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NEAR-RINGS OF HOMOTOPY CLASSES OF CONTINUOUS FUNCTIONS

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In this paper we show that for a compact connected abelian group G the near-ring $[G,G]$ of all homotopy classes of continuous selfmaps of G is an abstract affine near-ring, and investigate the ideal structure of these near-rings.

1. INTRODUCTION

Let G be a topological group. Under pointwise addition and under composition of functions the set $\mathrm{N}(G)$ of all continuous selfmaps of G is a near-ring. In [9] we showed that for compact abelian groups G with nontrivial connected components the intersection of all nonzero ideals of $\mathrm{N}(G)$ is the ideal of all functions in $\mathrm{N}(G)$ which are homotopic to the constant mapping which carries all of G onto the neutral element of G. Therefore the ideal structure of $\mathrm{N}(G)$ is completely determined by the ideals in the near-ring $[G,G]$ of all homotopy classes of continuous selfmaps of G where the operations are induced by those of $\mathrm{N}(G)$. Therefore, the following investigation of the ideal structure of $[G,G]$ for connected compact abelian groups G is at the same time a study of the ideals of the near-ring $\mathrm{N}(G)$. We show that for a connected compact abelian group G the near-ring $[G,G]$ is an abstract affine near-ring, which is isomorphic to the near-rings $\mathrm{Hom}(G, G) \oplus \pi_0(G)$ and $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$. Using this information we determine the ideals of $[G,G]$ for some important examples of connected compact abelian groups G.

2. BASIC DEFINITIONS AND RESULTS

For details on near-rings we refer the reader to [10]. An *abstract affine near-ring* is a near-ring whose additive group is abelian and where all zero-symmetric elements are distributive. Informations on abstract affine near-rings can be found in [4] and [10]. Examples of abstract affine near-rings can be constructed in the following way: let R be a ring and M be a R-module. Then the direct product

$$R \oplus M = \{(r,m) \mid r \in R,\ m \in M\}$$

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is an abstract affine near-ring by the operations

$$(r,m)+(r',m') := (r+r', m+m')$$
$$(r,m)\cdot(r',m') := (rr', rm'+m).$$

The set $R\oplus\{0\}$ is the zero-symmetric, the set $\{0\}\oplus M$ the constant part of $R\oplus M$. Conversely, any abstract affine near-ring N is isomorphic to a near-ring $R\oplus M$.

The following two statements have a direct proof which will not be given.

LEMMA 2.1. *Let M be an R-module, M' an R'-Module, α : a ring homomorphism from R into R' and β a group homomorphism from M into M' with*

$$\alpha(r)\beta(m)=\beta(rm)$$

for all $r\in R$ and $m\in M$. Then the mapping

$$\varphi : R\oplus M \to R'\oplus M' : (r,m)\to(\alpha(r),\beta(m))$$

is a homomorphism of near-rings. If α and β are isomorphisms, then φ is an isomorphism.

COROLLARY 2.2. *Let M be a R-module and M' a R'-module by the ring homomorphisms $\psi : R\to \mathrm{Hom}(M,M)$ respectively $\psi' : R'\to\mathrm{Hom}(M',M')$. Furthermore, let $\alpha : R\to R'$ be an isomorphism of rings and $\beta : M\to M'$ an isomorphism of groups. Finally, let*

$$\beta^{\#} : \mathrm{Hom}(M,M)\to\mathrm{Hom}(M',M') : f\mapsto \beta\circ f\circ\beta^{-1}$$

be the isomorphism of the rings $\mathrm{Hom}(M,M)$ and $\mathrm{Hom}(M',M')$ induced by β. If the diagram

$$\begin{array}{ccc} R & \xrightarrow{\psi} & \mathrm{Hom}(M,M) \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta^{\#}} \\ R' & \xrightarrow{\psi'} & \mathrm{Hom}(M',M') \end{array}$$

is commutative, then

$$\varphi : R\oplus M \to R'\oplus M' : (r,m)\to(\alpha(r),\beta(m))$$

is an isomorphism of near-rings.

The structure of the ideals in an abstract affine near-ring is well-known by the following theorem of Gonshor in [4].

THEOREM 2.3. *The ideals of an abstract affine near-ring $R\oplus M$ are precisely the sets $I_1\oplus M_1$, where I_1 is an ideal of the ring R and M_1 is a submodule of M with $I_1M\subseteq M_1$.*

3. The near-ring $[G,G]$ for a connected compact abelian group G

In this section let $\mathrm{Hom}(G, G)$ denote the ring of all continuous endomorphisms of a connected compact abelian group G and let $[G,G]_*$ denote the near-ring of all pointed homotopy classes of continuous selfmaps f of G with $f(0) = 0$, where 0 is the neutral element of G.

As an immediate consequence of [11, p.106], we have the following

Lemma 3.1. *If G is a connected compact abelian group, then the mapping*

$$\pi_* : \mathrm{Hom}(G, G) \to [G,G]_* : f \mapsto [f]_*$$

is an isomorphism of near-rings. In particular, $[G,G]_*$ is a ring.

Therefore, the group $\pi_0 G$ of all arc components of a connected compact abelian group G is both a $\mathrm{Hom}(G, G)$-module and a $[G,G]_*$-module, where the operation of $\mathrm{Hom}(G, G)$ respectively of $[G,G]_*$ on $\pi_0 G$ is given by

$$\pi_0 f(x + G_a) = f(x) + G_a$$

respectively

$$[f]_*(x + G_a) = f(x) + G_a,$$

where G_a denotes the arc component of the neutral element of G. Using Lemma 2.1 we can conclude

Corollary 3.2. *For a connected compact abelian group G the abstract affine near-rings $\mathrm{Hom}(G, G) \oplus \pi_0 G$ and $[G,G]_* \oplus \pi_0 G$ are isomorphic near-rings.*

Henceforth, for an element $c \in G$ let $\langle c \rangle$ denote the continuous function which carries all of G onto c.

Theorem 3.3. *Let G be a connected compact abelian group. Then the near-ring $[G,G]$ is an abstract affine near-ring. In particular, $[G,G]$ is isomorphic to the near-ring $\mathrm{Hom}(G, G) \oplus \pi_0 G$.*

Proof: By [11, p.104], the mapping $\varphi = (\varphi_1, \varphi_2) : [G,G] \to [G,G]_* \times \pi_0 G$, defined by $\varphi_1[f] = [f - \langle f(0) \rangle]_*$ and $\varphi_2[f] = f(0) + G_a$, is an isomorphism of groups. Thus, using the isomorphism π_* of Lemma 3.1, the mapping $\pi_*^{-1} \circ \varphi$ is an isomorphism of the groups $[G,G]$ and $\mathrm{Hom}(G, G) \oplus \pi_0 G$. The inverse mapping ψ of this isomorphism is given by

$$\psi : \mathrm{Hom}(G, G) \oplus \pi_0 G \to [G,G] : (f, c + G_a) \mapsto [f + \langle c \rangle]$$

It remains to show that ψ is a multiplicative homomorphism. Let f_1 and f_2 be in $\mathrm{Hom}(G, G)$ and let c_1, c_2 be elements of G. Then

$$\begin{aligned}\psi(f_1, c_1 + G_a) \circ \psi(f_2, c_2 + G_a) &= [f_1 + \langle c_1 \rangle] \circ [f_2 + \langle c_2 \rangle] \\ &= [f_1 \circ (f_2 + \langle c_2 \rangle) + \langle c_1 \rangle \circ (f_2 + \langle c_2 \rangle)] \\ &= [f_1 \circ f_2 + \langle c_1 \rangle + f_1 \circ \langle c_2 \rangle] \\ &= \psi(f_1 \circ f_2, c_1 + f_1(c_2)) + G_a \\ &= \psi(f_1, c_1 + G_a) \circ (f_2, c_2 + G_a)\end{aligned}$$

Therefore, ψ is an isomorphism of near-rings. □

In order to determine the ideal structure of $[G, G]$ for some concrete examples of connected compact abelian groups G we need some homology theory of discrete abelian groups. For definitions, notations and results of this theory we refer the reader to [**2**] and [**3**].

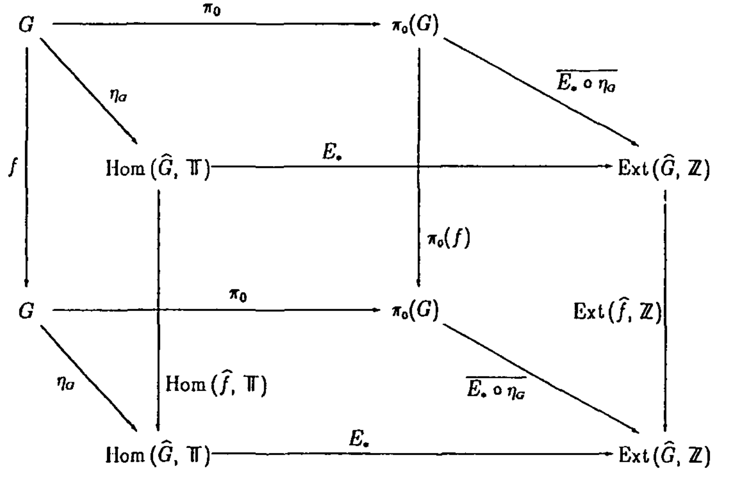
By [**5**, Theorem 23.17], the character group $\widehat{G}$ of a connected compact abelian group G is a discrete abelian group. Moreover, by [**2**, p.213 and p.221], the group $\mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ is a $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$-module by

$$\mathrm{Ext}(\,\cdot\,, \mathbb{Z}) : \mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \to \mathrm{Hom}\left(\mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right), \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)\right) : \gamma \mapsto \mathrm{Ext}(\gamma, \mathbb{Z}),$$

where $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$ is the opposite ring of $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)$. Now we are in position to prove

THEOREM 3.4. *Let G be a connected compact abelian group. Then the near-rings $[G, G]$ and $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ are isomorphic near-rings.*

PROOF: We consider the following diagram:



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By [7, p.285] the left diagram is a commutative diagram of abelian groups. In [8] it is shown, that the upper and the lower plane of the cube are commutative diagrams. Furthermore, by the remarks following Lemma 3.1 the diagram in the background is also commutative. Since by [2, p.217] the front diagram is commutative, too, we can conclude, that the right diagram is a commutative diagram of abelian groups.

We shall show now that the mapping

$$\varphi = (\varphi_1, \varphi_2) : \mathrm{Hom}(G, G) \oplus \pi_0 G \to \mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$$

given by $\varphi_1 : \mathrm{Hom}(G, G) \to \mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} : f \mapsto \widehat{f}$

and $\varphi_2 : \pi_0 G \to \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right) : c + G_a \mapsto \overline{E_* \circ \eta_G}(c + G_a)$

is an isomorphism of near-rings.

By [7] the mapping φ_1 is an isomorphism of rings, by [8] the mapping φ_2 is an isomorphism of abelian groups. Since the right diagram is commutative, we have for all $f \in \mathrm{Hom}(G, G)$ and $c + G_a \in \pi_0 G$:

$$\begin{aligned}\varphi_1(f) \cdot \varphi_2(c + G_a) &= \mathrm{Ext}\left(\widehat{f}, \mathbb{Z}\right)\left(\overline{E_* \circ \eta_G}(c + G_a)\right) = \overline{E_* \circ \eta_G}\left(\pi_0 f(c + G_a)\right) \\ &= \varphi_2\left(\pi_0(f)(c + G_a)\right) = \varphi_2\left(f \cdot (c + G_a)\right).\end{aligned}$$

Thus, by Lemma 2.1 the mapping φ is an isomorphism of near-rings. Hence the assertion of the theorem follows by Theorem 3.3. ∎

Using Theorem 2.3 we can conclude

Theorem 3.5. *Let G be a connected compact abelian group. Then the ideals of the near-ring $[G,G] \cong \mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ are precisely the sets $I \oplus M$, where I is an ideal of the ring $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$ and M is a submodule of the $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$-module $\mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ with $I \cdot \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right) \subseteq M$.*

Corollary 3.6. *If all nontrivial endomorphisms of $\widehat{G}$ are injective, then the ideals of the near-ring $[G,G] \cong \mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op} \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ are precisely the sets $I \oplus \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ for ideals I of the ring $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$ and the sets $\{0\} \oplus M$ for submodules M of the $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$-module $\mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$.*

Proof: By Theorem 3.5 an ideal of $[G,G]$ has the form $I \oplus M$, where I is an ideal of the ring $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$ and M is a submodule of the $\mathrm{Hom}\left(\widehat{G}, \widehat{G}\right)^{op}$-module $\mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ with $I \cdot \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right) \subseteq M$. If $I = \{0\}$, then $\{0\} \cdot \mathrm{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ is obviously a subset of M.

If $I \neq \{0\}$, there exists an injective endomorphism $\widehat{f} \in I$. By [**2**, Proposition 24.6], the mapping $\operatorname{Ext}\left(\widehat{f}, \mathbb{Z}\right) : \operatorname{Ext}\left(\widehat{G}, \mathbb{Z}\right) \to \operatorname{Ext}\left(\widehat{G}, \mathbb{Z}\right)$ is surjective. This implies $I \cdot \operatorname{Ext}\left(\widehat{G}, \mathbb{Z}\right) = \operatorname{Ext}\left(\widehat{G}, \mathbb{Z}\right)$. Thus we have $M = \operatorname{Ext}\left(\widehat{G}, \mathbb{Z}\right)$. ∎

The results of this section can be extended to connected locally compact abelian groups G. In this case, by [**5**, Theorem 9.14], G is isomorphic to a direct product of a connected compact abeliian group K and a vector group $\mathbb{R}^n$. It can be shown that the near-rings $[G, G]$ and $[K, K]$ are isomorphic. Furthermore, using the results of Hofer in [**6**] it is not difficult to show that for a locally compact abelian group G with more than two connected components the near-ring $[G, G]$ is not an abstract affine near-ring. Therefore, these near-rings must be investigated in another way.

4. Examples

Example 4.1. Let $G = \mathbb{T}^n$ be a finite-dimensional torus. Then $[G, G]$ is isomorphic to the complete matrix ring $M_n(\mathbb{Z})$ over the integers.

Proof: Since $\mathbb{T}^n$ is arcwise connected, by Theorem 3.3, by [**7**, p.285], and by [**3**, Theorem 106.1] we have the following isomorphisms:

$$[\mathbb{T}^n, \mathbb{T}^n] \cong \operatorname{Hom}(\mathbb{T}^n, \mathbb{T}^n) \cong \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}^n)^{op} \cong M_n(\mathbb{Z})^{op} \cong M_n(\mathbb{Z}).$$

The ideal structure of these matrix rings is well-known. As mentioned above, by this information on the ideals of $[\mathbb{T}^n, \mathbb{T}^n]$ at the same time the ideal structure of the near-rings $N(\mathbb{T}^n)$ of all continuous selfmaps of $\mathbb{T}^n$ is completely determined. ∎

Example 4.2. Let $G = \widehat{\mathbb{Q}}$ be the character group of the discrete group $\mathbb{Q}$ of the rational numbers. Then the near-ring $[G, G]$ is isomorphic to the abstract affine near-ring $\mathbb{Q} \oplus \mathbb{Q}^{\aleph_0}$, where the ring $\mathbb{Q}$ operates on the group $\mathbb{Q}^{\aleph_0}$ by the usual scalar multiplication.

The ideals of $[G, G]$ are precisely the sets $\{0\} \oplus V$, where V is a subspace of the $\mathbb{Q}$-vector space $\mathbb{Q}^{\aleph_0}$. In particular, there exists exactly one maximal ideal M, namely $M = \{0\} \oplus \mathbb{Q}^{\aleph_0}$

Proof: By Example 4 in [**3**, p.216] the mapping

$$\alpha : \operatorname{Hom}(\mathbb{Q}, \mathbb{Q})^{op} \to \mathbb{Q} : f \mapsto f(1)$$

is an isomorphism of rings. Moreover, by Exercise 7 in [**2**, p.221] there exists an isomorphism $\beta : \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \to \mathbb{Q}^{\aleph_0}$ of abelian groups. This isomorphism induces by

$$\beta^{\#} : \operatorname{Hom}(\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}), \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})) \to \operatorname{Hom}\left(\mathbb{Q}^{\aleph_0}, \mathbb{Q}^{\aleph_0}\right) : f \mapsto \beta \circ f \circ \beta^{-1}$$

an isomorphism of the rings Hom(Ext($\mathbb{Q}$, $\mathbb{Z}$), Ext($\mathbb{Q}$, $\mathbb{Z}$)) and Hom$\left(\mathbb{Q}^{\aleph_0}, \mathbb{Q}^{\aleph_0}\right)$ [3, p.217]. Then the mapping

$$\psi : \mathbb{Q} \to \operatorname{Hom}\left(\mathbb{Q}^{\aleph_0}, \mathbb{Q}^{\aleph_0}\right) : q \mapsto \beta^{\#} \circ \operatorname{Ext}(\,\cdot\,, \mathbb{Z}) \circ \alpha^{-1}(q)$$

is a homomorphism of rings with $\psi(1) = \mathrm{id}$, and the following diagram is commutative:

$$\begin{array}{ccc}
\operatorname{Hom}(\mathbb{Q}, \mathbb{Q})^{op} & \xrightarrow{\operatorname{Ext}(\,\cdot\,,\mathbb{Z})} & \operatorname{Hom}(\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}), \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})) \\
{\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta^{\#}} \\
\mathbb{Q} & \xrightarrow{\qquad\psi\qquad} & \operatorname{Hom}(\operatorname{Ext}(\mathbb{Q}, \mathbb{Z}), \operatorname{Ext}(\mathbb{Q}, \mathbb{Z})).
\end{array}$$

Thus, by Corollary 2.2 the mapping

$$\varphi : \operatorname{Hom}(\mathbb{Q}, \mathbb{Q})^{op} \oplus \operatorname{Ext}(\mathbb{Q}, \mathbb{Z}) \to \mathbb{Q} \oplus \mathbb{Q}^{\aleph_0} : (f, E) \mapsto (\alpha(f), \beta(E))$$

an isomorphism of near-rings.

Since $\psi : \mathbb{Q} \to \operatorname{Hom}\left(\mathbb{Q}^{\aleph_0}, \mathbb{Q}^{\aleph_0}\right)$ is a homomorphism of rings, we have for all numbers $m, n \in \mathbb{N}$ with $n \neq 0$:

$$n \cdot \psi\left(\frac{\pm m}{n}\right) = \pm\psi\Big(\underbrace{1 + \cdots + 1}_{m\ summands}\Big) = \pm m \cdot \psi(1) = \pm m \cdot \mathrm{id}\,.$$

Hence $\psi(\pm m)/n = \pm(m/n) \cdot \mathrm{id}$. Thus, $\mathbb{Q}$ operates on $\mathbb{Q}^{\aleph_0}$ by the usual scalar multiplication.

Since all nontrivial endomorphisms of $\mathbb{Q}$ are injective, by Corollary 3.6 the remaining assertions of the example follow. □

In the following, for a prime number $p \in \mathbb{N}$ let Σ_p denote the p-adic solenoid and let $1/(p^\infty)\mathbb{Z}$ denote its character group $1/(p^\infty)\mathbb{Z} = \widehat{\Sigma_p} = \{m/(p^n) \in \mathbb{Q} \mid m \in \mathbb{Z}, n \in \mathbb{N} \cup \{0\}\}$ (see [5, p.403]). Moreover, Δ_p denotes the group of the p-adic integers.

Example 4.3. The near-ring $[\Sigma_p, \Sigma_p]$ is isomorphic to the abstract affine near-ring $1/(p^\infty)\mathbb{Z} \oplus \Delta_p/\mathbb{Z}$, where the ring $1/(p^\infty)\mathbb{Z}$ operates on the group $\Delta_p/\mathbb{Z}$ by

$$\mu : \frac{1}{p^\infty}\mathbb{Z} \times \Delta_p/\mathbb{Z} \to \Delta_p/\mathbb{Z} : \left(\frac{z}{p^n}, \sum_{i=0}^{\infty} a_i p^i + \mathbb{Z}\right) \mapsto \sum_{i=0}^{\infty} z a_i \frac{p^i}{p^n} + \mathbb{Z}.$$

The ideals of $[\Sigma_p, \Sigma_p]$ are precisely the sets $I \oplus \operatorname{Ext}(1/(p^\infty)\mathbb{Z}, \mathbb{Z})$, where I is an ideal of the ring $1/(p^\infty)\mathbb{Z}$, and the sets $\{0\} \oplus M$, where M is a submodule of the $1/(p^\infty)\mathbb{Z}$-module $\operatorname{Ext}(1/(p^\infty)\mathbb{Z}, \mathbb{Z})$.

PROOF: Since $1/(p^\infty)\mathbb{Z}$ is the character group of the connected compact abelian group Σ_p, by Theorem 3.4 the near-ring $[\Sigma_p, \Sigma_p]$ is isomorphic to the abstract affine near-ring

$$\operatorname{Hom}\left(\frac{1}{p^\infty}\mathbb{Z}, \frac{1}{p^\infty}\mathbb{Z}\right)^{op} \oplus \operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right).$$

By Example 4 in [**3**, p.216] the mapping

$$\alpha : \operatorname{Hom}\left(\frac{1}{p^\infty}\mathbb{Z}, \frac{1}{p^\infty}\mathbb{Z}\right)^{op} \to \frac{1}{p^\infty}\mathbb{Z} : f \mapsto f(1)$$

is an isomorphism of rings. Furthermore, by [**1**, p.829ff] there exists an isomorphism $\beta : \operatorname{Ext}(1/(p^\infty)\mathbb{Z}, \mathbb{Z}) \to \Delta_p/\mathbb{Z}$ of abelian groups, where $\mathbb{Z}$ is the subgroup $\{\sum_{i=0}^{n} a_i p^i \mid n \in \mathbb{N},\ a_i \in \{0, \ldots, p-1\}\}$. This isomorphism induces

$$\beta^\# : \operatorname{Hom}\left(\operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right), \operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right)\right) \to \operatorname{Hom}(\Delta_p/\mathbb{Z}, \Delta_p/\mathbb{Z}) : f \mapsto \beta \circ f \circ \beta^{-1}$$

by [**3**, p.217], an isomorphism of rings from $\operatorname{Hom}(\operatorname{Ext}((1/p^\infty)\mathbb{Z}, \mathbb{Z}), \operatorname{Ext}(1/(p^\infty)\mathbb{Z}, \mathbb{Z}))$ onto $\operatorname{Hom}(\Delta_p/\mathbb{Z}, \Delta_p/\mathbb{Z})$. Then the mapping

$$\psi : \frac{1}{p^\infty}\mathbb{Z} \to \operatorname{Hom}(\Delta_p/\mathbb{Z}, \Delta_p/\mathbb{Z}) : q \mapsto \beta^\# \circ \operatorname{Ext}(\,\cdot\,, \mathbb{Z}) \circ \alpha^{-1}(q)$$

is a homomorphism of rings with $\psi(1) = \mathrm{id}$, and the following diagram is commutative:

$$\begin{array}{ccc}
\operatorname{Hom}\left(\frac{1}{p^\infty}\mathbb{Z}, \frac{1}{p^\infty}\mathbb{Z}\right)^{op} & \xrightarrow{\operatorname{Ext}(\,\cdot\,,\mathbb{Z})} & \operatorname{Hom}\left(\operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right), \operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right)\right) \\
\alpha \downarrow & & \downarrow \beta^\# \\
\frac{1}{p^\infty}\mathbb{Z} & \xrightarrow{\psi} & \operatorname{Hom}(\Delta_p/\mathbb{Z}, \Delta_p/\mathbb{Z}).
\end{array}$$

By Corollary 2.2 the mapping

$$\varphi : \operatorname{Hom}\left(\frac{1}{p^\infty}\mathbb{Z}, \frac{1}{p^\infty}\mathbb{Z}\right)^{op} \oplus \operatorname{Ext}\left(\frac{1}{p^\infty}\mathbb{Z}, \mathbb{Z}\right) \to \frac{1}{p^\infty}\mathbb{Z} \oplus \Delta_p/\mathbb{Z} : (f, E) \mapsto (\alpha(f), \beta(E)).$$

is an isomorphism of near-rings.

Since the mapping $\psi : (1/p^\infty)\mathbb{Z} \to \operatorname{Hom}(\Delta_p/\mathbb{Z}, \Delta_p/\mathbb{Z})$ is a ring homomorphism, we have for all numbers $n \in \mathbb{N}$:

$$p^n \cdot \psi\left(\frac{1}{p^n}\right) = \psi\Bigg(\underbrace{1 + \cdots + 1}_{p^n \text{ summands}}\Bigg) = \psi(1) = \mathrm{id},$$

hence $\psi(1/p^n) = (1/p^n) \cdot \mathrm{id}$. Therefore $1/(p^\infty)\mathbb{Z}$ operates on $\Delta_p/\mathbb{Z}$ by

$$\mu : \frac{1}{p^\infty}\mathbb{Z} \times \Delta_p/\mathbb{Z} \to \Delta_p/\mathbb{Z} : \left(\frac{z}{p^n}, \sum_{i=0}^{\infty} a_i p^i + \mathbb{Z}\right) \mapsto \sum_{i=0}^{\infty} z a_i \frac{p^i}{p^n} + \mathbb{Z}.$$

Since all nontrivial endomorphisms of $1/(p^\infty)\mathbb{Z}$ are injective, the remaining assertions of the example follow by Corollary 3.6. ∎

Again, these examples give by [9] at the same time complete information on the ideal structure of the near-rings $N\left(\widehat{\mathbb{Q}}\right)$ respectively $N(\Sigma_p)$.

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