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A CONTINUUM OF C*-NORMS ON $\mathbb{B}(H) \otimes \mathbb{B}(H)$ AND RELATED TENSOR PRODUCTS

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Abstract. For any pair M, N of von Neumann algebras such that the algebraic tensor product $M \otimes N$ admits more than one C*-norm, the cardinal of the set of C*-norms is at least $2^{\aleph_0}$. Moreover, there is a family with cardinality $2^{\aleph_0}$ of injective tensor product functors for C*-algebras in Kirchberg's sense. Let $\mathbb{B} = \prod_n M_n$. We also show that, for any non-nuclear von Neumann algebra $M \subset \mathbb{B}(\ell_2)$, the set of C*-norms on $\mathbb{B} \otimes M$ has cardinality equal to $2^{2^{\aleph_0}}$.

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1. Introduction. A norm α on an involutive algebra A is called a C*-norm if it satisfies

$$\forall x \in A \quad \alpha(x^*x) = \alpha(x)^2$$

in addition to $\alpha(x^*) = \alpha(x)$ and $\alpha(xy) \leq \alpha(x)\alpha(y)$ for all $x, y \in A$. After completion, (A, α) yields a C*-algebra. While it is well known that C*-algebras have a unique C*-norm, it is not so for involutive algebras before completion. For example, it is well known that the algebraic tensor product $A \otimes B$ of two C*-algebras may admit distinct C*-norms, in particular a minimal one and a maximal one denoted respectively by $\| \ \|_{\min}$ and $\| \ \|_{\max}$. When two C*-norms on $A \otimes B$ are equivalent, they must coincide since the completion has a unique C*-norm. The C*-algebras A such that $\| \ \|_{\min} = \| \ \|_{\max}$ on $A \otimes B$ (or equivalently $A \otimes B$ has a unique C*-norm) for any other C*-algebra B are called nuclear. Since they were introduced in the 1950's, they have been extensively studied in the literature, notably in the works of Takesaki, Lance, Effros and Lance, Choi and Effros, Connes, Kirchberg, and many more. We refer to **[16]** or to **[3]** for an account of these developments.

In his 1976 paper **[19]**, Simon Wassermann proved that $\mathbb{B}(H)$ is not nuclear when $H = \ell_2$ (or any infinite dimensional Hilbert space H). Here, $\mathbb{B}(H)$ denotes the C*-algebra formed of all the bounded linear operators on H. This left open the question (see **[9]**) whether $\| \ \|_{\min} = \| \ \|_{\max}$ on $\mathbb{B}(H) \otimes \mathbb{B}(H)$. The latter was answered negatively in

[7]. Curiously however, the proofs in [7] only establish the existence of two inequivalent C*-norms on $\mathbb{B}(H) \otimes \mathbb{B}(H)$, namely the minimal and maximal ones, leaving open the likely existence of many more, which is the main result of this note.

It follows from [7] that the min and max norms are not equivalent on $M \otimes N$ for any pair M, N of von Neumann algebras except if either M or N is nuclear, in which case, of course, the min and max norms are equal. In [19], S. Wassermann showed that a von Neumann algebra M is nuclear if and only if it is "finite type I of bounded degree". This means that M is (isomorphic to) a finite direct sum of tensor products of a commutative algebra with a matrix algebra. Equivalently, this means that M does not contain the von Neumann algebra $\prod_n M_n$ as a C*-subalgebra.

In the first part of this note, we prove that there is at least a continuum of different (and hence inequivalent) C*-norms on the algebraic tensor product $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$. As a corollary, we obtain a continuum of injective tensor product functors for C*-algebras in the sense of Kirchberg [10].

Let $\mathbb{B} = \prod_n M_n$. This is the von Neumann algebra the unit ball of which is the product of the unit balls of the matrix algebras M_n. The assertion that there are at least two distinct C*-norms on $\mathbb{B}(H) \otimes \mathbb{B}(H)$ (or on $M \otimes N$ with M, N not nuclear) reduces to the same assertion on $\mathbb{B} \otimes \mathbb{B}$, and this is used in [7]. It turns out to be immediate to deduce from [7] (see Lemma 8) that the cardinality of the set of C*-norms on $\mathbb{B} \otimes \mathbb{B}$ is $\geq c$. Unfortunately, however, we do not see how to pass from $\mathbb{B} \otimes \mathbb{B}$ to $\mathbb{B}(H) \otimes \mathbb{B}(H)$ in the case of more than two C*-norms. In any case we will show in Section 2 that the cardinality of the set of C*-norms on $\mathbb{B} \otimes \mathbb{B}$ (or $\mathbb{B} \otimes M$ with M non-nuclear) is 2^c with c denoting the continuum.

We end this introduction with some background remarks.

REMARK 1. It is easy to see that any unital simple C*-algebra is what algebraists call "central simple". A unital algebra over a field is called central simple (or centrally simple) if it is simple and its centre is reduced to the field of scalars. It is classical (see e.g. [4, p. 151]) that the tensor product of two such algebras is again central simple, and a fortiori simple. The kernel of a C*-seminorm on (the algebraic tensor product) $A \otimes B$ of two C*-algebras is clearly an ideal. Therefore, if A, B are both simple and unital, any C*-seminorm on (the algebraic tensor product) $A \otimes B$ is a norm as soon as it induces a norm on each of its two factors.

REMARK 2. Let I be a closed ideal in a C*-algebra A. It is well known that the maximal C*-norm is "projective" in the following sense (see e.g. [3, p. 92] or [11, p. 237]): for any other C*-algebra B, $I \otimes_{\max} B$ embeds naturally (isometrically) in $A \otimes_{\max} B$ and we have a natural (isometric) identification

$$(A/I) \otimes_{\max} B = (A \otimes_{\max} B)/(I \otimes_{\max} B). \tag{1}$$

Let $Q(H) = \mathbb{B}(H)/K(H)$ be the Calkin algebra. By Kirchberg's well-known work [8, 10] (see [11, p. 289] or [3, p. 105] for more details), a C*-algebra A is exact if and only if

$$Q(H) \otimes_{\min} A = (\mathbb{B}(H) \otimes_{\min} A)/(K(H) \otimes_{\min} A). \tag{2}$$

Note that $K(H) \otimes_{\min} A = K(H) \otimes_{\max} A$ since $K(H)$ is nuclear. Thus, by (1), if A is not exact, the minimal and maximal C*-norms must differ on $Q(H) \otimes A$.

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REMARK 3. Let A, B, I be as in the preceding Remark. We can define a C*-norm on $(A/I) \otimes B$ by setting, for any $x \in (A/I) \otimes B$,

$$\alpha(x) = \|x\|_{(A \otimes_{\min} B)/(I \otimes_{\min} B)}. \tag{3}$$

More precisely, if $y \in A \otimes B$ is any element lifting x i.e. such that $(q \otimes Id)(y) = x$ where $q: A \to A/I$ denotes the quotient map, we have

$$\alpha(x) = \inf\{\|y + z\|_{\min} \mid z \in I \otimes_{\min} B\}.$$

Since $(I \otimes_{\min} B) \cap (A \otimes B) = I \otimes B$, this is indeed a norm on $(A/I) \otimes B$.

Let $G \subset B$ be any finite dimensional subspace. Then, for any $x \in (A/I) \otimes G$ we have

$$\alpha(x) = \inf\{\|y\|_{\min} \mid y \in A \otimes_{\min} G, \ (q \otimes Id)(y) = x\}. \tag{4}$$

Moreover, the infimum is actually attained. See [**11**, Section 2.4].

Now assume that I is nuclear or merely such that the min and max norms coincide on $I \otimes B$. Then,

$$(A/I) \otimes_{\min} B = (A/I) \otimes_{\max} B \Rightarrow A \otimes_{\min} B = A \otimes_{\max} B. \tag{5}$$

More precisely, it suffices to assume that $\alpha = \| \ \|_{\max}$, i.e. we have

$$(A \otimes_{\min} B)/(I \otimes_{\min} B) = (A/I) \otimes_{\max} B \Rightarrow A \otimes_{\min} B = A \otimes_{\max} B. \tag{6}$$

Indeed, this follows from (1) and $I \otimes_{\min} B = I \otimes_{\max} B$.

2. C*-norms on $M \otimes N$ and $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$. We recall the operator space duality which states that $F \otimes_{\min} E^* \subset CB(E, F)$ isometrically (see Theorem B.13 in [**3**] or [**11**, p. 40]). Namely, for any operator spaces E, F and any tensor $z = \sum_k f_k \otimes e_k^* \in F \otimes E^*$, the corresponding map $\varphi_z: E \to F$ given by $\varphi_z(x) = \sum_k e_k^*(x) f_k$ satisfies $\|z\|_{\min} = \|\varphi_z\|_{cb}$. For a finite dimensional operator space E, we denote by j_E the element of $E \otimes E^*$ which goes to the identity map on E under this correspondence. We note that $\|j_E\|_{\min} = 1$ and that $\|z\|_{\min}$ is independent of the (completely isometric) embeddings $F \hookrightarrow \mathbb{B}(\ell_2)$ and $E^* \hookrightarrow \mathbb{B}(\ell_2)$.

For each $d \in \mathbb{N}$, let $\mathcal{OS}_d$ denote the metric space of all d-dimensional operator spaces, equipped with the cb Banach–Mazur distance. We recall that by [**7**] the metric space $\mathcal{OS}_d$ is non-separable whenever $d \geq 3$. Incidentally, the case $d = 2$ remains an open problem. If A is a separable C*-algebra, then the set $\mathcal{OS}_d(A)$ of all d-dimensional operator subspaces of A is a separable subset of $\mathcal{OS}_d$.

Let M, N be any pair of non-nuclear von Neumann algebras, and let α be a C*-norm on $M \otimes N$. Since $\mathbb{B}$ embeds in both M and N, any $E \in \mathcal{OS}_d$ admits a completely isometric embedding in both. We denote by $\mathcal{M}_d^\alpha$ the subset of $\mathcal{OS}_d$ that consists of all $E \in \mathcal{OS}_d$ admitting (completely isometric) realizations $E \subset M$ and $E^* \subset N$ with respect to which $\|j_E\|_\alpha = 1$.
For example, one has $\mathcal{M}_d^{\max} = \mathcal{OS}_d(C^*(\mathbb{F}_\infty))$ (see [**7**]).

THEOREM 4. *Let M, N be any pair of von Neumann algebras such that $M \otimes_{\min} N \neq M \otimes_{\max} N$. For every $d \in \mathbb{N}$ and every countable subset $\mathcal{L} \subset \mathcal{OS}_d$, there is a* C*-*norm α*

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on $M \otimes N$ such that $\mathcal{M}_d^\alpha$ is separable and contains $\mathcal{L}$. Consequently, there is a family of C*-*norms on $M \otimes N$ with the cardinality of the continuum.*

Proof. First note that our assumption ensures that M, N are not nuclear and hence (by [**19**]) contain a copy of $\mathbb{B}$. For each $E \in \mathcal{L}$, we may assume $E \subset M$ and $E^* \subset N$ completely isometrically. Let $A_E \subset M$ be a separable unital C*-subalgebra containing E completely isometrically. Let $\mathbb{F}$ be a large enough free group so that M is a quotient of $C^*(\mathbb{F})$. Consider the C*-algebraic free product

$$A = \mathop{\ast}_{E\in\mathcal{L}} A_E \ast C^*(\mathbb{F}).$$

Let $Q\colon A \to M$ denote the free product of the inclusions $A_E \subset M$ and the quotient map $C^*(\mathbb{F}) \to M$, and let $I = \ker(Q)$, so that we have $M \simeq A/I$. Let α be the C*-norm defined in (3) with $B = N$. Using $M \simeq A/I$ we view α as a norm on $M \otimes N$. Then for any $E \in \mathcal{L}$, we have $\alpha(j_E) = 1$. Indeed, the inclusion map $E \to A_E \to A$ has *cb* norm 1 and hence defines an element $z \in A \otimes E^*$ with $\|z\|_{\min} = 1$ such that $(Q \otimes I)(z) = j_E$. In the converse direction, let $F \subset M$ be any d-dimensional subspace such that, viewing $F^* \subset N$ we have $\alpha(j_F) = 1$. Then, by (4) (applied to $G = F^*$) j_F admits a lifting $z \in A \otimes F^*$ with $\|z\|_{\min} = 1$. This yields a completely isometric mapping $F \to A$, showing that F is completely isometric to a subspace of A, equivalently $F \in \mathcal{OS}_d(A)$. But it is easy to check that, for any d, the latter set is separable, since any $F \in \mathcal{OS}_d(A)$ is also a subspace of $\ast_{E\in\mathcal{L}} A_E \ast C^*(\mathbb{F}_\infty)$ which is separable (since we assume $\mathcal{L}$ countable). Thus, we have $\mathcal{L} \subset \mathcal{M}_d^\alpha$ and $\mathcal{M}_d^\alpha$ is separable.

For any d-dimensional $E \subset M$, let α_E be the C*-norm associated to the singleton $\mathcal{L} = \{E\}$, and let $\mathcal{C}_E = \mathcal{M}_d^{\alpha_E}$, so that $E \in \mathcal{C}_E$. Let $d'(E, F) = \max\{d_{cb}(E, \mathcal{C}_F), d_{cb}(F, \mathcal{C}_E)\}$, where $d_{cb}(E, \mathcal{C}_F) = \inf\{d_{cb}(E, G) \mid G \in \mathcal{C}_F\}$. By what precedes, if $d'(E, F) > 1$ then necessarily $\alpha_E \neq \alpha_F$ since $\alpha_E(j_F) = \alpha_F(j_E) = 1$ implies $d'(E, F) = 1$.

By [**7**], for some $\varepsilon > 0$, there is a subset $\mathcal{F} \subset \mathcal{OS}_d$ with cardinality $2^{\aleph_0}$ such that $d_{cb}(E, F) > 1 + \varepsilon$ for any $E \neq F \in \mathcal{F}$. Fix ξ such that $1 < \xi < (1+\varepsilon)^{1/2}$. Since all the $\mathcal{C}_E$'s are separable, we claim that there is a subset $\mathcal{F}' \subset \mathcal{F}$ still with cardinality $2^{\aleph_0}$ such that $d'(E, F) > \xi$ for any $E \neq F \in \mathcal{F}'$, and hence the set of C*-norms $\{\alpha_E \mid E \in \mathcal{F}'\}$ has cardinality $2^{\aleph_0}$.

Indeed, let $\mathcal{F}' \subset \mathcal{F}$ be maximal with this property. Then, for any $E \in \mathcal{F}$ there is $F \in \mathcal{F}'$ such that $d'(E, F) \leq \xi$. Now for any E let $\mathcal{D}_E \subset \mathcal{C}_E$ be a dense countable subset. Let $\mathcal{F}'' = \cup_{E\in\mathcal{F}'} \mathcal{D}_E$. For any $E \in \mathcal{F}'$, there is $G = f(E) \in \mathcal{F}''$ such that $d_{cb}(E, G) < (1+\varepsilon)^{1/2}$. This defines a function $f\colon \mathcal{F} \to \mathcal{F}''$. Assume by contradiction that $|\mathcal{F}'| < |\mathcal{F}| = 2^{\aleph_0}$, then also $|\mathcal{F}''| < |\mathcal{F}|$, and hence the function cannot be injective ("pigeon hole"). Therefore, there are $E \neq F \in \mathcal{F}$ such that $f(E) = f(F)$ and hence $d_{cb}(E, F) \leq d_{cb}(E, f(E)) d_{cb}(F, f(E)) < 1 + \varepsilon$ and we reach a contradiction, proving the claim. Thus, we obtain a family of C*-norms $\{\alpha_E \mid E \in \mathcal{F}'\}$ with cardinality $2^{\aleph_0}$. $\square$

We now turn to admissible norms on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$.

We say a C*-norm $\|\cdot\|_\alpha$ on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ is *admissible* if it is invariant under the flip and tensorizes unital completely positive maps (i.e., for every unital completely positive maps $\varphi\colon \mathbb{B}(\ell_2) \to \mathbb{B}(\ell_2)$ the corresponding map $\varphi \otimes \mathrm{id}$ extends to a completely positive map on the C*-algebra $\mathbb{B}(\ell_2) \otimes_\alpha \mathbb{B}(\ell_2)$). Let an admissible C*-norm $\|\cdot\|_\alpha$ be given. We note that for every completely bounded map ψ on $\mathbb{B}(\ell_2)$ one has

$$\|\psi \otimes \mathrm{id}\colon \mathbb{B}(\ell_2) \otimes_\alpha \mathbb{B}(\ell_2) \to \mathbb{B}(\ell_2) \otimes_\alpha \mathbb{B}(\ell_2)\|_{\mathrm{cb}} = \|\psi\|_{\mathrm{cb}}$$

(and likewise for $\mathrm{id} \otimes \psi$), since ψ can be written as $\|\psi\|_{\mathrm{cb}} S_1^* \varphi(S_1 \cdot S_2^*) S_2$ for some unital completely positive map φ on $\mathbb{B}(\ell_2)$ and isometries S_1, S_2 on ℓ_2 (see Theorem 1.6 in [**11**]).

We recall that the density character of a metric space X is the smallest cardinality of a dense subset. Let $\mathfrak{c}$ denote the cardinality of the continuum.

LEMMA 5. *Let $\mathcal{H}$ be the Hilbert space with density character $\mathfrak{c}$ and consider $\ell_2 \subset \mathcal{H}$. Accordingly, let $\mathbb{B}(\ell_2) \subset \mathbb{B}(\mathcal{H})$ (non-unital embedding) and $\theta \colon \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\ell_2)$ be the compression. Then, for every unital completely positive map $\varphi \colon \mathbb{B}(\ell_2) \to \mathbb{B}(\ell_2)$, there are a $*$-homomorphism $\pi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ and an isometry $V \in \mathbb{B}(\ell_2, \mathcal{H})$ such that $\varphi(\theta(a)) = V^*\pi(a)V$ for every $a \in \mathbb{B}(\mathcal{H})$.*

Proof. By Stinespring's Dilation Theorem (see [**11**, p. 24] or [**3**, p. 10]), there are a $*$-representation π of $\mathbb{B}(\mathcal{H})$ on a Hilbert space $\mathcal{K}$ and an isometry $V \in \mathbb{B}(\ell_2, \mathcal{K})$ such that $\varphi(\theta(a)) = V^*\pi(a)V$ for every $a \in \mathbb{B}(\mathcal{H})$. We may assume that $\pi(\mathbb{B}(\mathcal{H}))V\ell_2$ is dense in $\mathcal{K}$. Since $\varphi(\theta(P_{\ell_2})) = 1$, one has $\pi(\mathbb{B}(\mathcal{H}))V\ell_2 = \pi(\mathbb{B}(\ell_2, \mathcal{H}))V\ell_2$. We claim that the density character of $\mathbb{B}(\ell_2, \mathcal{H})$ is $\mathfrak{c}$. Indeed, if we write $\mathcal{H} = \ell_2(I)$ with $|I| = \mathfrak{c}$, then $\mathbb{B}(\ell_2, \mathcal{H}) = \bigcup_{J \in [I]^{\mathbb{N}}} \mathbb{B}(\ell_2, \ell_2(J))$, where $[I]^{\mathbb{N}}$ is the family of countable subsets of I. Since $|[I]^{\mathbb{N}}| = \mathfrak{c}$ and $\mathbb{B}(\ell_2)$ has density character $\mathfrak{c}$, our claim follows. It follows that $\mathcal{K}$ has density character $\mathfrak{c}$ and hence we may identify $\mathcal{K}$ with $\mathcal{H}$. $\square$

Note that, when α is admissible, $\mathcal{M}_d^\alpha$ is a closed subset of $\mathcal{OS}_d$.

THEOREM 6. *For every $d \in \mathbb{N}$ and every separable subset $\mathcal{L} \subset \mathcal{OS}_d$, there is an admissible* C*-*norm α on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ such that $\mathcal{M}_d^\alpha$ is separable and contains $\mathcal{L}$. Consequently, there is a family of admissible* C*-*norms on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$ with the cardinality of the continuum.*

Proof. Let $\mathcal{L}^* = \{E^* : E \in \mathcal{L}\}$ and take a separable unital C*-algebra A_0 such that $\mathcal{OS}_d(A_0)$ contains a dense subset of $\mathcal{L} \cup \mathcal{L}^*$. Let $A = \mathrm{C}^*(\mathbb{F}_\infty) * \mathop{\text{\Large$*$}}_{\mathbb{N}} A_0$ be the unital full free product of the full free group algebra $\mathrm{C}^*(\mathbb{F}_\infty)$ and countably many copies of A_0. Let $\{\sigma_i\}$ be the set of all unital $*$-homomorphisms from A into $\mathbb{B}(\mathcal{H})$ and $\sigma = \mathop{\text{\Large$*$}}_i \sigma_i$ be the $*$-homomorphism from $\tilde{A} = \mathop{\text{\Large$*$}}_i A$ to $\mathbb{B}(\mathcal{H})$, which is surjective. Note that $\mathcal{OS}_d(\tilde{A}) = \mathcal{OS}_d(A)$ and hence it is separable. Denote $J = \ker \sigma$. As in (3), we induce the C*-norm β on $\mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\ell_2)$ from $\tilde{A} \otimes_{\min} \mathbb{B}(\ell_2)$ through $\sigma \otimes \mathrm{id}$, i.e., for every $z \in \mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\ell_2)$ one defines

$$\|z\|_\beta = \inf\{\|\tilde{z}\|_{\tilde{A} \otimes_{\min} \mathbb{B}(\ell_2)} : (\sigma \otimes \mathrm{id})(\tilde{z}) = z\}.$$

Since the infimum is attained, there is a lift $\tilde{z} \in A \otimes F$ such that $\|\tilde{z}\|_{\min} = \|z\|_\beta$.

Consider $\ell_2 \hookrightarrow \mathcal{H}$ and restrict β to $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$, which is still denoted by β. We claim that for every unital completely positive φ on $\mathbb{B}(\ell_2)$, the corresponding maps $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$ are completely positive on $\mathbb{B}(\ell_2) \otimes_\beta \mathbb{B}(\ell_2)$. The latter is trivial. For the former, we use the above lemma. The $*$-homomorphism π on $\mathbb{B}(\mathcal{H})$ induces a map on $\{\sigma_i\}$ and thus a $*$-homomorphism $\tilde{\pi}$ from $\tilde{A}$ into $\tilde{A}$ such that $\sigma \circ \tilde{\pi} = \pi \circ \sigma$. It follows that $\pi \otimes \mathrm{id}$ is a continuous $*$-homomorphism on $\mathbb{B}(\mathcal{H}) \otimes_\beta \mathbb{B}(\ell_2)$ and hence that $\varphi \otimes \mathrm{id}$ is completely positive.

We note that $\beta = \min$ on $E \otimes \mathbb{B}(\ell_2)$ for any $E \in \mathcal{OS}_d(A)$. Let $E \subset \mathbb{B}(\ell_2) \subset \mathbb{B}(\mathcal{H})$ and consider the element $\jmath_E \in E \otimes E^* \subset \mathbb{B}(\mathcal{H}) \otimes \mathbb{B}(\ell_2)$. If $\|\jmath_E\|_\beta = 1$, then $\mathrm{id}_E \colon E \to \mathbb{B}(\ell_2)$ has a completely contractive lift into $\tilde{A}$. Indeed, an isometric lifting $\tilde{\jmath}_E \in \tilde{A} \otimes_{\min} E^*$ corresponds to a complete contraction $\theta \colon E \to \tilde{A}$ for which

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$\sigma \circ \theta = \mathrm{id}_E \colon E \hookrightarrow \mathbb{B}(\mathcal{H})$. It follows that $\mathcal{M}_d^\beta \subset \mathcal{OS}_d(A)$. Finally, take the flip β^{op} of β and let $\alpha = \max\{\beta, \beta^{\mathrm{op}}\}$. □

We recall that a *tensor product functor* is a bifunctor $(A, B) \mapsto A \otimes_\alpha B$ which assigns in a functorial way a C*-completion of each algebraic tensor product $A \otimes B$ of C*-algebras A and B. It is said to be *injective* if $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$ gives rise to a faithful embedding $A_0 \otimes_\alpha B_0 \hookrightarrow A_1 \otimes_\alpha B_1$. See **[10]**. For example, the spatial tensor product functor min is injective, while the maximal one max is not.

COROLLARY 7. *There is a family with cardinality $2^{\aleph_0}$ of different injective tensor product functors.*

Proof. Let α be an admissible C*-norm $\|\cdot\|_\alpha$ on $\mathbb{B}(\ell_2) \otimes \mathbb{B}(\ell_2)$. We extend it to a tensor product functor. For every finite dimensional operator spaces E and F, the norm $\|\cdot\|_\alpha$ is unambiguously defined via embeddings $E \hookrightarrow \mathbb{B}(\ell_2)$ and $F \hookrightarrow \mathbb{B}(\ell_2)$. For every C*-algebras A and B and $z \in A \otimes B$, we find finite dimensional operator subspaces E and F such that $z \in E \otimes F$ and define $\|z\|_\alpha$ to be the α-norm of z in $E \otimes F$. □

3. C*-norms on $\mathbb{B} \otimes \mathbb{B}(\ell_2)$ or $\mathbb{B} \otimes M$.

Let $(N(m))$ be any sequence of positive integers tending to ∞ and let

$$B = \prod_m M_{N(m)}.$$

Actually, the existence of a continuum of distinct C*-norms on $B \otimes B$ can be proved very simply, as a consequence of **[7]**.

LEMMA 8. *Let M be any C*-algebra such that $B \otimes_{\min} M \neq B \otimes_{\max} M$. Then, there is a continuum of distinct C*-norms on $B \otimes M$.*

Proof. For any infinite subset $s \subset \mathbb{N}$, we can define a C*-norm γ_s on $B \otimes M$ by setting

$$\gamma_s(x) = \max\{\|x\|_{\min}, \|(q_s \otimes Id)(x)\|_{B_s \otimes_{\max} B}\},$$

where $B_s = \prod_{m \in s} M_{N(m)}$ and where $q_s : B \to B_s$ denotes the canonical projection (which is a $*$-homomorphism). Let $\hat{B}_s = B_s \oplus \{0\} \subset B$ be the corresponding ideal in B. We claim that if $s' \subset \mathbb{N}$ is another infinite subset such that $s \cap s' = \phi$, or merely such that $t = s \setminus s'$ is infinite, then $\gamma_s \neq \gamma_{s'}$. Indeed, otherwise we would find that the minimal and maximal norms coincide on $\hat{B}_t \otimes M$, and hence (since B embeds in $\hat{B}_t$ and is the range of a unital completely positive projection) on $B \otimes M$, contradicting our assumption. □

By **[7]**, this gives a continuum (γ_s) of distinct C*-norms on $B \otimes B$ or on $B \otimes M$ whenever M is not nuclear. Apparently, producing a family of cardinality $2^{2^{\aleph_0}}$ requires a bit more.

THEOREM 9. *There is a family of cardinality $2^{2^{\aleph_0}}$ of mutually distinct (and hence inequivalent) C*-norms on $M \otimes B$ for any von Neumann algebra M that is not nuclear.*

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REMARK 10. Assuming $M \subset \mathbb{B}(\ell_2)$ non-nuclear, we note that the cardinality of $\mathbb{B}(\ell_2)$ and hence of $M \otimes \mathbb{B}(\ell_2)$ is $c = 2^{\aleph_0}$, so the set of *all* real valued functions of $M \otimes \mathbb{B}(\ell_2)$ into $\mathbb{R}$ has the same cardinal $2^{2^{\aleph_0}}$ as the set of C*-tensor norms.

REMARK 11. In the sequel, the complex conjugate $\bar{a}$ of a matrix a in M_N is meant in the usual way, i.e. $(\bar{a})_{ij} = \overline{a_{ij}}$. In general, we will need to consider the conjugate $\bar{A}$ of a C*-algebra A. This is the same object but with the complex multiplication changed to $(\lambda, a) \to \bar{\lambda}a$, so that $\bar{A}$ is anti-isomorphic to A. For any $a \in A$, we denote by $\bar{a}$ the same element viewed as an element of $\bar{A}$. Note that $\bar{A}$ can also be identified with the opposite C*-algebra A^{op} which is defined as the same object but with the product changed to $(a, b) \to ba$. It is easy to check that the mapping $\bar{a} \to a^*$ is a (linear) $*$-isomorphism from $\bar{A}$ to A^{op}. The distinction between A and $\bar{A}$ is necessary in general, but not when $A = \mathbb{B}(H)$ since in that case, using $H \simeq \overline{H}$, we have $\overline{\mathbb{B}(H)} \simeq \mathbb{B}(\overline{H}) \simeq \mathbb{B}(H)$, and in particular $\overline{M_N} \simeq M_N$. Note however that $H \simeq \overline{H}$ depends on the choice of a basis so the isomorphism $\overline{\mathbb{B}(H)} \simeq \mathbb{B}(H)$ is not canonical.

As in [**7**, **12**] (see also [**17**]), our main ingredient will be the fact that the numbers $C(n)$ defined below are smaller than n. More precisely, it was proved in [**6**] that $C(n) = 2\sqrt{n-1}$ for any n. However, it suffices to know for our present purpose that $C(n) < n$ for infinitely many n's or even merely for some n. This can be proved in several ways for which we refer the reader to [**7**] or [**11**]. See also [**13**] for a more recent-somewhat more refined-approach.

For any integer $n \geq 1$, the constant $C(n)$ is defined as follows: $C(n)$ is the smallest constant C such that for each $m \geq 1$, there is $N_m \geq 1$ and an n-tuple $[u_1(m), \ldots, u_n(m)]$ of unitary $N_m \times N_m$ matrices such that

$$\sup_{m \neq m'} \left\| \sum_{i=1}^{n} u_k(m) \otimes \overline{u_k(m')} \right\|_{\min} \leq C. \tag{7}$$

Throughout the rest of this note, we fix $n > 2$ and a constant $C < n$ and we assume given a sequence of n-tuples $[u_1(m), \ldots, u_n(m)]$ of unitary $N_m \times N_m$ matrices satisfying (7).

By compactness (see e.g. [**12**]), we may assume (after passing to a subsequence) that the n-tuples $[u_1(m), \ldots, u_n(m)]$ converge in distribution (i.e. in moments in the sense of [**18**]) to an n-tuple $[u_1, \ldots, u_n]$ of unitaries in a von Neumann algebra M equipped with a faithful normal trace τ. In fact, if ω is any ultrafilter refining the selected subsequence, we can take for M, τ the associated ultraproduct M_ω of the family $\{M_{N(m)}\}$ $(m \to \infty)$ equipped with normalized traces.

For any subset $s \subset \mathbb{N}$ and any $1 \leq k \leq n$, we denote by $u_k(s) = \oplus_m u_k(s)(m)$ the element of B defined by $u_k(s)(m) = u_k(m)$ if $m \in s$ and $u_k(m) = 0$ otherwise.

Let τ_N denote the normalized trace on M_N. To any free ultrafilter ω on $\mathbb{N}$ is associated a tracial state on B defined for any $x = (x_m) \in B$ by $\varphi_\omega(x) = \lim_\omega \tau_{N(m)}(x_m)$. The GNS construction applied to that state produces a representation

$$\pi_\omega : B \to \mathbb{B}(H_\omega).$$

It is classical (see [**1**]) that $M_\omega = \pi_\omega(B)$ is a II_1-factor and that φ_ω allows to define a trace τ_ω on M_ω such that $\tau_\omega(\pi_\omega(b)) = \varphi_\omega(b)$ for any $b \in B$.

REMARK 12. Let M be a finite von Neumann algebra. Then for any n-tuple $(u_1, \dots, u_n)$ of unitaries in M

$$\left\| \sum_1^n u_k \otimes \bar{u}_k \right\|_{M \otimes_{\max} \overline{M}} = \left\| \sum_1^n u_k \otimes u_k^* \right\|_{M \otimes_{\max} M^{op}} = n. \tag{8}$$

This is a well-known fact. See e.g. [**3**] or [**11**].

LEMMA 13. *Let $\omega \neq \omega'$. Consider disjoint subsets $s \subset \mathbb{N}$ and $s' \subset \mathbb{N}$ with $s \in \omega$ and $s' \in \omega'$, and let*

$$t(s, s') = \sum_{k=1}^n u_k(s) \otimes \overline{u_k(s')} \in B \otimes \bar{B}.$$

Then

$$\|t(s, s')\|_{B \otimes_{\min} \bar{B}} \le C \quad \text{and} \quad \|[\pi_\omega \otimes \overline{\pi_{\omega'}}](t(s, s'))\|_{M_\omega \otimes_{\max} \overline{M_{\omega'}}} = n.$$

Proof. We have obviously

$$\|t\|_{\min} = \sup_{(m,m') \in s \times s'} \left\| \sum u_k(m) \otimes \overline{u_k(m')} \right\|$$

hence $\|t\|_{\min} \le C$. We now turn to the max tensor product. We follow [**12**]. Let $u_k = \pi_\omega(u_k(s))$ and $v_k = \pi_{\omega'}(u_k(s'))$ so that we have

$$\|[\pi_\omega \otimes \overline{\pi_{\omega'}}](t(s, s'))\|_{M_\omega \otimes_{\max} \overline{M_{\omega'}}} = \left\| \sum u_k \otimes \bar{v}_k \right\|_{M_\omega \otimes_{\max} \overline{M_{\omega'}}}.$$

Now, since we assume that $[u_1(m), \dots, u_n(m)]$ converges in distribution, $(u_1, \dots, u_n)$ and $(v_1, \dots, v_n)$ must have the same distribution relative respectively to τ_ω and $\tau_{\omega'}$. But this implies that there is a $*$-isomorphism π from the von Neumann algebra $M(v) \subset M_{\omega'}$ generated by $(v_1, \dots, v_n)$ to the one $M(u) \subset M_\omega$ generated by $(u_1, \dots, u_n)$, defined simply by $\pi(v_k) = u_k$. Moreover, since we are dealing here with *finite* traces, there is a conditional expectation P from $M_{\omega'}$ onto $M(v)$. Therefore, the composition $Q = \pi P$ is a unital completely positive map from $M_{\omega'}$ to $M(u)$ such that $Q(v_k) = u_k$. Since such maps preserve the max tensor products (see e.g. [**3**] or [**11**]), we have

$$\left\| \sum u_k \otimes \bar{v}_k \right\|_{\max} \ge \left\| \sum u_k \otimes \overline{Q(v_k)} \right\|_{M(u) \otimes_{\max} \overline{M(u)}} = \left\| \sum u_k \otimes \bar{u}_k \right\|_{M(u) \otimes_{\max} \overline{M(u)}}.$$

But then by (8), we conclude that $\|t(s, s')\|_{\max} = n$. □

For any free ultrafilter ω on $\mathbb{N}$, we denote by α_ω the norm defined on $B \otimes \bar{B}$ by

$$\forall t \in B \otimes \bar{B} \qquad \alpha_\omega(t) = \max\{\|t\|_{B \otimes_{\min} \bar{B}}, \|[\pi_\omega \otimes Id](t)\|_{M_\omega \otimes_{\max} \bar{B}}\}.$$

THEOREM 14. *There is a family of cardinality $2^{2^{\aleph_0}}$ of mutually distinct (and hence inequivalent) C^*-norms on $B \otimes \bar{B}$. More precisely, the family $\{\alpha_\omega\}$ indexed by free ultrafilters on $\mathbb{N}$ is such a family on $B \otimes \bar{B}$.*

Proof. Let (ω, ω') be two distinct free ultrafilters on $\mathbb{N}$. Let $s \subset \mathbb{N}$ and $s' \subset \mathbb{N}$ be disjoint subsets such that $s \in \omega$ and $s' \in \omega'$. By Lemma 13, we have

$$\alpha_\omega(t(s, s')) \ge \|[\pi_\omega \otimes \overline{\pi_{\omega'}}](t(s, s'))\|_{M_\omega \otimes_{\max} \overline{M_{\omega'}}} = n$$

but since $(\pi_{\omega'} \otimes Id)(t(s, s')) = 0$ we have $\alpha_{\omega'}(t(s, s')) \leq C < n$. This shows α_ω and $\alpha_{\omega'}$ are different, and hence (automatically for C*-norms) inequivalent. Lastly, it is well known (see e.g. [**5**, p. 146]) that the cardinality of the set of free ultrafilters on $\mathbb{N}$ is $2^{2^{\aleph_0}}$. □

Proof of Theorem 9. If M is not nuclear, by [**19**, Corollary 1.9] there is an embedding $B \subset M$. Moreover, since B is injective, there is a conditional expectation from M to B, which guarantees that, for any A, the max norm on $A \otimes \bar{B}$ coincides with the restriction of the max norm on $A \otimes \bar{M}$. Thus, we can extend α_ω to a C*-norm $\widetilde{\alpha}_\omega$ on $B \otimes \bar{M}$ by setting

$$\forall t \in B \otimes \bar{M} \qquad \widetilde{\alpha}_\omega(t) = \max\{\|t\|_{B\otimes_{\min}\bar{M}}, \|[\pi_\omega \otimes Id](t)\|_{M_\omega \otimes_{\max} \bar{M}}\}.$$

Of course, we can replace M by $\bar{M}$. □

REMARK 15. It is easy to see that Theorem 9 remains valid for any choice of the sequence $(N(m))$ and in particular it holds if $N(m) = m$ for all m, i.e. for $B = \mathbb{B}$.

4. Additional remarks. REMARK 16. Let G be a discrete group such that its reduced C*-algebra A is simple. We can associate to any unitary representation $\pi : G \to \mathbb{B}(H_\pi)$ a C*-norm α_π on $A \otimes A$ as follows. Let $\lambda : A \to \mathbb{B}(\ell_2(G))$ and $\rho : A \to \mathbb{B}(\ell_2(G))$ be the left and right regular representations of G linearly extended to A. This gives us a pair of commuting representations of A on $\ell_2(G)$. By the Fell absorption principle (see e.g. [**3**, p. 44] or [**11**, p. 149]), the representation $\pi \otimes \lambda : G \to \mathbb{B}(H_\pi \otimes \ell_2(G))$ is unitarily equivalent to $I \otimes \lambda$, and hence (since A is assumed simple) it extends to a faithful representation on A. Similarly, $I \otimes \rho : G \to \mathbb{B}(H_\pi \otimes \ell_2(G))$ extends to a faithful representation on A. We define

$$\forall a, b \in A \times A \quad \widetilde{\pi}(a \otimes b) = (\pi \otimes \lambda)(a).(I \otimes \rho)(b),$$

and we denote by $\widetilde{\pi}$ the canonical extension to $A \otimes A$. Then, for any $x \in A \otimes A$ we set

$$\alpha_\pi(x) = \|\widetilde{\pi}(x)\|.$$

By Remark 1, this is a C*-*norm* on $A \otimes A$. However, if we restrict it to the diagonal subalgebra $D \subset A \otimes A$ spanned by $\{\lambda(t) \otimes \lambda(t) \mid t \in G\}$, we find for any $x = \sum x(t)\lambda(t) \otimes \lambda(t)$

$$\|\widetilde{\pi}(x)\| = \|\sum x(t)\pi(t) \otimes \sigma(t)\|$$

where $\sigma(t)\delta_s = \delta_{tst^{-1}}$.

Now, if G is any non-Abelian free group, σ is weakly equivalent to $1 \oplus \lambda$ (see [**2**]), so we have for any such diagonal x (using again $\pi \otimes \lambda \simeq I \otimes \lambda$)

$$\|\widetilde{\pi}(x)\| = \max\{\|\sum x(t)\pi(t)\|, \|\sum x(t)\lambda(t)\|\}. \tag{9}$$

But it is known (see [**14**, **15**]) that there is a continuum of unitary representations on a non-Abelian free group G that are "intermediate" between λ and the universal unitary representation of G. More precisely, let $G = \mathbb{F}_k$ be the free group with $k > 1$ generators $g_1, \ldots, g_k$. Let $S_k = \sum_1^k \delta_{g_j} + \delta_{g_j^{-1}}$. By [**15**, Theorem 5], for any number

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$r \in ((2k-1)^{-1/2}, 1)$, G admits a unitary representation π_r such that

$$\|\pi_r(S_k)\| = (2k-1)r + 1/r > 2\sqrt{2k-1}.$$

By (9), we have

$$\|\widetilde{\pi}_r(S_k)\| = (2k-1)r + 1/r,$$

and hence if we define $x_k = \sum \lambda(g_k) \otimes \lambda(g_k) \in A \otimes A$ we find

$$\alpha_{\pi_r}(x_k) = (2k-1)r + 1/r$$

which shows that the family of C*-norms $\{\alpha_{\pi_r} \mid (2k-1)^{-1/2} < r < 1\}$ are mutually distinct. Thus, we obtain in this case a continuum of distinct C*-norms on $A \otimes A$.
Let M denote the von Neumann algebra generated by A in $\mathbb{B}(\ell_2(G))$. Since G is i.c.c. M is a finite factor and hence (see [**16**, p. 349]) is a simple C*-algebra, thus again automatically central simple. The representation $\widetilde{\pi}$ clearly extends to a $*$-homomorphism on $M \otimes M$ which is isometric when restricted either to $M \otimes 1$ or $1 \otimes M$. Thus, we also obtain a continuum of distinct C*-norms on $M \otimes M$, extending the preceding ones on $A \otimes A$.

Remark 17. Let $I \subset A$ and $J \subset B$ be (closed two-sided) ideals in two arbitrary C*-algebras A, B. Assume that there is only one C*-norm both on $I \otimes B$ and on $A \otimes J$. Let $K = I \otimes_{\min} B + A \otimes_{\min} J$. Then, for any pair α, β of distinct C*-norms on $A \otimes B$, the quotient spaces $(A \otimes_\alpha B)/K$ must be different (note that $I \otimes_{\min} B$, $A \otimes_{\min} J$ and hence also K are closed in both $A \otimes_\alpha B$ and $A \otimes_\beta B$). Therefore, the C*-norms naturally induced on $(A/I) \otimes (B/J)$ are also distinct.

For instance, for the Calkin algebra $Q(H)$, we deduce that there are at least $2^{\aleph_0}$ C*-norms on $Q(H) \otimes \mathbb{B}(H)$ or on $Q(H) \otimes Q(H)$.

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