22

Separability and Entanglement

22.1 Introduction

In this chapter, we discuss the structure of signal amplitudes in terms of the separability and entanglement of quantum register states. We review the concepts of *subregisters*, *splits*, *partitions*, factorizable Hilbert spaces, the separability of states, *separations*, and *entanglements*.

All Hilbert spaces in this chapter are taken to be complex and finite dimensional, and denoted by capital calligraphic Latin letters, such as $\mathcal{H}$. We do not restrict the discussion to quantum bits, but ultimately, it is those in which we are most interested.

Suppose $\mathcal{X}$ and $\mathcal{Y}$ are two finite-dimensional Hilbert spaces with a surjective linear map U from $\mathcal{X}$ to $\mathcal{Y}$.[1] Suppose further that U has the property that it preserves norm, that is, if for any element x in $\mathcal{X}$ we have $\|Ux\|_{\mathcal{Y}} = \|x\|_{\mathcal{X}}$, where $\|x\|_{\mathcal{X}} \equiv \sqrt{(x,x)_{\mathcal{X}}}$ is the Hilbert space "length" of x in $\mathcal{X}$ and $\|y\|_{\mathcal{Y}} \equiv \sqrt{(y,y)_{\mathcal{Y}}}$ is the Hilbert space "length" of y in $\mathcal{Y}$. Then U is called an *isometric isomorphism*. Under these conditions, it is necessarily the case that $\mathcal{X}$ and $\mathcal{Y}$ have the same dimension and, as far as basic Hilbert space properties are concerned, are identical. We denote this form of equality by $\mathcal{X} \simeq \mathcal{Y}$.

22.2 Quantum Registers

Hilbert spaces per se are central to quantized detector networks (QDN) but that is on the mathematical side. Physics brings in empirical context that adds an extra flavor to the discussion, requiring the mathematics of tensor products and quantum registers. We define a quantum register as the tensor product of two or more finite-dimensional Hilbert spaces, referred to as *subregisters*. In principle,

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[1] Here *surjective* means that $U\mathcal{X} = \mathcal{Y}$, where $U\mathcal{X}$ is the image of $\mathcal{X}$ in $\mathcal{Y}$ under the linear map U. This condition is not imposed in our definition of semi-unitary operators (which are linear maps).

subregisters can have any dimension, including one. In the following, we shall rule out one-dimensional Hilbert spaces as being of no empirical interest to us.

Example 22.1   Let $\mathcal{U}$, $\mathcal{V}$, and $\mathcal{W}$ be three complex, finite-dimensional Hilbert spaces of dimensions $3, 2$, and 4, respectively. Consider the tensor product $\mathcal{X}$ defined as

$$\mathcal{X} \equiv \mathcal{U}^{(1)} \otimes \mathcal{W} \otimes \mathcal{U}^{(2)} \otimes \mathcal{V}, \tag{22.1}$$

where $\mathcal{U}^{(1)}$ and $\mathcal{U}^{(2)}$ are copies of $\mathcal{U}$. Then $\mathcal{X}$ is a complex Hilbert space of dimension $3 \times 4 \times 3 \times 2 = 72$.

The number of subregisters in a given quantum register $\mathcal{H}$ is called the *rank* of that register and denoted $rank\mathcal{H}$. In Example 22.1, $\mathcal{X}$ has rank four. A rank-one Hilbert space will be called an *atom*.

The dimension of a rank-r register must have at least r non-trivial factors. For example, a rank-two register of dimension 63 is the tensor product of a 7-dimensional atom with a nine-dimensional atom, or the tensor product of a 21-dimensional atom with a 3-dimensional atom.

The subregister concept, and by implication, that of rank, is contextual, because it is possible to encounter situations where a subregister could be thought of as a register itself, with its own subregisters. In other words, atoms can have constituents.

Example 22.2   *Positronium* is generally described as an unstable bound state of an electron and a positron. When viewed as a single particle, positronium comes in two forms: *para-positronium* and *ortho-positronium*. The former has a particle spin angular momentum classification $j = 0$ (spin zero), described by a singlet spin state in a one-dimensional Hilbert space $\mathcal{H}^{\mathrm{para}}$, while the latter has spin $j = 1$ (spin one) described by spin states in a three-dimensional Hilbert space $\mathcal{H}^{\mathrm{ortho}}$.

On the other hand, a nonrelativistic analysis of positronium in terms of its two constituents would lead to its spin being modeled in terms of a four-dimensional Hilbert space $\mathcal{H}^{\mathrm{electron}} \otimes \mathcal{H}^{\mathrm{positron}}$, a rank-two tensor product of an electron spin space $\mathcal{H}^{\mathrm{electron}}$, and a positron spin space $\mathcal{H}^{\mathrm{positron}}$.

Mathematically, we may write

$$\mathcal{H}^{\mathrm{para}} \oplus \mathcal{H}^{\mathrm{ortho}} \simeq \mathcal{H}^{\mathrm{electron}} \otimes \mathcal{H}^{\mathrm{positron}}, \tag{22.2}$$

where the symbol $\oplus$ denotes a *direct sum* of vector spaces. The direct sum of two Hilbert spaces is a Hilbert space with dimension equal to the sum of the two Hilbert spaces being summed. The group theory of such bound states and their decomposition leads in this case to the rule $\mathbf{1} \oplus \mathbf{3} \simeq \mathbf{2} \otimes \mathbf{2}$ (Lichtenberg, 1970).

Throughout this chapter, we adhere to our convention that subscripts label stages while superscripts label subregisters. Stages concern dynamics, which is not the focus in the present chapter, so we avoid subscripts in this chapter.

As we have seen before in this book, the ordering of subregisters in a tensor product will not usually be regarded as significant; i.e., $\mathcal{H}^a \otimes \mathcal{H}^b$ will mean the same mathematically as $\mathcal{H}^b \otimes \mathcal{H}^a$, as well as physically. What is important is the fact that each subregister can be identified, meaning that labels are physically significant.

The reason for this is based on the physics of observation. For instance, given two detectors D^a and D^b, we may represent them by two Hilbert spaces $\mathcal{H}^a$ and $\mathcal{H}^b$, respectively. There will in general be no natural way of ordering these detectors, and so there is no logical reason to order $\mathcal{H}^a$ and $\mathcal{H}^b$ in a tensor product. What matters is the labeling, which can be regarded as part of the empirical contextual information always available to observers.

With these comments in mind, we henceforth suppress the tensor product symbol $\otimes$, so that $\mathcal{H}^a \mathcal{H}^b$ and $\mathcal{H}^b \mathcal{H}^a$ both mean $\mathcal{H}^a \otimes \mathcal{H}^b$.

Given two Hilbert spaces $\mathcal{H}^a$ and $\mathcal{H}^b$ of dimension d^a and d^b, respectively, the tensor product $\mathcal{H}^a \mathcal{H}^b$ is a Hilbert space of rank two and dimension $d^a d^b$. We define $\mathcal{H}^{[ab]}$ as a rank-one Hilbert space (that is, an atom) of dimension $d^a d^b$ that is isometrically isomorphic to $\mathcal{H}^a \mathcal{H}^b$, that is,

$$\mathcal{H}^{[ab]} \simeq \mathcal{H}^a \mathcal{H}^b. \tag{22.3}$$

Specifically, $\mathcal{H}^{[ab]}$ is the same mathematically as $\mathcal{H}^a \mathcal{H}^b$, but its physical context, that it is a tensor product, is now ignored.

Likewise, given three Hilbert spaces $\mathcal{H}^a$, $\mathcal{H}^b$, $\mathcal{H}^c$, we define $\mathcal{H}^{[abc]}$ to be an atom isometrically isomorphic to $\mathcal{H}^a \mathcal{H}^b \mathcal{H}^c$, and so on for higher rank registers. Note the trivial identity $\mathcal{H}^{[a]} \simeq \mathcal{H}^a$.

As we have indicated previously, the order of superscripts in the above is not significant, so $\mathcal{H}^{[ab]} = \mathcal{H}^{[ba]}$, and so on.

22.3 Splits

A *split* is any convenient way of grouping the subregisters in a quantum register into two or more factor registers, or *atoms*, each of which is regarded for the purposes of that split as a Hilbert space of rank one, that is, a Hilbert space not itself split into two or more factor registers. For large-rank quantum registers, very many different splits will be possible.

Example 22.3 The rank-three register $\mathcal{H}^a \mathcal{H}^b \mathcal{H}^c$ may be split in five distinct ways:

$$\mathcal{H}^a \mathcal{H}^b \mathcal{H}^c \simeq \mathcal{H}^a \mathcal{H}^{[bc]} \simeq \mathcal{H}^b \mathcal{H}^{[ac]} \simeq \mathcal{H}^c \mathcal{H}^{[ab]} \simeq \mathcal{H}^{[abc]}. \tag{22.4}$$

Note that one of these ways is the original register itself.

The number of ways of splitting a rank-n quantum register is the same as the number of ways B_n of partitioning a given set of cardinality n, a historically important topic in combinatorial mathematics. The B_n are called *Bell numbers*[2] and satisfy many curious and interesting relations in diverse fields, such as probability, game theory, and number theory. For instance,

$$B_{n+1} = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} B_k, \quad B_0 = 1, \tag{22.5}$$

from which we find the sequence $\{B_n, n = 1, 2, \ldots\} = \{1, 2, 5, 15, 52, \ldots\}$. An explicit formula for B_n is given by Dobinski's formula,

$$B_n = \frac{1}{e} \sum_{k=0}^{\infty} \frac{k^n}{k!}. \tag{22.6}$$

22.4 Partitions

Consider a set $\{\mathcal{H}^a : a = 1, 2, \ldots, r\}$ of Hilbert spaces, denoting the dimension of $\mathcal{H}^a$ by d^a. Then the rank-r quantum register $\mathcal{H}^{[r]} \equiv \mathcal{H}^1 \mathcal{H}^2 \ldots \mathcal{H}^r$ is a vector space of dimension $d^{[r]} \equiv d^1 d^2 \ldots d^r$ that contains both entangled and separable states.

The classification of states in such a register into separable or entangled types is too limited for us, so we introduce the more useful concepts of *separations* and *entanglements*. We explain our terminology starting with the separable sets.

Separations
For any two subregisters $\mathcal{H}^a, \mathcal{H}^b$ of the quantum register $\mathcal{H}^{[r]}$, such that $a \neq b$, we define the *rank-two separation* $\mathcal{H}^{ab}$ to be the subset of the tensor product $\mathcal{H}^a \mathcal{H}^b$ consisting of all separable elements in that tensor product, that is,

$$\mathcal{H}^{ab} \equiv \left\{ \phi^a \psi^b : \phi^a \in \mathcal{H}^a, \ \psi^b \in \mathcal{H}^b \right\}. \tag{22.7}$$

By definition, $\mathcal{H}^{ab}$ includes the zero vector 0^{ab} of the tensor product $\mathcal{H}^a \mathcal{H}^b$. Note that $\mathcal{H}^{ab} = \mathcal{H}^{ba}$.

The separation concept readily generalizes to higher rank separations as follows. Pick an integer k in the interval $[1, r]$ and then select k different elements $a^1, a^2, \ldots, a^k$ of this interval. Then the rank-k separation $\mathcal{H}^{a^1 a^2 \ldots a^k}$ is the subset of $\mathcal{H}^{a^1} \mathcal{H}^{a^2} \ldots \mathcal{H}^{a^k}$ given by

$$\mathcal{H}^{a^1 a^2 \ldots a^k} \equiv \left\{ \psi^{a^1} \psi^{a^2} \ldots \psi^{a^k} : \psi^{a^i} \in \mathcal{H}^{a^i}, \ \ 1 \leqslant i \leqslant k \right\}. \tag{22.8}$$

Every element of a rank-k separation has k factors. A rank-one separation of a subregister is by definition equal to that subregister, and is therefore a Hilbert space in its own right. Separations of rank greater than one, however, cannot

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[2] After E. T. Bell (Bell, 1938).

be Hilbert spaces because they do not contain entangled states, which all tensor products of rank two or more necessarily do.

Rank-Two Entanglements

We can now construct the *entanglements*, which are defined in terms of complements. Starting with the lowest rank possible, we define the rank-two entanglement $\mathcal{H}^{\overline{ab}}$ to be the complement of the separation $\mathcal{H}^{ab}$ in the tensor product $\mathcal{H}^a \mathcal{H}^b$, that is,

$$\mathcal{H}^{\overline{ab}} \equiv \mathcal{H}^a \mathcal{H}^b - \mathcal{H}^{ab}. \tag{22.9}$$

Note that $\mathcal{H}^{\overline{ab}}$ cannot be a vector space because it does not contain the zero vector.

The original tensor product space considered as a set is therefore the union

$$\mathcal{H}^a \mathcal{H}^b = \mathcal{H}^{ab} \cup \mathcal{H}^{\overline{ab}} \tag{22.10}$$

of the set $\mathcal{H}^{ab}$ of all separable states and the set $\mathcal{H}^{\overline{ab}}$ of all entangled states. These two subsets are disjoint and neither is a vector space.

Separation Products

The generalization of the above to larger rank entanglements is straightforward, but first it will be useful to extend our notation to include the concept of *separation product*.

Suppose A and B are subsets of Hilbert spaces $\mathcal{H}^a$ and $\mathcal{H}^b$, respectively, where $a \neq b$. We define the *separation product* $A \bullet B$ to be the subset of $\mathcal{H}^a \mathcal{H}^b$ given by

$$A \bullet B \equiv \{\psi\phi : \psi \in A, \ \phi \in B\}. \tag{22.11}$$

Properties of the separation product are:

1. The separation product is symmetric, that is, $A \bullet B = B \bullet A$. This means that the separation product is not equivalent to the Cartesian product $A \times B$, which is the set of all ordered pairs of elements.
2. $\mathcal{H}^{ab} = \mathcal{H}^a \bullet \mathcal{H}^b$. Note that this is not the same thing as $\mathcal{H}^a \mathcal{H}^b$, which in our notation is the tensor product of $\mathcal{H}^a$ and $\mathcal{H}^b$.
3. The separation product is associative, commutative, and cumulative, i.e.

$$\left(\mathcal{H}^a \bullet \mathcal{H}^b\right) \bullet \mathcal{H}^c = \mathcal{H}^a \bullet \left(\mathcal{H}^b \bullet \mathcal{H}^c\right) \equiv \mathcal{H}^{abc}$$
$$\mathcal{H}^{ab} \bullet \mathcal{H}^c = \mathcal{H}^{abc}. \tag{22.12}$$

The separation product can also be defined to include entanglements. For example,

$$\mathcal{H}^{\overline{ab}} \bullet \mathcal{H}^c = \left\{\phi\psi : \phi \in \mathcal{H}^{\overline{ab}}, \psi \in \mathcal{H}^c\right\}. \tag{22.13}$$

Significantly, while the separation product of two separations is a separation, the separation product of two entanglements is *not* an entanglement, that is,

$$\mathcal{H}^{\overline{ab}} \bullet \mathcal{H}^{\overline{cd}} \neq \mathcal{H}^{\overline{abcd}}. \tag{22.14}$$

A further notational simplification is to use a single $\mathcal{H}$ symbol, incorporating the separation product symbol $\bullet$ with indices directly, as the following example illustrates.

Example 22.4 Given Hilbert spaces $\mathcal{H}^a, \mathcal{H}^b, \dots, \mathcal{H}^i$, we may write

$$\mathcal{H}^{\overline{cd}\bullet\overline{hi}\bullet abefg} \equiv \mathcal{H}^{ab} \bullet \mathcal{H}^{\overline{cd}} \bullet \mathcal{H}^{efg} \bullet \mathcal{H}^{\overline{hi}}. \tag{22.15}$$

Other expansions are possible, given that separations such as $\mathcal{H}^{efg}$ can be expanded further as separation products.

Associativity of the separation product applies to both separations and entanglements, as can be readily proved.

Higher Rank Entanglements

We can now define higher rank entanglements, such as $\mathcal{H}^{\overline{abc}}$, $\mathcal{H}^{\overline{abcd}}$, and so on. These are defined in terms of complements, in the same way that $\mathcal{H}^{\overline{ab}}$ was defined.

Example 22.5 Consider the rank-three tensor product $\mathcal{H}^a\mathcal{H}^b\mathcal{H}^c$ and the following disjoint subsets: $\mathcal{H}^{abc}, \mathcal{H}^{a\bullet\overline{bc}}, \mathcal{H}^{b\bullet\overline{ac}}, \mathcal{H}^{c\bullet\overline{ab}}$. These are all separable in one way or another. For instance, $\mathcal{H}^{abc}$ is completely separable, while the other three subsets referred to are partially separable. If we remove all those subsets from $\mathcal{H}^a\mathcal{H}^b\mathcal{H}^c$, then what is left will be completely entangled, which is what we want to define. Hence we define the rank-three entanglement $\mathcal{H}^{\overline{abc}}$ as the complement

$$\mathcal{H}^{\overline{abc}} \equiv \mathcal{H}^a\mathcal{H}^b\mathcal{H}^c - \mathcal{H}^{abc} \cup \mathcal{H}^{a\bullet\overline{bc}} \cup \mathcal{H}^{b\bullet\overline{ac}} \cup \mathcal{H}^{c\bullet\overline{ab}}. \tag{22.16}$$

We will refer to a set such as $\mathcal{H}^{\overline{abc}}$ as a rank-three entanglement, and so on. In general, higher rank entanglements such as $\mathcal{H}^{\overline{abcd}}$ in the above require a deal of filtering out of separations and cross-entanglements from the original tensor product Hilbert space for their definition to be possible, which accounts partly for the fact that entanglements are generally not as conceptually simple or intuitive as separations. From this we can appreciate just how complicated the entanglement structure of qubit registers can be.

Exercise 22.6 Show that the rank-four entanglement $\mathcal{H}^{\overline{abcd}}$ is given by

$$\begin{aligned}
\mathcal{H}^{\overline{abcd}} \equiv\ & \mathcal{H}^a\mathcal{H}^b\mathcal{H}^c\mathcal{H}^d \\
&- \mathcal{H}^{abcd} \cup \mathcal{H}^{ab\bullet\overline{cd}} \cup \mathcal{H}^{ac\bullet\overline{bd}} \cup \mathcal{H}^{ad\bullet\overline{bc}} \cup \mathcal{H}^{bc\bullet\overline{ad}} \cup \mathcal{H}^{bd\bullet\overline{ac}} \cup \mathcal{H}^{cd\bullet ab} \\
&\cup \mathcal{H}^{\overline{ab}\bullet\overline{cd}} \cup \mathcal{H}^{\overline{ac}\bullet\overline{bd}} \cup \mathcal{H}^{\overline{ad}\bullet\overline{bc}} \cup \mathcal{H}^{a\bullet\overline{bcd}} \cup \mathcal{H}^{b\bullet\overline{acd}} \cup \mathcal{H}^{c\bullet\overline{abd}} \cup \mathcal{H}^{d\bullet\overline{abc}}.
\end{aligned} \tag{22.17}$$

Entanglement Partitions

The decomposition of a quantum register $\mathcal{H}$ into the union of disjoint separations and entanglements will be called an *entanglement partition* of $\mathcal{H}$, and each element of that partition will be referred to as a *partition element*. We can now list the various entanglement partitions that we have encountered, along with the number of partition elements in each:

$$
\begin{array}{ccc}
\text{rank} & \text{register} & \begin{array}{c}\text{number of}\\ \text{partition elements}\end{array} \\
1 & \mathcal{H}^a & 1 = B_1 \\
2 & \mathcal{H}^a\mathcal{H}^b & 2 = B_2 \\
3 & \mathcal{H}^a\mathcal{H}^b\mathcal{H}^c & 5 = B_3 \\
4 & \mathcal{H}^a\mathcal{H}^b\mathcal{H}^c\mathcal{H}^d & 15 = B_4
\end{array}
\tag{22.18}
$$

A calculation gives 52 partition elements in the entanglement partition of a rank-five quantum register, which we recognize as B_5, the 5th Bell number. The conjecture, then, is that the entanglement partition of a rank-r quantum register has B_r partition elements. This was proved by Eakins (Eakins, 2004), and shows that the entanglement structure of large-rank quantum registers will be too complicated to deal with without computer assistance.

The relationships between separations and entanglements are subtle, as are the relationships between splits and entanglement partitions. Although the number of partitions in the entanglement partition of a rank-r quantum register is the same as the number of splits and given by the Bell numbers, splits and partitions cannot coincide for $r > 1$. Every factor register in a split is a vector space, whereas no partition element is a vector space. Both splits and partitions are essential and unavoidable features in quantum mechanics and hence in the QDN account of quantum measurement and causal set structure.

A final simplification in this line of investigation is to use the above superscript notation to label the various elements of entanglements and separations. So, for example, $\Psi^{abc\bullet\overline{de}\bullet\overline{fgh}}$ denotes a state in the partition element $\mathcal{H}^{abc\bullet\overline{de}\bullet\overline{fgh}}$ and so on. Translated into direct terms, this means the following. First, this state is an element of the rank-eight quantum register

$$
\mathcal{H}^{[8]} \equiv \mathcal{H}^a\mathcal{H}^b\mathcal{H}^c\mathcal{H}^d\mathcal{H}^e\mathcal{H}^f\mathcal{H}^g\mathcal{H}^h.
\tag{22.19}
$$

Second, we may write this state in the factorized form

$$
\Psi^{abc\bullet\overline{de}\bullet\overline{fgh}} = \psi^a\psi^b\psi^c\psi^{\overline{de}}\psi^{\overline{fgh}},
\tag{22.20}
$$

where $\psi^a \in \mathcal{H}^a, \psi^b \in \mathcal{H}^b, \psi^c \in \mathcal{H}^c, \psi^{\overline{de}} \in \mathcal{H}^{\overline{de}}$, and $\psi^{\overline{fgh}} \in \mathcal{H}^{\overline{fgh}}$.

Exercise 22.7 Investigate the superposition (vector addition) properties of separations and entanglements.

22.5 Quantum Zipping

We may avoid the material presented in this chapter when dealing with most situations discussed so far in this book, because those tend to involve relatively low-rank quantum registers with subregisters of relatively low dimension. However, we envisage the application of QDN or its analogues to large-rank, large-dimensionality contexts. Then it will be necessary to have some organization principles available. The split, partition, separation, and entanglement concepts discussed above appear unavoidable in this respect.

Consider the inner product between two pure labstates in a quantum register $\mathcal{H}$. Each state will come from a specific partition. Depending on the details of the two partitions concerned, this inner product may or may not factorize. This is because factor states can only take inner products in combinations that lie in the same factor Hilbert space of some split of the $\mathcal{H}$, a process we refer to as *quantum zipping*. Using the notation for splits, partitions, separations, and entanglements given in the previous chapter, the following example illustrates the point.

Example 22.8   Consider the rank-eight quantum register $\mathcal{H}^{[1\ldots8]}$. By inspection, the inner product of the states $\boldsymbol{\Psi}^{123\bullet\overline{456}\bullet\overline{78}}$ and $\boldsymbol{\Phi}^{145\bullet\overline{23}\bullet\overline{678}}$ takes the factorized form

$$\overline{\boldsymbol{\Phi}}^{145\bullet\overline{23}\bullet\overline{678}}\boldsymbol{\Psi}^{123\bullet\overline{456}\bullet\overline{78}} = (\overline{\phi}^1\psi^1)(\overline{\phi}^{23}\psi^2\psi^3)(\overline{\phi}^4\overline{\phi}^5\overline{\phi}^{678}\psi^{\overline{456}\bullet\overline{78}}), \quad (22.21)$$

which cannot be simplified further. Figure 22.1 illustrates the factorization structure of this inner product. The numbers 1 to 8 represent the individual atoms of the quantum register, not necessarily detectors.

This example is reasonably simple, as the two labstates involved can be ordered as shown. In general, more complex patters will occur, and then the corresponding zip diagrams will be more complicated.

The separation and entanglement structure of labstates should be of importance in any experiment. A significant application of these concepts is to *causal sets*, studied in the next chapter.
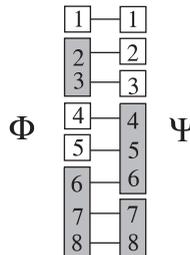
Figure 22.1. Quantum zipping.