ON CHERN CLASSES OF STABLY FIBRE HOMOTOPIC TRIVIAL BUNDLES

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Introduction. Let ξ be a stably fibre homotopic trivial vector bundle. A classical result of Thom states that the Stiefel–Whitney classes of ξ vanish, and one way to prove this is as follows. Let u be the Thom class of ξ in mod 2 cohomology. Then u is stably spherical by **[2]** and therefore all stable cohomology operations vanish on u, showing that $w_i(\xi)u = \mathrm{Sq}^i u = 0$. In this note we shall apply this same method using complex cobordism and Landweber–Novikov operations to study relations among Chern classes of a stably fibre homotopic trivial complex vector bundle. We will thus obtain in a unified way certain strong mod p conditions for every prime p.

The results. Let MU denote the complex cobordism spectrum. Then for every finite sequence E of nonnegative integers there is the Landweber–Novikov operation s_E and we may define the total operation s_t to be $\sum_E s_E t^E$, where $t_1, t_2, \ldots$ are variables, t_i has degree $-2i$, and t^E is the monomial corresponding to the sequence E. If $c_1, c_2, \ldots$ denote the universal Chern classes in MU cohomology and c_E is the polynomial associated with E as in Chapter 16 of [**3**], write $c_t = \sum_E c_E t^E$. Then, if u is the canonical MU cohomology Thom class of a complex vector bundle ξ, we have the formula $s_t u = c_t(\xi)u$; see Part I of [**1**].

Let ξ be a complex vector bundle of complex dimension k over a connected finite CW-complex X and assume that ξ is stably fibre homotopic trivial as a real vector bundle. Then by [**2**] there exists a stable map $f: M(\xi) \to S^{2k}$ of degree one on the bottom cell. Then if σ is the canonical generator in $MU^{2k}(S^{2k})$ we have that $f^*(\sigma)$ is a Thom class for ξ. Therefore $f^*(\sigma) = (1+x)u$, where u is the canonical Thom class of ξ and x belongs to $\widetilde{MU}^0(X)$. Applying s_t to this equation we obtain $s_t(1+x)c_t(\xi) = 1+x$. Now let μ be the canonical map of ring spectra $MU \to H$, where H denotes the Eilenberg–MacLane spectrum for singular integral cohomology. Mapping our last equation into $H^*(X)$ via μ we obtain $\mu(s_t(1+x))c_t(\xi) = 1$, where c_t is the series described above but with coefficients the singular cohomology Chern classes c_E. We now state our theorem.

If E is a nonzero finite sequence of nonnegative integers and $F \leqq E$ is also nonzero, let $\lambda_{E,F}$ be the index of the subgroup $\mathrm{im}(s_F)$ of $MU^0(\text{point})$ and let λ_E be the greatest common divisor of all $\lambda_{E,F}$ for all nonzero $F \leqq E$.

THEOREM. *If ξ is a complex vector bundle as above, then $c_E(\xi)$ is divisible by λ_E.*

Recall from [**4**] that $\widetilde{MU}^*(X)$ is generated over $MU^*(\text{point})$ by a finite set $x_1, \ldots, x_m$ of positive dimensional elements. Write $x = \sum a_i x_i$, where $a_i \in MU^*(\text{point})$. Observe that if $G \neq 0$, then $s_G(1+x) = s_G(x) = \sum\sum s_I(a_i)s_J(x_i)$, where the first sum is over i and the second over $I+J=G$. Since $\mu(s_I(a_i))=0$ unless $s_I(a_i)$ has dimension zero, applying μ to the previous equation we see that $\mu(s_G(1+x))$ is divisible by λ_E. Now from the last equation before the statement of the theorem we deduce that $c_E(\xi) = -\sum \mu(s_G(1+x))c_H(\xi)$, where the sum is over $G+H=E$ with $G \neq 0$, and this implies $c_E(\xi)$ is divisible by λ_E, which is our result.

There remains the problem of calculating λ_E. We leave to the interested reader the exercise of showing that $\lambda_E = 1$ unless there is a prime p such that $i+1$ is a power of p for all nonzero coordinates e_i of E, and that $\lambda_E = p$ in this case.

AUTHORS' NOTE. The reader may have noticed that the philosopy behind this paper is not unlike Antony's crocodile (Shakespeare, Antony and Cleopatra, Act II, Scene VII, lines 47–56).

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