Yet more characterisations of parallelograms

MOWAFFAQ HAJJA and PANAGIOTIS T. KRASOPOULOS

1. *Introduction*

This article, like our previous one [1], combines known and new characterisations of parallelograms. Both can be thought of as additions to Martin Josefsson's series on 'characterisations of' and 'properties of' various types of quadrilaterals – a series that does not include parallelograms. Josefsson's publications can be found listed in [2], [3] and [4]. For the importance of characterisations in geometry, see [5].

Sections 3 and 4 contain several characterisations of parallelograms based on two problems of Victor Thébault that are referred to as *Thébault's first and second problems* in [6] and [7] and that we shall refer to by these names throughout the paper. Since complex numbers are used extensively in that treatment, their basic properties are presented in Section 2. For using complex numbers in geometry we refer the readers to the excellent books [8], [9] and [10]. Section 5 treats a Pompeiu-like property of parallelograms that seemed to us at first to be a characterisation of these figures. Section 6 provides three more characterisations of parallelograms that are based on material scattered in the literature.

2. *Some geometric properties expressed in terms of complex numbers*

In this section we identify the Euclidean plane with the plane of complex numbers, and we let $i = \sqrt{-1}$ as usual. For brevity, we refer to a *positively oriented figure* as *simply positive*.

Figure 1 shows a positive angle $\theta = \angle ZXY$ having equal sides XY and XZ. Using the fact that Z is obtained from Y by rotating Y about X by a positive (i.e. a counter-clockwise) angle of magnitude θ, we obtain $Z - X = (Y - X)e^{i\theta}$, and therefore

$$Z = X + (Y - X)e^{i\theta}. \tag{1}$$

Letting ω be the primitive third root of 1 given by

$$\omega = e^{2i\pi/3},$$

and using (1) and the simple facts that $e^{i\pi/2} = i$, $e^{i\pi/3} = -\omega^2$ and $1 + \omega + \omega^2 = 0$, we obtain

$$Z = \begin{Bmatrix} (1-i)X + iY & \text{if} \quad \theta = \frac{1}{2}\pi, \\ -\omega X - \omega^2 Y & \text{if} \quad \theta = \frac{1}{3}\pi. \end{Bmatrix}. \tag{2}$$

Thus if UVW is a positive regular (i.e. equilateral) triangle, and if $ABCD$ is a positive square, as shown in Figures 2 and 3, then it follows from (2) that

$$\begin{aligned} W &= -\omega U - \omega^2 V, \\ D &= (1-i)A + iB, \\ C &= (1-i)D + iA = (1-i)\big((1-i)A + iB\big) + iA \\ &= -2iA + (1-i)iB + iA = -iA + (1+i)B. \end{aligned}$$

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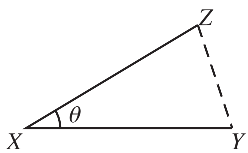

FIGURE 1

$Z = X + (Y - X)e^{i\theta}$

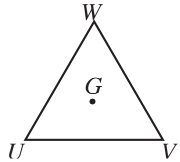

FIGURE 2

$W = -\omega U - \omega^2 V$

$G = \frac{1}{3}\big((1-\omega)U + (1-\omega^2)\big)V$

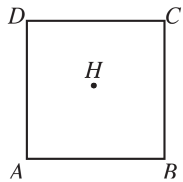

FIGURE 3

$D = (1 - i)A + iB$

$C = -iA + (1 + i)B$

$H = \frac{1}{2}\big((1 - i)A + (1 + i)B\big)$

Also, if the centres of UVW and $ABCD$ are G and H, respectively, then

$$G = \frac{U + V + W}{3} = \frac{(1 - \omega)U + (1 - \omega^2)V}{3}$$

$$H = \frac{A + B + C + D}{4} = \frac{A + B + (-i)A + (1 + i)B + (1 - i)A + iB}{4}$$

$$= \frac{(1 - i)A + (1 + i)B}{2}.$$

These will be very useful and efficient in proving the characterisations to be given in the next sections.

3. *Characterisations based on Thébault's first problem*

Thébault's first problem states that if squares are built outwardly on the sides of a parallelogram $ABCD$, then their centres α, β, γ and δ form a square; see Figure 4. A variant of this that appears in [6] refers to the same configuration and states that if the neighbouring free vertices of the squares are joined, then the midpoints U, V, W and X of the resulting line segments form a square; see Figure 5.

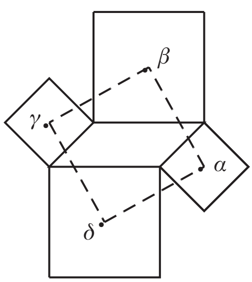

FIGURE 4: Illustrating Thébault's first problem

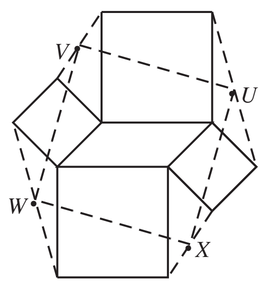

FIGURE 5: Illustrating Viglione's variant

Figure 6 repeats Figures 4 and 5, except that $ABCD$ is now *any* convex quadrilateral, not necessarily a parallelogram. For this general setting, a theorem of van Aubel that first appeared in [11] states that the line segments $\alpha\gamma$ and $\beta\delta$ are perpendicular to each other and have the same length; see [12, Theorem 1]. We can add to this a new variant stating that the line

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segments UW and VX are also perpendicular to each other and have the same length; see Figure 7.

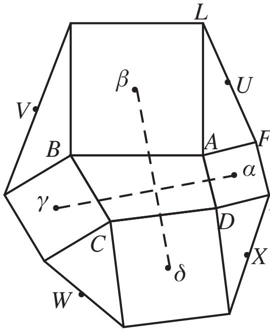

FIGURE 6: Illustrating van Aubel's theorem

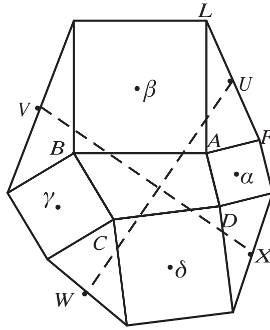

FIGURE 7: Illustrating a new variant

Since a parallelogram whose diagonals are perpendicular to each other and equal in length is a square, it follows that

$$\alpha\beta\gamma\delta \text{ (likewise } UVWX\text{) is a parallelogram} \iff \text{it is a square.} \tag{3}$$

Also, a quadrilateral $PQRS$ is a parallelogram if, and only if, the midpoints $(P + R)/2$ and $(Q + S)/2$ of its diagonals coincide. Thus

$$PQRS \text{ is a parallelogram} \iff P + R = Q + S.$$

We shall now give in Theorem 1 short proofs of van Aubel's theorem and the new variant. The proof of van Aubel's theorem does not differ essentially from the one given in [12], which in turn is reproduced in [10, p. 16].

Theorem 1: As shown in Figure 6, let squares with centres α, β, γ and δ be drawn outwardly on the sides of a convex quadrilateral $ABCD$, and let the midpoints of the line segments joining the neighbouring free vertices of these squares be U, V, W, X. Let L, E and F be as shown. Then the line segments $\alpha\gamma$ and $\beta\delta$ are perpendicular and equal in length. Similarly for the line segments UW and VX.

Proof: It is easy to see that the permutation

$$(A \to B \to C \to D \to A)$$

induces the permutations

$$(U \to V \to W \to X \to U) \text{ and } (\alpha \to \beta \to \gamma \to \delta \to \alpha).$$

Also, using Figure 3, we can find E, F, α and L in terms of A, B, C, D, and we can then find $U = \frac{1}{2}(L + F)$. Thus we have

$$E = -iA + (1 + i)D, \quad F = (1 - i)A + iD, \quad L = (1 + i)A - iB$$
$$\alpha = \frac{(1 - i)A + (1 + i)D}{2}, \quad U = \frac{2A - iB + iD}{2}. \tag{4}$$

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It follows from these that

$$\alpha - \gamma = \frac{(1 - i)A + (1 + i)D}{2} - \frac{(1 - i)C + (1 + i)B}{2}$$

$$= \frac{(1 + i)D - (1 - i)C - (1 + i)B + (1 - i)A}{2}$$

$$\beta - \delta = \frac{(1 + i)A - (1 - i)D - (1 + i)C + (1 - i)B}{2} = i(\alpha - \gamma).$$

Thus if we rotate the line segment $\gamma\alpha$ by 90°, we obtain a line segment perpendicular and equal in length to $\delta\beta$. This proves that the line segments $\gamma\alpha$ and $\delta\beta$ are perpendicular and equal in length.

The proof that the line segments UW and VX are also perpendicular and equal in length follows in a similar manner from the calculations

$$U - W = \frac{-iB + 2A + iD}{2} - \frac{-iD + 2C + iB}{2}$$

$$= iD - C - iB + A$$

$$V - X = iA - D - iC + B = i(U - W).$$

The next theorem gives two characterisations of parallelograms based on Figure 6.

Theorem 2: Referring to Figures 6 and 7, let *ABCD* be a positive convex quadrilateral, and let four positive squares be built outwardly on its sides. Let α, β, γ and δ be their centres. Let U, V, W and X be the midpoints of the four segments that join neighbouring free vertices of the squares. Then

$$\begin{aligned} ABCD \text{ is a parallelogram } &\Leftrightarrow \alpha\beta\gamma\delta \text{ is a square,} \\ &\Leftrightarrow UVWX \text{ is a square.} \end{aligned}$$

Proof: These follow from the formulas (4) and their permutations and the following routine calculations.

$$\begin{aligned} \alpha\beta\gamma\delta \text{ is a square } &\Leftrightarrow \alpha\beta\gamma\delta \text{ is a parallelogram, by (3)} \\ &\Leftrightarrow \alpha + \gamma = \beta + \delta \\ &\Leftrightarrow \frac{(1 - i)A + (1 + i)D}{2} + \frac{(1 - i)C + (1 + i)B}{2} \\ &\qquad = \frac{(1 - i)B + (1 + i)A}{2} + \frac{(1 - i)D + (1 + i)C}{2} \\ &\Leftrightarrow -iA + iD - iC + iB = 0 \\ &\Leftrightarrow A + C = B + D \\ &\Leftrightarrow ABCD \text{ is a parallelogram.} \\ UVWX \text{ is a square } &\Leftrightarrow UVWX \text{ is a parallelogram, by (3)} \\ &\Leftrightarrow U + W = V + X \\ &\Leftrightarrow \frac{-iB + 2A + iD}{2} + \frac{-iD + 2C + iB}{2} \end{aligned}$$

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$$= \frac{-iC + 2B + iA}{2} + \frac{-iA + 2D + iC}{2}$$

$$\Leftrightarrow D + B = C + A$$

$\Leftrightarrow$ *ABCD* is a parallelogram.

4. *A characterisation based on Thébault's second problem*

The second problem of Thébault, as it is called in [7], states that if *ABCD* is a positive parallelogram, and if *ADE* and *BAF* are equilateral triangles built outwardly on sides *AD* and *BA*, then *CEF* is an equilateral triangle; see [13, Problem 18, p. 314] and [7]. The converse is not difficult to see. In fact, let *ABCD* be a positive convex quadrilateral and let *ADE* and *BAF* be equilateral triangles built outwardly on sides *AD* and *BA*, as shown in Figure 8. Supposing that triangle *CEF* is equilateral, we complete *DAB* to a parallelogram *ABC′D*, and observe that triangle *C′EF* is also equilateral (by the aforementioned Thébault theorem). Since one cannot build two different equilateral triangles UVW and TVW on the same side of a line segment VW, it follows that $C' = C$, and hence *ABCD* is a parallelogram.

We state this characterisation of parallelograms in the following theorem, and we give a proof using complex numbers. We shall use the fact that if XYZ is a positive triangle placed in the complex plane, then

$$XYZ \text{ is equilateral if, and only if, } X + \omega Y + \omega^2 Z = 0. \tag{5}$$

This appears as Proposition 2 (p. 71) of [9] and [8, Example, p. 60], and a simple proof follows from the formula for W in Figure 2.

Theorem 3: Let *ABCD* be a positive convex quadrilateral, as shown in Figure 8. Then *ABCD* is a parallelogram if, and only if, the positive equilateral triangles *ADE* and *BAF* that are drawn outwardly on *AD* and *BA* have the property that the triangle *CEF* is equilateral.

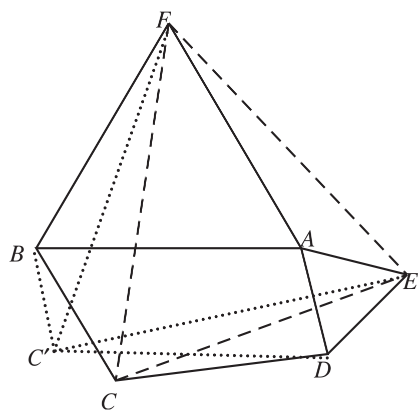

FIGURE 8

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Proof: Since EAD and FBA are positive and equilateral, it follows from (5) that

$$E = -\omega A - \omega^2 D, \qquad F = -\omega B - \omega^2 A.$$

Using (5) again, we obtain

$$\begin{aligned}\text{triangle } CEF \text{ is equilateral } &\Leftrightarrow C + \omega E + \omega^2 F = 0 \\ &\Leftrightarrow C + \omega\left(-\omega A - \omega^2 D\right) + \omega^2\left(-\omega B - \omega^2 A\right) = 0 \\ &\Leftrightarrow A + C = B + D \\ &\Leftrightarrow ABCD \text{ is a parallelogram}.\end{aligned}$$

This proves the theorem.

5. *The Pompeiu property for parallelograms*

A theorem of Pompeiu states that if ABC is an equilateral triangle, and if P is any point in its plane that is not a vertex, then the distances PA, PB and PC can serve as the side-lengths of a triangle. Proofs of this and of possible generalisations to higher dimensions have attracted many authors, as seen in [14] and [15]. It is easy to see that if ABC is not equilateral, say $AB > AC$, then points P very close to A would have the property that $PA + PC < PB$, because $PA \approx 0$, $PC \approx AC$, $PB \approx AB$. Thus the distances PA, PB and PC cannot serve as the side-lengths of a triangle. This observation appears in [16], and shows that the Pompeiu property described above is a characterisation of equilateral triangles, and ought to be added to the list of characterisations of equilateral triangles compiled in [17].

Related to the previous paragraph, Problem 6 on p. 52 (with a solution on p. 229) of [18] states that squares also have the Pompeiu property. It states that if P is a point in the plane of a square that is not a vertex, then its distances from the vertices of the square can serve as the side lengths of a quadrilateral. A negative answer to the possible conjecture that this property characterises squares is given in Theorem 1.2.3 on p. 12 of [19], which tells us that the same property holds also for parallelograms. This raises the question whether it characterises parallelograms. The answer turns out to be negative, and lavishly so, as Theorem 4 below shows. However, a full characterisation of those convex quadrilaterals which have the Pompeiu property is still being investigated, and so far we know that Theorem 4 is very far from being such.

Before stating our next theorem, we remind the reader that if $ABCD$ is a convex quadrilateral having

$$AB = a, BC = b, CD = c, DA = d, \tag{6}$$

then a, b, c and d satisfy the triangle inequalities. Conversely, it is known that if a, b, c and d are positive numbers, then the following are equivalent.

There exists a quadrilateral $ABCD$ satisfying (6).

$\Leftrightarrow$ There exists a convex quadrilateral $ABCD$ satisfying (6).

$\Leftrightarrow$ There exists a cyclic quadrilateral $ABCD$ satisfying (6).

$\Leftrightarrow$ The sum of every three of these numbers is greater than the fourth.

For a reference, see, for example, [20, p. 8, ll. 19-24] and [19, Theorem 1.2.1, p. 10].

Theorem 4: Let $Q = ABCD$ be a convex quadrilateral in which a pair of opposite sides are equal, as shown in Figure 9. Let P be a point in its plane different from the vertices of Q. Then the distances PA, PB, PC and PD can serve as the side lengths of a quadrilateral.

Proof: Without loss of generality we assume $AB = CD$. Then it follows from the triangle inequality that

$$PA < PB + AB = PB + DC \leqslant PB + PC + PD,$$

where the last inequality is an equality if, and only if, P is on the side CD. Thus

$$PA < PB + PC + PD.$$

Similarly for the remaining three inequalities

$$PB < PA + PC + PD, \quad PC < PA + PB + PD, \quad PD < PA + PB + PC,$$

and the proof is complete.

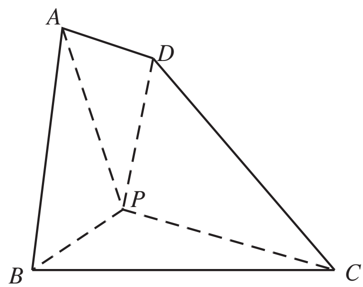

FIGURE 9

6. *A few more characterisations*

We end this article with few sparse characterisations of parallelograms that did not find a place in the previous sections.

The first such characterisation appears in Exercise 8.20 (p. 207) of [21] and states that parallelograms are the only quadrilaterals that are affinely equivalent to a square. To affinely transform a given parallelogram $Q = ABCD$ lying in the Cartesian plane into a square, one first assumes that A is the origin $(0, 0)$ (which is obtained by applying a shift), then applies the linear transformation that takes B and D to $(1, 0)$ and $(0, 1)$, respectively. Since $C = A + B$, C goes to $(1, 1)$, showing that $ABCD$ is affinely transformed into the square with vertices

$$(0,\ 0),\ (1,\ 0),\ (1,\ 1),\ (0,\ 1).$$

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Conversely, it is easy to see that every affine transformation takes a square into a parallelogram.

The second characteristic property of a parallelogram is given in Problem 9.1.11 (p. 175) of [22], and it states that parallelograms are the only *polygons*, not only quadrilaterals, which cannot be enclosed in a triangle whose sides lie along three sides of the given polygon. Interestingly, the proof uses mathematical induction, a tool rather alien to the field of geometry.

The third characterisation appears as Problem 7.1.20 (p. 187) of [19]. It says that if $ABCD$ is a convex quadrilateral, and if

$$[ABC] \leqslant [CDA] \leqslant [BCD] \leqslant [DAB], \tag{7}$$

then $ABCD$ is a parallelogram, where [.] denotes the area. To see this, let M be the point of intersection of the diagonals of $ABCD$, as shown in Figure 10, let

$$x = AM,\ y = BM,\ z = CM,\ w = DM,\ t = \sin\angle AMD.$$

Then (7) translates into

$$\frac{y(x+z)t}{2} \leqslant \frac{w(x+z)t}{2} \leqslant \frac{z(y+w)t}{2} \leqslant \frac{x(y+w)t}{2},$$

which in turn simplifies to

$$y \leqslant w,\ wx \leqslant zy,\ z \leqslant x.$$

This implies that $yz \leqslant wx \leqslant zy$, and hence equality in each case. Thus $y = w$ and $z = x$, and hence the diagonals bisect each other, and $ABCD$ is a parallelogram, as desired. Of course, the converse it trivial, since (7) holds (in fact with equalities) for parallelograms.

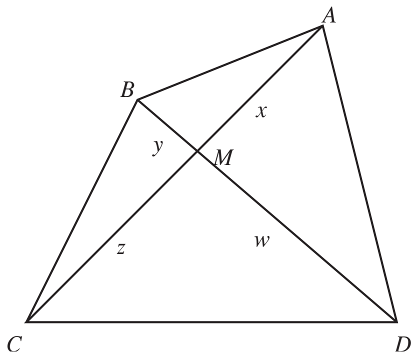

FIGURE 10

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MOWAFFAQ HAJJA
P. O. Box 388 (Al-Husun),
21510 Irbid – Jordan
e-mail: *mowhajja1234@gmail.com*, *mowhajja@yahoo.com*
PANAGIOTIS T. KRASOPOULOS
Department of Informatics, KEAO
Electronic National Social Security Fund
12 Patision St. 10677 Athens – Greece
e-mail: *pan_kras@yahoo.gr*, *pankras@teemail.gr*