each $f_j(x) = x_j^r$, $j = \overline{1,m}$.

Proof: For each $0 \leqslant h \leqslant \delta$, we have

$$\Delta_h^r\big[P_\xi(f)\big](x) = \left[\prod_{i=1}^m (2\xi_i)\right]^{-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Delta_h^r f(x+t)\left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right)dt_1\dots dt_m,$$

$$\Delta_h^r\big[Q_\xi(f)\big](x) = \left[\prod_{i=1}^m\left(\frac{\xi_i}{\pi}\right)\right]\int_{-\pi}^{\pi}\cdots\int_{\pi}^{\pi}\frac{\Delta_h^r f(x+t)}{\prod_{i=1}^m\left(t_i^2+\xi_i^2\right)}\,dt_1\dots dt_m$$

and

$$\Delta_h^r\big[W_\xi(f)\big](x) = \left[\prod_{i=1}^m\sqrt{\pi\xi_i}\right]^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\Delta_h^r f(x+t)\left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right)dt_1\dots dt_m,$$

where as subsequently $t = (t_1,\dots,t_m)$.

We now take absolute values, using

$$\left|\int_{-\pi}^{\pi}\cdots\int_{\pi}^{\pi}F(x,t)\,dt_1\dots dt_m\right| \leqslant \int_{-\pi}^{\pi}\cdots\int_{\pi}^{\pi}\big|F(x,t)\big|\,dt_1\dots dt_m$$

and

$$\left|\Delta_h^r f(x+t)\right| \leqslant \omega_r(f;\delta)_\infty.$$

Inequalities (16)-(18) now follow from (5)-(7).

If $f_j(x) := x_j^r$, we have

$$\Delta_h^r f_j(x) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (x_j + ih_j)^r = r! h_j^r,\ j = \overline{1,m},$$

which implies $\omega_r(f_j;\delta)_\infty = r!\delta_j^r < +\infty$ for any $\delta > 0$.

Similarly we see that

$$\Delta_h^r\left[P_\xi(f_j)\right](x) = r! h_j^r,$$
$$\Delta_h^r\left[Q_\xi(f_j)\right](x) = r! h_j^r \prod_{i=1}^{m} \left(\frac{2}{\pi}\cdot \tan^{-1}\frac{\pi}{\xi_i}\right)$$

and

$$\Delta_h^r\left[W_\xi(f_j)\right](x) = r! h_j^r \prod_{i=1}^{m} \left(\operatorname{Erf}\left(\frac{\pi}{\sqrt{\xi_i}}\right)\right).$$

It is apparent that (16)-(18) are attained for each function f_j, $j = \overline{1,m}$.

Finally it is easy to show that $P_\xi(f_j)(x) \in \mathbf{R}$ for $0 < \xi < 1/r$, and $Q_\xi(f_j)(x)$, $W_\xi(f_j)(x) \in \mathbf{R}$ for $\xi > 0$, for all $x \in \mathbf{R}^m$. ∎

The following theorem is related.

Theorem 2. *Let $f : \mathbf{R}^m \to \mathbf{R}$ satisfy $\omega_r(f;\delta)_\infty < +\infty$ for any $\delta > 0$ such that $P_{n,\xi}(f)(x)$, $Q_{n,\xi}(f)(x)$, $W_{n,\xi}(f)(x) \in \mathbf{R}$, for all $x \in \mathbf{R}^m$, $n \in \mathbf{N}$ and $\xi > 0$. Then for any $\delta > 0$,*

$$\omega_r\left(P_{n,\xi}(f);\delta\right)_\infty \leqslant \left(2^{n+1}-1\right)\omega_r(f;\delta)_\infty, \tag{19}$$

$$\omega_r\left(Q_{n,\xi}(f);\delta\right)_\infty \leqslant \left(2^{n+1}-1\right)\omega_r(f;\delta)_\infty \tag{20}$$

and

$$\omega_r\left(W_{n,\xi}(f);\delta\right)_\infty \leqslant \left(2^{n+1}-1\right)\omega_r(f;\delta)_\infty. \tag{21}$$

Proof: For $0 \leqslant h \leqslant \delta$ we have

$$\Delta_h^r\left[P_{n,\xi}(f)\right](x) = -\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \Delta_h^r f(x+kt)\left(\prod_{i=1}^{m} e^{-|t_i|/\xi_i}\right) dt_1\ldots dt_m,$$

$$\Delta_h^r\left[Q_{n,\xi}(f)\right](x) = -\left\{\prod_{i=1}^{m}\left[\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\right\}^{-1} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \cdot \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{\Delta_h^r f(x+kt)}{\prod_{i=1}^{m}\left(t_i^2+\xi_i^2\right)} dt_1\ldots dt_m$$

https://doi.org/10.1017/S0004972700022516 Published online by Cambridge University Press

and

$$\Delta_h^r\left[W_{n,\xi}(f)\right](x) = -\left[\prod_{i=1}^m C(\xi_i)\right]^{-1}\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}$$
$$\cdot\Delta_h^r f(x+kt)\left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right)dt_1\dots dt_m.$$

Reasoning as in the proof of Theorem 1 and using

$$\sum_{k=1}^{n+1}\binom{n+1}{k} = 2^{n+1}-1$$

gives (19)-(21). ∎

Next we present results on global smoothness preservation, with respect first to the L^1-norm and then the L^p-norm for $p > 1$.

Theorem 3. *Suppose either $f \in L^1(\mathbf{R}^m)$ (for $P_\xi(f)$) or $f \in L^1_{2\pi}(\mathbf{R}^m)$ (for $Q_\xi(f)$, $W_\xi(f)$), $\xi > 0$ and $r \in \mathbf{N}$. Then for any $\delta > 0$ we have*

$$\omega_r\Big(P_\xi(f);\delta\Big)_1 \leqslant \omega_r(f;\delta)_1, \tag{22}$$

$$\omega_r^*\Big(Q_\xi(f);\delta\Big)_1 \leqslant \left[\prod_{i=1}^m\left(\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right)\right]\omega_r^*(f;\delta)_1 \tag{23}$$

and

$$\omega_r^*\Big(W_\xi(f);\delta\Big)_1 \leqslant \left[\prod_{i=1}^m\left(\operatorname{Erf}\left(\frac{\pi}{\sqrt{\xi_i}}\right)\right)\right]\omega_r^*(f;\delta)_1. \tag{24}$$

Proof: From the proof of Theorem 1 we have for $0 \leqslant h \leqslant \delta$ that

$$\left|\Delta_h^r\left[P_\xi(f)\right](x)\right| \leqslant \left[\prod_{i=1}^m(2\xi_i)\right]^{-1}\cdot\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\Delta_h^r f(x+t)\right|$$
$$\cdot\left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right)dt_1\dots dt_m, \tag{25}$$

$$\left|\Delta_h^r\left[Q_\xi(f)\right](x)\right| \leqslant \left[\prod_{i=1}^m\left(\frac{\xi_i}{\pi}\right)\right]\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{\left|\Delta_h^r f(x+t)\right|}{\prod_{i=1}^m\left(t_i^2+\xi_i^2\right)}dt_1\dots dt_m \tag{26}$$

and

$$\left|\Delta_h^r\left[W_\xi(f)\right](x)\right| \leqslant \left[\prod_{i=1}^m\sqrt{\pi\xi_i}\right]^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|$$
$$\cdot\left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right)dt_1\dots dt_m. \tag{27}$$

We now integrate m times, in (25) from $-\infty$ to $+\infty$ and in (26), (27) from $-\pi$ to π. Use of a Fubini-type result provides

$$\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\Delta_h^r\left[P_\xi(f)\right](x)\right|dx_1\ldots dx_m$$
$$\leqslant\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\Delta_h^r f(x+t)\right|dx_1\ldots dx_m\right\}$$
$$\cdot\prod_{i=1}^{m}\left(e^{-|t_i|/\xi_i}\right)dt_1\ldots dt_m\leqslant\omega_r(f;\delta),$$

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r\left[Q_\xi(f)\right](x)\right|dx_1\ldots dx_m$$
$$\leqslant\left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|dx_1\ldots dx_m\right\}\cdot\frac{dt_1\ldots dt_m}{\prod_{i=1}^{m}\left(t_i^2+\xi_i^2\right)}$$
$$\leqslant\left[\prod_{i=1}^{m}\left(\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right)\right]\omega_r^*(f;\delta)_1$$

and

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r\left[W_\xi(f)\right](x)\right|dx_1\ldots dx_m$$
$$\leqslant\left[\prod_{i=1}^{m}\sqrt{\pi\xi_i}\right]^{-1}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|dx_1\ldots dx_m\right\}$$
$$\cdot\left(\prod_{i=1}^{m}e^{-t_i^2/\xi_i}\right)dt_1\ldots dt_m$$
$$\leqslant\left[\prod_{i=1}^{m}\left(\operatorname{Erf}\left(\frac{\pi}{\sqrt{\xi_i}}\right)\right)\right]\omega_r^*(f;\delta)_1.$$

Here we have used

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|dx_1\ldots dx_m$$
$$=\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+h)\right|d(x_1+t_1)\ldots d(x_m+t_m)$$
$$=\|\Delta_h^r f\|_{L^1_{2\pi}([t-\pi,t+\pi])}=\|\Delta_h^r f\|_{L^1_{2\pi}([-\pi,\pi]^m)}=\|\Delta_h^r f\|_{L^1_{2\pi}(\mathbf{R})},$$

where $[t-\pi,t+\pi]=[t_1-\pi,t_1+\pi]\times\ldots\times[t_m-\pi,t_m+\pi]$. Relations (22)-(24) follow from these inequalities. □

In the case of the singular integrals given by (11)-(13), we derive the following.

Theorem 4. *Suppose either* $f\in L^1(\mathbf{R}^m)$ *(for* $P_{n,\xi}(f)$*) or* $f\in L^1_{2\pi}(\mathbf{R}^m)$ *(for* $Q_{n,\xi}(f)$*,* $W_{n,\xi}(f)$*),* $\xi>0$ *and* $n,r\in\mathbf{N}$*. Then for any* $\delta>0$*, we have*

(28) $$\omega_r\left(P_{n,\xi}(f);\delta\right)_1\leqslant\left(2^{n+1}-1\right)\omega_r(f;\delta)_1,$$

(29) $$\omega_r^*\left(Q_{n,\xi}(f);\delta\right)_1\leqslant\left(2^{n+1}-1\right)\omega_r^*(f;\delta)_1$$

https://doi.org/10.1017/S0004972700022516 Published online by Cambridge University Press

and

(30) $$\omega_r^*\left(W_{n,\xi}(f);\delta\right)_1 \leqslant \left(2^{n+1}-1\right)\omega_r^*(f;\delta)_1.$$

PROOF: By the proof of Theorem 2 we have if $0 \leqslant h \leqslant \delta$ that

$$\left|\Delta_h^r\left[P_{n,\xi}(f)\right](x)\right| \leqslant \left[\prod_{i=1}^m (2\xi_i)\right]^{-1} \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} \cdot\left|\Delta_h^r f(x+kt)\right| \left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right) dt_1 \ldots dt_m,$$

$$\left|\Delta_h^r\left[Q_{n,\xi}(f)\right](x)\right| \leqslant \left\{\prod_{i=1}^m \left[\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\right\}^{-1} \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \cdot \frac{\left|\Delta_h^r f(x+kt)\right|}{\prod_{i=1}^m \left(t_i^2+\xi_i^2\right)} dt_1 \ldots dt_m,$$

and

$$\left|\Delta_h^r\left[W_{n,\xi}(f)\right](x)\right| \leqslant \left[\prod_{i=1}^m C(\xi_i)\right]^{-1} \sum_{k=1}^{n+1} \binom{n+1}{k} \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} \cdot\left|\Delta_h^r f(x+kt)\right| \left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right) dt_1 \ldots dt_m.$$

We integrate m times, from $-\infty$ to $+\infty$ in the first inequality and from $-\pi$ to π in the next two. Reasoning exactly as in the proof of Theorem 3 and using

$$\sum_{k=1}^{n+1} \binom{n+1}{k} = 2^{n+1}-1,$$

we obtain (28)-(30). ∎

We now extend Theorem 3 to the case $1 < p < \infty$.

THEOREM 5. *Suppose either $f \in L^p(\mathbf{R}^m)$ (for $P_\xi(f)$) or $f \in L^p_{2\pi}(\mathbf{R}^m)$ (for $Q_\xi(f)$, $W_\xi(f)$), $1 < p < \infty$. Let $\xi > 0$ and $q > 1$, with $1/p + 1/q = 1$. Then for any $\delta > 0$ we have*

(31) $$\omega_r(P_\xi(f);\delta)_p \leqslant \left[\frac{2}{p^{1/p}\cdot q^{1/q}}\right]^m \cdot \omega_r(f;\delta)_p,$$

(32) $$\omega_r^*(Q_\xi(f);\delta)_p \leqslant \prod_{i=1}^m \left[\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right] \omega_r^*(f;\delta)_p,$$

and

(33) $$\omega_r^*\left(W_\xi(f);\delta\right)_p \leqslant \left[\frac{\sqrt{2}}{p^{1/(2p)}\cdot q^{1/(2q)}}\right]^m \cdot \prod_{i=1}^m \left[\left(\operatorname{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right)^{1/p} \left(\operatorname{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right)^{1/q}\right] \omega_r^*(f;\delta)_p.$$

PROOF: Let $0 \leqslant h \leqslant \delta$. We have

$$\Delta_h^r\big[P_\xi(f)\big](x) = \left[\prod_{i=1}^{m}(2\xi_i)\right]^{-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} \cdot\Delta_h^r f(x+t)\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)dt_1\dots dt_m$$

and

$$\begin{aligned}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}&\left|\Delta_h^r\big[P_\xi(f)\big](x)\right|^p dx_1\dots dx_m\\ &=\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-p}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left\{\left|\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Delta_h^r f(x+t)\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)\right.\right.\\ &\qquad\left.\left.\cdot\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)dt_1\dots dt_m\right|\right\}^p dx_1\dots dx_m\\ &\leqslant\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-p}\cdot\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\Delta_h^r f(x+t)\right|\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)\right.\\ &\qquad\left.\cdot\left(\prod_{i=1}^{m}e^{-|t_i|/(2\xi_i)}\right)dt_1\dots dt_m\right\}^p dx_1\dots dx_m.\end{aligned}$$

Hence by Hölder's inequality for multivariate integrals and a Fubini-type result we get

$$\begin{aligned}\left\|\Delta_h^r\big[P_\xi(f)\big]\right\|_{L^p(\mathbf{R}^m)}^p &\leqslant\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-p}\\ &\quad\cdot\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left\{\left(\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\left|\Delta_h^r f(x+t)\right|^p\left[\prod_{i=1}^{m}e^{-|t_i|p/(2\xi_i)}\right]dt_1\dots dt_m\right)^{1/p}\cdot\right.\\ &\qquad\left.\cdot\left(\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{m}e^{-|t_i|q/(2\xi_i)}dt_1\dots dt_m\right)^{1/q}\right\}^p dx_1\dots dx_m\\ &\leqslant\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-p}\omega_r(f;\delta)_p^p\left(\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{m}\left(e^{-|t_i|p/(2\xi_i)}\right)dt_1\dots dt_m\right)\\ &\qquad\cdot\left(\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\prod_{i=1}^{m}\left(e^{-|t_i|q/(2\xi_i)}\right)dt_1\dots dt_m\right)^{p/q}\\ &=\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-p}\cdot\prod_{i=1}^{m}\left(\frac{4\xi_i}{p}\right)\left(\prod_{i=1}^{m}\left(\frac{4\xi_i}{q}\right)\right)^{p/q}\omega_r(f;\delta)_p^p,\end{aligned}$$

that is,

$$\left\|\Delta_h^r\big[P_\xi(f)\big]\right\|_{L^p(\mathbf{R}^m)}\leqslant\left[\prod_{i=1}^{m}(2\xi_i)\right]^{-1}\left[\prod_{i=1}^{m}\left(\frac{4\xi_i}{p}\right)\right]^{1/p}\left[\prod_{i=1}^{m}\left(\frac{4\xi_i}{q}\right)\right]^{1/q}\omega_r(f;\delta)_p$$

$$= \frac{1}{[p^{1/p}q^{1/q}]^m} \cdot \frac{\prod_{i=1}^{m}(4\xi_i)}{\prod_{i=1}^{m}(2\xi_i)} \omega_r(f;\delta)_p = \left[\frac{2}{p^{1/p}\cdot q^{1/q}}\right]^m \cdot \omega_r(f;\delta)_p,$$

which implies (31).

In the case of $Q_\xi(f)(x)$, we use the formula

$$\frac{1}{t_i^2+\xi_i^2} = \left(\frac{1}{t_i^2+\xi_i^2}\right)^{1/p}\left(\frac{1}{t_i^2+\xi_i^2}\right)^{1/q}.$$

From the formula for $\Delta_h^r\big[Q_\xi(f)\big](x)$ in the proof of Theorem 1, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r\big[Q_\xi(f)\big](x)\right|^p dx_1\ldots dx_m \\
&= \left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]^p \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\left|\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\Delta_h^r f(x+t)\right.\right. \\
&\qquad \left.\left.\cdot\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right)^{1/p}\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right)^{1/q} dt_1\ldots dt_m\right|\right\}^p dx_1\ldots dx_m \\
&\leqslant \left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]^p \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right)^{1/p}\right. \\
&\qquad \left.\cdot\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right)^{1/q} dt_1\ldots dt_m\right\}^p dx_1\ldots dx_m.
\end{aligned}$$

Again by Hölder's inequality and a Fubini-type result we obtain

$$\begin{aligned}
&\left\|\Delta_h^r\big[Q_\xi(f)\big]\right\|_{L_{2\pi}^p(\mathbf{R}^m)}^p \leqslant \left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]^p \\
&\quad\cdot\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left\{\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left|\Delta_h^r f(x+t)\right|^p\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right) dt_1\ldots dt_m\right)\right. \\
&\qquad\left.\cdot\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right) dt_1\ldots dt_m\right)^{p/q}\right\} dx_1\ldots dx_m \\
&\leqslant \left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]^p \omega_r^*(f;\delta)_p^p\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right) dt_1\ldots dt_m\right)^{p/q+1} \\
&= \left[\prod_{i=1}^{m}\left(\frac{\xi_i}{\pi}\right)\right]^p \omega_r^*(f;\delta)_p^p\left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\left(\prod_{i=1}^{m}\frac{1}{t_i^2+\xi_i^2}\right) dt_1\ldots dt_m\right)^{p},
\end{aligned}$$

which implies

$$\begin{aligned}
\left\|\Delta_h^r\big[Q_\xi(f)\big]\right\|_{L_{2\pi}^p(\mathbf{R}^m)} &\leqslant \left[\prod_{i=1}^{m}\frac{\xi_i}{\pi}\right]\left[\prod_{i=1}^{m}\left(\frac{2}{\xi_i}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right)\right]\omega_r^*(f;\delta)_p \\
&= \prod_{i=1}^{m}\left[\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\omega_r^*(f;\delta)_p.
\end{aligned}$$

This immediately proves (32). In the case of $W_\xi(f)(x)$, we use the formula

$$e^{-t_i^2/\xi_i} = e^{-t_i^2/(2\xi_i)}e^{-t_i^2/(2\xi_i)}, \quad i = \overline{1,m}$$

and the formula for $\Delta_h^r\big[W_\xi(f)\big](x)$ in the proof of Theorem 1. By Hölder's inequality, we obtain as above that

$$\left\|\Delta_h^r\big[W_\xi(f)\big]\right\|_{L_{2\pi}^p(\mathbf{R}^m)}^p \leqslant \left[\prod_{i=1}^m \sqrt{\pi\xi_i}\right]^{-p} \omega_r^*(f;\delta)_p^p \left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{i=1}^m e^{-t_i^2 p/(2\xi_i)}\,dt_1\ldots dt_m\right) \cdot \left(\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\prod_{i=1}^m e^{-t_i^2 q/(2\xi_i)}\,dt_1\ldots dt_m\right)^{p/q}.$$

But for all $i = \overline{1,m}$,

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-t_i^2 q/(2\xi_i)}\,dt_i &= 2\int_0^{\pi} e^{-t_i^2/(2\xi_i/q)}dt_i = 2\int_0^{\pi} e^{-\left(t_i/\sqrt{2\xi_i/q}\right)^2}\,dt_i \\ &= 2\sqrt{2\xi_i/q}\int_0^{\pi} e^{-\left(t_i/\sqrt{2\xi_i/q}\right)^2}\,d\left[t_i/\sqrt{2\xi_i/q}\right] \\ &= 2\sqrt{2\xi_i/q}\cdot\int_0^{\pi\sqrt{q/(2\xi_i)}} e^{-t_i^2}dt_i = \sqrt{2\xi_i/q}\cdot\sqrt{\pi}\,\mathrm{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right) \\ &= \sqrt{2\pi\xi_i/q}\cdot\mathrm{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right).\end{aligned}$$

Thus

$$\begin{aligned}&\left\|\Delta_h^r\big[W_\xi(f)\big]\right\|_{L_{2\pi}^p(\mathbf{R}^m)} \\ &\leqslant \left[\prod_{i=1}^m \sqrt{\pi\xi_i}\right]^{-1}\left[\prod_{i=1}^m\left(\sqrt{2\pi\xi_i/p}\cdot\mathrm{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right)\right]^{1/p} \\ &\quad\cdot\left[\prod_{i=1}^m \sqrt{2\pi\xi_i/q}\cdot\mathrm{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right]^{1/q}\cdot\omega_r^*(f;\delta)_p \\ &= \frac{\left(\prod_{i=1}^m\sqrt{2\pi\xi_i}\right)^{1/p+1/q}}{\prod_{i=1}^m\sqrt{\pi\xi_i}}\cdot\prod_{i=1}^m\left[\left(\mathrm{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right)^{1/p}\cdot\left(\mathrm{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right)^{1/q}\right] \\ &\quad\cdot\left[\prod_{i=1}^m\left(p^{1/(2p)}\cdot q^{1/(2q)}\right)\right]^{-1}\omega_r^*(f;\delta)_p = \left[\frac{\sqrt{2}}{p^{1/(2p)}\cdot q^{1/(2q)}}\right]^m \\ &\quad\cdot\prod_{i=1}^m\left[\left(\mathrm{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right)^{1/p}\cdot\left(\mathrm{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right)^{1/q}\right]\cdot\omega_r^*(f;\delta)_p,\end{aligned}$$

giving (33). □

We now generalise Theorem 4 to the case $p > 1$.

https://doi.org/10.1017/S0004972700022516 Published online by Cambridge University Press

Theorem 6. *Suppose either $f \in L^p(\mathbf{R}^m)$ (for $P_{n,\xi}(f)$) or $f \in L^p_{2\pi}(\mathbf{R}^m)$ (for $Q_{n,\xi}(f)$, $W_{n,\xi}(f)$), $1 < p < \infty$. Let $\xi > 0$ and $q > 1$, $1/p + 1/q = 1$. Then for any $\delta > 0$, we have*

$$\omega_r(P_{n,\xi}(f);\delta)_p \leqslant \left(2^{n+1}-1\right)\left[\frac{2}{p^{1/p}\cdot q^{1/q}}\right]^m \omega_r(f;\delta)_p, \tag{34}$$

$$\omega_r^*(Q_{n,\xi}(f);\delta)_p \leqslant \left(2^{n+1}-1\right)\omega_r^*(f;\delta)_p \tag{35}$$

and

$$\omega_r^*(W_{n,\xi}(f);\delta)_p \leqslant \left(2^{n+1}-1\right)\left[\frac{\sqrt{2}}{p^{1/(2p)}\cdot q^{1/(2q)}}\right]^m \cdot\prod_{i=1}^m\left\{\frac{\left[\operatorname{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right]^{1/p}\cdot\left[\operatorname{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right]^{1/q}}{\operatorname{Erf}\left(\pi/\sqrt{\xi_i}\right)}\right\}\omega_r^*(f;\delta)_p. \tag{36}$$

Proof: For $k = \overline{1, n+1}$, set

$$M_k := \left[\prod_{i=1}^m (2\xi_i)\right]^{-1}\cdot\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Delta_h^r f(x+kt)\left(\prod_{i=1}^m e^{-|t_i|/\xi_i}\right)dt_1\ldots dt_m.$$

Suppose $0 \leqslant h \leqslant \delta$. From the first equality in the proof of Theorem 2, we have

$$\Delta_h^r\left[P_{n,\xi}(f)\right](x) = -\sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k}M_k,$$

which implies

$$\left\|\Delta_h^r\left[P_{n,\xi}(f)\right]\right\|_{L^p(\mathbf{R}^m)} \leqslant \sum_{k=1}^{n+1}\binom{n+1}{k}\|M_k\|_{L^p(\mathbf{R}^m)} = \left(2^{n+1}-1\right)\|M_k\|_{L^p(\mathbf{R}^m)}.$$

Putting $\xi_i' = k\xi_i$, we have

$$M_k = \left[\prod_{i=1}^m (2\xi_i')\right]^{-1}\int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty}\Delta_h^r f(x+t)\left(\prod_{i=1}^m e^{-|t_i|/\xi_i'}\right)dt_1\ldots dt_m,$$

and by the proof of Theorem 5

$$\|M_k\|_{L^p(\mathbf{R}^m)} \leqslant \left[\frac{2}{p^{1/p}\cdot q^{1/q}}\right]^m\omega_r(f;\delta)_p.$$

Therefore we get

$$\left\|\Delta_h^r\left[P_{n,\xi}(f)\right]\right\|_{L^p(\mathbf{R}^m)} \leqslant \left(2^{n+1}-1\right)\left[\frac{2}{p^{1/p}\cdot q^{1/q}}\right]^m\omega_r(f;\delta)_p,$$

which establishes (34).

For $Q_{n,\xi}(f)$, we obtain from the second equality in the proof of Theorem 2 that

$$\Delta_h^r\Big[Q_{n,\xi}(f)\Big](x) = -\left\{\prod_{i=1}^m \left[\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\right\}^{-1} \sum_{k=1}^{n+1}(-1)^k \binom{n+1}{k} A_k,$$

with

$$A_k = \prod_{i=1}^m \left(\frac{\xi_i}{\pi}\right) \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \frac{\Delta_h^r f(x+kt)}{\prod_{i=1}^m \left(t_i^2+\xi_i^2\right)}\, dt_1\ldots dt_m.$$

Reasoning as for $\Delta_h^r\Big[Q_\xi(f)\Big]$) in the proof of Theorem 5 yields

$$\|A_k\|_{L^p_{2\pi}(\mathbf{R}^m)} \leqslant \prod_{i=1}^m \left[\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right] \left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \left|\Delta_h^r f(x+kt)\right|^p dx_1\ldots dx_m\right\}^{1/p},$$

where by the 2π-periodicity of f we have

$$\begin{aligned}&\left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \left|\Delta_h^r f(x+kt)\right|^p dx_1\ldots dx_m\right\}^{1/p}\\ &\qquad = \left\{\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \left|\Delta_h^r f(x+t)\right|^p dx_1\ldots dx_m\right\}^{1/p} \leqslant \omega_r^*(f;\delta)_p.\end{aligned}$$

As a consequence

$$\begin{aligned}\left\|\Delta_h^r\Big[Q_{n,\xi}(f)\Big]\right\|_{L^p_{2\pi}(\mathbf{R}^m)} &\leqslant \left\{\prod_{i=1}^m \left[\frac{2}{\pi}\tan^{-1}\left(\frac{\pi}{\xi_i}\right)\right]\right\}^{-1} \cdot \sum_{k=1}^{n+1}\binom{n+1}{k}\|A_k\|_{L^p_{2\pi}(\mathbf{R}^m)}\\ &\leqslant \left(2^{n+1}-1\right)\omega_r^*(f;\delta)_p,\end{aligned}$$

which establishes (35).

Finally, for $W_{n,\xi}(f)$, we get from the third equality in the proof of Theorem 2 that

$$\Delta_h^r\Big[W_{n,\xi}(f)\Big](x) = -\frac{\prod_{i=1}^m \sqrt{\pi\xi_i}}{\prod_{i=1}^m C(\xi_i)} \cdot \sum_{k=1}^{n+1}(-1)^k\binom{n+1}{k} B_k,$$

where

$$B_k := \left[\prod_{i=1}^m \sqrt{\pi\xi_i}\right]^{-1} \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \Delta_h^r f(x+kt)\left(\prod_{i=1}^m e^{-t_i^2/\xi_i}\right) dt_1\ldots dt_m.$$

Reasoning as for $\Delta_h^r\Big[W_\xi(f)\Big]$ in the proof of Theorem 5 we obtain

$$\begin{aligned}\|B_k\|_{L^p_{2\pi}(\mathbf{R}^m)} \leqslant \left[\frac{\sqrt{2}}{p^{1/(2p)}\cdot q^{1/(2q)}}\right]^m &\cdot \left(\prod_{i=1}^m \operatorname{Erf}\left(\pi\sqrt{p/(2\xi_i)}\right)\right)^{1/p}\\ &\cdot \left(\prod_{i=1}^m \operatorname{Erf}\left(\pi\sqrt{q/(2\xi_i)}\right)\right)^{1/q} \cdot \omega_r^*(f;\delta).\end{aligned}$$

Since

$$C(\xi_i) = \sqrt{\pi\xi_i}\,\mathrm{Erf}\left(\pi/\sqrt{\xi_i}\right),\ i = \overline{1,m},$$

we obtain

$$\left\|\Delta_h^r\left[W_{n,\xi}(f)\right]\right\|_{L^p_{2\pi}(\mathbf{R}^m)} \leqslant \frac{\prod_{i=1}^m \sqrt{\pi\xi_i}}{\left(\prod_{i=1}^m \sqrt{\pi\xi_i}\right)\left(\prod_{i=1}^m \mathrm{Erf}\left(\pi/\sqrt{\xi_i}\right)\right)} \cdot \sum_{k=1}^{n+1}\binom{n+1}{k}\|B_k\|_{L^p_{2\pi}(\mathbf{R}^m)},$$

which together the previous inequality proves (36). ∎

Next we establish a global smoothness preservation theory for the $L_\xi(f)(x)$ operators given by (14).

Theorem 7. *Let $f \in C\left([0,1]^m\right)$, $r \in \mathbf{N}$, $\xi > 0$. Then for any $\delta > 0$,*

$$\omega_r(L_\xi(f);\delta)_\infty \leqslant \omega_r(f;\delta)_\infty. \tag{37}$$

Inequality (37) is asymptotically sharp as $\xi \to 0$, that is, asymptoticaly attained for all $f_j(x) = x_j^r$, $j = \overline{1,m}$, $x \in [0,1]^m$.

Proof: Let $0 \leqslant h \leqslant \delta$ and $x_i \in [0, 1 - rh_i]$, $i = \overline{1,m}$. We see that

$$\begin{aligned}\Delta_h^r\left[L_\xi(f)\right](x) &= \sum_{i=0}^r (-1)^{r-i}\binom{r}{i} L_\xi(f)(x+ih)\\ &= \left[\prod_{i=1}^m \xi_i\right]^{-1} \int_0^\infty \cdots \int_0^\infty \left[\Delta^r_{h/e^t} f\left(x/e^t\right)\right]\left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1 \ldots dt_m,\end{aligned}$$

where for simplicity we employ the notation $e^t = (e^{t_1},\ldots,e^{t_m})$, $h/e^t = (h_1/e^{t_1},\ldots,h_m/e^{t_m})$, $x/e^t = (x_1/e^{t_1},\ldots,x_m/e^{t_m})$.

By (15) we have

$$\begin{aligned}\left|\Delta_h^r\left[L_\xi(f)\right](x)\right| &\leqslant \left[\prod_{i=1}^m \xi_i\right]^{-1} \int_0^\infty \cdots \int_0^\infty \left|\Delta^r_{h/e^t} f\left(\frac{x}{e^t}\right)\right| \left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1 \ldots dt_m\\ &\leqslant \left[\prod_{i=1}^m \xi_i\right]^{-1} \int_0^\infty \cdots \int_0^\infty \omega_r\left(f;\delta/e^t\right)_\infty \left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1 \ldots dt_m\\ &\leqslant \omega_r(f;\delta)_\infty. \end{aligned} \tag{38}$$

This implies (37).

Define $f_j(x) = x_j^r$, $j = \overline{1,m}$. Then

$$L_\xi(f_j)(x) = \frac{1}{\xi_j}\cdot\int_0^\infty \left(x_j/e^{t_j}\right)^r \cdot e^{-t_j/\xi_j}\,dt_j = \frac{x_j^r}{r\xi_j+1},\ x \in [0,1]^m.$$

On the other hand,

$$\Delta_h^r\left[L_\xi(f_j)\right](x) = \frac{r!h_j^r}{r\xi_j+1} \text{ and } \Delta_h^r f_j(x) = r!h_j^r.$$

https://doi.org/10.1017/S0004972700022516 Published online by Cambridge University Press

Consequently, from

$$\frac{r!\delta_j^r}{r\xi_j+1} = \omega_r(L_\xi(f_j);\delta)_\infty \leqslant \omega_r\left(x_j^r;\delta\right)_\infty = r!\delta_j^r$$

we derive equality as $\xi_j \to 0$, which completes the proof. □

The corresponding L^1 result is as follows.

THEOREM 8. *Let $f \in L^1\left([0,1]^m\right)$, $r \in \mathbb{N}$ and $0 < \xi < 1$. Then for any $\delta > 0$,*

$$\omega_r(L_\xi(f);\delta)_1 \leqslant \left[\prod_{i=1}^m (1-\xi_i)\right]^{-1} \omega_r(f;\delta)_1. \tag{39}$$

Inequality (39) is asymptotically sharp as $\xi \to 0$, that is, attained asymptotically for all $f_j(x) = x_j^r$, $j = \overline{1,m}$, $x \in [0,1]^m$.

PROOF: By integrating (38) and emloying a Fubini-type result, we get

$$\begin{aligned}
&\int_0^{1-rh_1}\cdots\int_0^{1-rh_m}\left|\Delta_h^r\left[L_\xi(f)\right](x)\right|dx_1\dots dx_m \\
&\leqslant \left[\prod_{i=1}^m \xi_i\right]^{-1}\int_0^\infty\cdots\int_0^\infty\left[\int_0^{1-rh_1}\cdots\int_0^{1-rh_m}\right. \\
&\qquad\qquad \left.\cdot\left|\Delta_{h/e^t}^r f\left(\frac{x}{e^t}\right)\right|\left(\prod_{i=1}^m e^{-t_i/\xi_i}\right)dx_1\dots dx_m\right]dt_1\dots dt_m \\
&= \left[\prod_{i=1}^m \xi_i\right]^{-1}\int_0^\infty\cdots\int_0^\infty\left[\int_0^{1-rh_1}\cdots\int_0^{1-rh_m}\left|\Delta_{h/e^t}^r f\left(\frac{x}{e^t}\right)\right|d\left(\frac{x_1}{e^{t_1}}\right)\dots d\left(\frac{x_m}{e^{t_m}}\right)\right] \\
&\quad\cdot\left(\prod_{i=1}^m e^{t_i-t_i/\xi_i}\right)dt_1\dots dt_m = \left[\prod_{i=1}^m \xi_i\right]^{-1}\int_0^\infty\cdots\int_0^\infty \\
&\quad\cdot\left[\int_0^{(1-rh_1)/e^{t_1}}\cdots\int_0^{(1-rh_m)/e^{t_m}}\left|\Delta_{h/e^t}^r f(u)\right|du_1\dots du_m\right]\left[\prod_{i=1}^m e^{-t_i(-1+1/\xi_i)}\right]dt_1\dots dt_m \\
&\leqslant \left[\prod_{i=1}^m \xi_i\right]^{-1}\int_0^\infty\cdots\int_0^\infty\left[\int_0^{1-rh_1/e^{t_1}}\cdots\int_0^{1-rh_m/e^{t_m}}\left|\Delta_{h/e^t}^r f(u)\right|du_1\dots du_m\right] \\
&\quad\cdot\left[\prod_{i=1}^m e^{-t_i(-1+1/\xi_i)}\right]dt_1\dots dt_m \leqslant \left[\prod_{i=1}^m \xi_i\right]^{-1}\int_0^\infty\cdots\int_0^\infty \omega_r\left(f;\delta/e^t\right)_1 \\
&\qquad\qquad\cdot\left(\prod_{i=1}^m e^{-t_i/(\xi_i/(1-\xi_i))}\right)dt_1\dots dt_m \leqslant \omega_r(f;\delta)_1\cdot\left[\prod_{i=1}^m\left(\frac{\xi_i}{1-\xi_i}\right)\right]\left[\prod_{i=1}^m \xi_i\right]^{-1} \\
&= \left(\prod_{i=1}^m \frac{1}{1-\xi_i}\right)\omega_r(f;\delta)_1,
\end{aligned}$$

that is,

$$\left\|\Delta_h^r\left[L_\xi(f)\right]\right\|_{L^1([0,1]^m)} \leqslant \left(\prod_{i=1}^m \frac{1}{1-\xi_i}\right)\omega_r(f;\delta)_1,$$

which provides (39).

By the proof of Theorem 7 we obtain

$$\int_0^{1-rh_1} \cdots \int_0^{1-rh_m} \left|\Delta_h^r\big[L_\xi(f_j)\big](x)\right| dx_1 \ldots dx_m = \prod_{i=1}^m (1-rh_i)\frac{r!h_j^r}{r\xi_j+1}$$

and

$$\omega_r\big(L_\xi(f_j);\delta\big)_1 = \frac{r!}{r\xi_j+1}\cdot \sup\left\{h_j^r\left[\prod_{i=1}^m (1-rh_i)\right];\ 0 \leqslant h_j \leqslant \delta_j,\ j=\overline{1,m}\right\},$$

while

$$\omega_r(f_j;\delta)_1 = r!\sup\left\{h_j^r\left[\prod_{i=1}^m (1-rh_i)\right];\ 0 \leqslant h_j \leqslant \delta_j,\ j=\overline{1,m}\right\}.$$

This shows that (39) is attained asymptotically by each $f_j(x) = x_j^r$, $j=\overline{1,m}$, as $\xi \to 0$. ∎

In what follows we present the L^p $(1<p<\infty)$ global smoothness preservation for $L_\xi(f)$ operators.

Theorem 9. *Suppose $f \in L^p\big([0,1]^m\big)$, $1<p<\infty$, $r \in \mathbb{N}$ and $0<\xi<p/2$. Let $q>1$, $1/p+1/q=1$. Then for any $\delta>0$ we have*

$$\omega_r\big(L_\xi(f);\delta\big)_p \leqslant 2^m q^{-m/q}\left[\prod_{i=1}^m (p-2\xi_i)\right]^{-1/p}\omega_r(f;\delta)_p. \tag{40}$$

Proof: From (38) we obtain for $0 \leqslant x_i \leqslant 1-rh_i$, $i=\overline{1,m}$, $0 \leqslant h \leqslant \delta$ that

$$\left|\Delta_h^r\big[L_\xi(f)\big](x)\right| \leqslant \left(\prod_{i=1}^m \xi_i\right)^{-1}\left[\int_0^\infty \cdots \int_0^\infty \left|\Delta_{h/e^t}^r f\left(\frac{x}{e^t}\right)\right|\left(\prod_{i=1}^m e^{-t_i/\xi_i}\right)\right] dt_1 \ldots dt_m.$$

By Hölder's inequality and a Fubini-type result

$$\begin{aligned}
&\left\|\Delta_h^r\big[L_\xi(f)\big]\right\|_{L^p(I_{r,h})}^p \\
&\leqslant \left(\prod_{i=1}^m \xi_i\right)^{-p}\int_0^{1-rh_1}\cdots\int_0^{1-rh_m}\left\{\left(\int_0^\infty\cdots\int_0^\infty\left|\Delta_{h/e^t}^r f\left(\frac{x}{e^t}\right)\right|^p\cdot\left(\prod_{i=1}^m e^{-t_ip/(2\xi_i)}\right)\right.\right. \\
&\qquad \left.\left.\cdot dt_1\ldots dt_m\right)\left(\int_0^\infty\cdots\int_0^\infty\left(\prod_{i=1}^m e^{-t_iq/(2\xi_i)}\right)dt_1\ldots dt_m\right)^{p/q}\right\}dx_1\ldots dx_m \\
&= \left[\prod_{i=1}^m\left(\frac{2\xi_i}{q}\right)\right]^{p/q}\left(\prod_{i=1}^m \xi_i\right)^{-p}\cdot\int_0^\infty\cdots\int_0^\infty\left[\int_0^{1-rh_1}\cdots\int_0^{1-rh_m}\left|\Delta_{h/e^t}^r f\left(\frac{x}{e^t}\right)\right|^p\right. \\
&\qquad \left.\cdot\left(\prod_{i=1}^m e^{-t_ip/(2\xi_i)}\right)\right]dt_1\ldots dt_m.
\end{aligned}$$

Reasoning as in the proof of Theorem 8 shows that the last expression is in turn less than or equal to

$$\left[\prod_{i=1}^{m}\left(\frac{2\xi_i}{q}\right)\right]^{p/q}\left(\prod_{i=1}^{m}\xi_i\right)^{-p}\int_0^{\infty}\cdots\int_0^{\infty}\left[\int_0^{1-rh_1/e^{t_1}}\cdots\int_0^{1-rh_m/e^{t_m}}\left|\Delta^r_{h/e^t}f(u)\right|^p du_1\ldots du_m\right]$$
$$\cdot\left(\prod_{i=1}^{m}e^{-t_i[-1+p/(2\xi_i)]}\right)dt_1\ldots dt_m\leqslant\left[\omega_r(f;\delta)_p\right]^p\cdot\left[\prod_{i=1}^{m}\left(\frac{2\xi_i}{q}\right)\right]^{p/q}\left(\prod_{i=1}^{m}\xi_i\right)^{-p}$$
$$\cdot\int_0^{\infty}\cdots\int_0^{\infty}\left(\prod_{i=1}^{m}e^{-t_i[-1+p/(2\xi_i)]}\right)dt_1\ldots dt_m$$
$$=[\omega_r(f;\delta)_p]^p\cdot\left[\prod_{i=1}^{m}\left(\frac{2\xi_i}{q}\right)\right]^{p/q}\left(\prod_{i=1}^{m}\xi_i\right)^{-p}\cdot\prod_{i=1}^{m}\left(\frac{2\xi_i}{p-2\xi_i}\right).$$

Consequently we get

$$\left[\omega_r\Big(L_\xi(f);\delta\Big)_p\right]^p\leqslant\left[\omega_r(f;\delta)_p\right]^p\cdot\left[\prod_{i=1}^{m}\left(\frac{2\xi_i}{q}\right)\right]^{p/q}\left(\prod_{i=1}^{m}\xi_i\right)^{-p}\cdot\prod_{i=1}^{m}\left(\frac{2\xi_i}{p-2\xi_i}\right),$$

that is,

$$\omega_r(L_\xi(f);\delta)_p\leqslant 2^m q^{-m/q}\left[\prod_{i=1}^{m}(p-2\xi_i)\right]^{-1/p}\omega_r(f;\delta)_p,$$

which proves (40). □

To conclude we give the Ditzian-Totik treatment for $L_\xi(f)$ operators for global smoothness preservation.

Theorem 10. *Let $f\in C\big([0,1]^m\big)$ and $\phi(x)=\big(\varphi(x_1),\ldots,\varphi(x_m)\big)$, $x\in[0,1]^m$, $\varphi(s)=\sqrt{s(1-s)}$, $s\in[0,1]$, $r\in\mathbb{N}$, $\xi>0$. Then for any $\delta>0$ we have*

$$\omega^r_\phi\Big(L_\xi(f);\delta\Big)_\infty\leqslant\omega^r_\phi(f;\delta)_\infty. \tag{41}$$

Proof: For $0\leqslant h\leqslant\delta$ we have

$$\widetilde{\Delta}^r_{h\phi(x)}f(x)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}f\left(x+\frac{rh\phi(x)}{2}-kh\phi(x)\right),$$

where

$$\omega^r_\phi(f;\delta)_\infty:=\sup_{0\leqslant h\leqslant\delta}\left\|\widetilde{\Delta}^r_{h\phi(x)}f(x)\right\|_{C([0,1]^m)}.$$

We see that

$$\Delta^r_{h\phi(x)}\big[L_\xi(f)\big](x)=\sum_{k=0}^{r}(-1)^k\binom{r}{k}L_\xi(f)\left(x+\frac{rh\phi(x)}{2}-kh\phi(x)\right)$$
$$=\sum_{k=0}^{r}(-1)^k\binom{r}{k}\cdot\left(\prod_{i=1}^{m}\xi_i\right)^{-1}\cdot\int_0^{\infty}\cdots\int_0^{\infty}$$
$$\cdot f\left[\frac{x}{e^t}+\frac{rh\phi(x)}{2e^t}-\frac{kh\phi(x)}{e^t}\right]\left(\prod_{i=1}^{m}e^{-t_i/\xi_i}\right)dt_1\ldots dt_m,$$

https://doi.org/10.1017/S0004972700022516 Published online by Cambridge University Press

with e^t, x/e^t as in Theorem 7 and $kh = (kh_1, \dots, kh_m)$, $h\phi(x) = \big(h_1\varphi(x_1), \dots, h_m\varphi(x_m)\big)$.

Therefore

$$\left|\widetilde{\Delta}^r_{h\phi(x)}\big[L_\xi(f)\big](x)\right| \leqslant \left(\prod_{i=1}^m \xi_i\right)^{-1} \int_0^\infty \dots \int_0^\infty \left|\sum_{k=0}^r (-1)^k \binom{r}{k} f\left[\frac{x}{e^t} + \frac{rh\phi(x)}{2e^t} - \frac{kh\phi(x)}{e^t}\right]\right| \cdot \left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1 \dots dt_m.$$

Put $x_i/e^{t_i} = y_i$. If $t_i \in [0, \infty)$ and $x_i \in [0, 1]$, then $y_i \in [0, 1]$. Thus

$$\frac{x}{e^t} + \frac{rh\phi(x)}{2e^t} - \frac{kh\phi(x)}{e^t} = y + \frac{rh'\phi(y)}{2} - kh'\phi(y),$$

where $y = (y_1, \dots, y_m)$ and $h' = (h'_1, \dots, h'_m)$, with $h'_i = h_i\sqrt{\dfrac{1-x_i}{e^{t_i} - x_i}} \leqslant h_i \leqslant \delta_i$, $i = \overline{1, m}$. Therefore

$$\left|\sum_{k=0}^r (-1)^k \binom{r}{k} f\left[\frac{x}{e^t} + \frac{rh\phi(x)}{2e^t} - \frac{kh\phi(x)}{e^t}\right]\right| = \left|\widetilde{\Delta}^r_{h'\phi(y)} f(y)\right| \leqslant \omega^r_\phi(f;\delta)_\infty,$$

which implies by (15) that

$$\left|\widetilde{\Delta}^r_{h\phi(x)}\big[L_\xi(f)\big](x)\right| \leqslant \left(\prod_{i=1}^m \xi_i\right)^{-1} \cdot \int_0^\infty \dots \int_0^\infty \omega^r_\phi(f;\delta)_\infty \left(\prod_{i=1}^m e^{-t_i/\xi_i}\right) dt_1 \dots dt_m = \omega^r_\phi(f;\delta)_\infty.$$

Relation (41) follows. ∎

REMARK. The convergence to unity of the above multivariate singular integrals will be studied elsewhere.

References

[1] G.A. Anastassiou and S.G. Gal, 'General theory of global smoothness preservation by singular integrals, univariate case', *J. Comput. Anal. Appl.* 1 (1999), 289–317.

[2] Z. Ditzian and V. Totik, *Moduli of smoothness* (Springer-Verlag, Berlin, Heidelberg, New York, 1987).

[3] A.F. Timan, *Theory of approximation of functions of a real variable*, (in Russian) (Fizmat-giz, Moscow, 1960).

Department of Mathematical Sciences
The University of Memphis
Memphis TN 38152
United States of America
e-mail: ANASTASG@msci.memphis.edu

Department of Mathematics
University of Oradea
Str. Armatei Romane 5
3700 Oradea
Romanis
e-mail: galso@math.uoradea.ro