ON FINITE POLARIZED PARTITION RELATIONS

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1. Theorems. Call an $m \times n$ array an $m \times n; k$ array if its mn entries come from a set of k elements. An $m \times n; 1$ array has mn like entries. We write

$$(1) \qquad (m, n; k) \longrightarrow (p, q; 1)$$

if every $m \times n; k$ array contains a $p \times q; 1$ sub-array. The negation of (1) is written

$$(m, n; k) \not\longrightarrow (p, q; 1)$$

and means that there is an $m \times n; k$ array containing no $p \times q; 1$ sub-array. Relations (1) are called "polarized partition relations among cardinal numbers" by P. Erdős and R. Rado [2]. In this note we prove the following theorems.

THEOREM 1. *If* $n\binom{m/k}{p} > (q-1)\binom{m}{p}$ *then* $(m, n; k) \longrightarrow (p, q; 1)$.

THEOREM 2. $(k^2 + k + 1, k^2 + k + 1; k) \longrightarrow (2, 2; 1)$.

THEOREM 3. *If* $k \geq 2$ *then*

$$\left(\left\{(p-1)\frac{kt-1}{t-1}\right\}, \left\{t^p (q-1) k^q\right\}; k\right) \longrightarrow (p, q; 1)$$

for every real $t > 1$, *where* $\{s\}$ *denotes the least integer* $\geq s$.

C. Frasnay [3] proved that

$$(1 + pk^{1+pk}, 1 + pk^{1+pk}; k) \longrightarrow (p, p; 1).$$

We improve this in

THEOREM 4. $(pk^p, pk^p; k) \longrightarrow (p, p; 1)$.

THEOREM 5. *If* $\binom{m}{p}\binom{n}{q} < k^{pq-1}$ then $(m, n; k) \not\longrightarrow (p, q; 1)$.

In particular

$$([(p!)^{1/p} k^{(p^2-1)/2p}], [(p!)^{1/p} k^{(p^2-1)/2p}]; k) \not\longrightarrow (p, p; 1).$$

P. Erdős and R. Rado [2, p. 485] proved

$$(kp - k + 1, k^{kp-k+1}(q - 1) + 1; k) \longrightarrow (p, q; 1).$$

We improve this in

THEOREM 6. $(kp - k + 1, k\binom{kp-k+1}{p}(q - 1) + 1; k) \longrightarrow (p, q; 1)$,

$$(kp - k + 1, k\binom{kp-k+1}{p}(q - 1); k) \not\longrightarrow (p, q; 1).$$

Theorem 6 has been proved independently by Scott Niven.

THEOREM 7. $(4p - 3, \binom{2p-1}{p} + 2; 2) \longrightarrow (p, 2; 1)$.

2. Proofs. We first prove two lemmas. K. Zarankiewicz [5] asked for the least positive integer $k(m, n; p, q)$ such that every choice of k entries of an $m \times n$ array contains a $p \times q$ sub-array. For a detailed survey of this problem, see [4].

LEMMA 1. *If* $n\binom{N/n}{p} > (q - 1)\binom{m}{p}$ *then* $k(m, n; p, q) \leq N$.

Proof. Let $A = (a_{ij})$ be an $m \times n$ array and S be a choice of N of the mn entries of A. Then $c_j = \sum_{a_{ij} \in S} 1$ is the number of elements of S which come from column j of A, so

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$$\sum_{j=1}^{n} c_j = N \text{ and } \sum_{j=1}^{n} \binom{c_j}{p} \geq n\binom{N/n}{p} > (q-1)\binom{m}{p}.$$

By the pigeon-hole principle, q of the choices (of p rows) counted in $\sum_{j=1}^{n} \binom{c_j}{p}$ must be the same, so S contains a $p \times q$ sub-array.

LEMMA 2. *If* $k \geq 2$, $p \geq 3$ *and* $t = (p/(p-1))^{1/p}$ *then* $t^p k^p > \frac{kt-1}{t-1}$.

Proof. When $k = 2$, $p = 3, 4, 5$, the inequality is easily verified. In other cases we have

$$t^p k^{p-1} - 1 = \frac{p}{p-1} k^{p-1} - 1 \geq p^2,$$

so

$$\left(1 + \frac{1}{t^p k^{p-1} - 1}\right)^{p^2} < e < \left(1 + \frac{1}{p-1}\right)^{p} = t^{p^2},$$

or

$$1 + \frac{1}{t^p k^{p-1} - 1} < t,$$

and the desired inequality follows.

Proof of Theorem 1. Let A be an $m \times n; k$ array with entries from the set $\{1, 2, \ldots, k\}$, and, for $j = 1, 2, \ldots, k$, let r_j denote the number of entries equal to j. Then $r_1 + r_2 + \ldots + r_k = mn$, so $\max(r_1, \ldots, r_k) \geq mn/k$, say $r_j \geq mn/k$. Now, $(r_j/n) \geq (m/k)$, so by the hypothesis

$$n\binom{r_j/n}{p} > (q-1)\binom{m}{p}.$$

By Lemma 1, the r_j entries of A (all equal to j) contain a $p \times q$ sub-array. This sub-array is a $p \times q; 1$ sub-array of A.

Proof of Theorem 2. Let A be a $(k^2+k+1)\times(k^2+k+1); k$ array, $m = n = k^2+k+1$, $p = q = 2$. We have

$$n\binom{m/k}{p} = (k^2+k+1)\binom{(k^2+k+1)/k}{2} > (k^2+k+1)\binom{k+1}{2} = \binom{k^2+k+1}{2} = (p-1)\binom{b}{q}$$

so the hypothesis of Theorem 1 is satisfied. By Theorem 1, A contains a $2\times 2; 1$ sub-array.

Proof of Theorem 3. Let t be a real ≥ 1 and let p, q be positive integers. Let A be an $m\times n; k$ array where $m = \{(p-1)(kt-1)/(t-1)\}$, $n = \{t^p(q-1)k^p\}$. We have $m - k(p-1) \geq (m-(p-1))/t$, $k \geq 2$, so

$$n\cdot\frac{m}{k}\left(\frac{m}{k}-1\right)\ldots\left(\frac{m}{k}-p+1\right) > t^p(q-1)\,k^p\,\frac{m}{kt}\cdot\frac{m-1}{kt}\cdot\ldots\cdot\frac{m-p+1}{kt} =$$

$$= (q-1)\,m(m-1)\ldots(m-p+1)$$

or $n\binom{m/k}{p} > (q-1)\binom{m}{p}$. By Theorem 1, A contains a $p\times q; 1$ sub-array which is the desired result.

Proof of Theorem 4. Let A be a $pk^p\times pk^p; k$ array. We have to show that A contains a $p\times p; 1$ sub-array. This is trivial when $p = 1$ or $k = 1$. When $p = 2$, $k \geq 2$, the conclusion follows by Theorem 2 as $2k^2 \geq k^2+k+1$. When $p = 3$, $k \geq 2$, we have

$$(\{(p-1)(kt-1)/(t-1)\},\ \{t^p(p-1)k^p\};\ k) \longrightarrow (p, p; 1)$$

by Theorem 3. Set $t = (p/(p-1))^{1/p}$. Then by Lemma 2, $t^pk^p > (kt-1)/(t-1)$ or $pk^p = t^p(p-1)k^p > (p-1)(kt-1)/(t-1)$. Therefore A contains a $p\times p; 1$ sub-array and the proof is complete.

Proof of Theorem 5. We shall use a method developed by P. Erdös [1]. Consider $m\times n; k$ arrays with entries from the set $\{1, 2, \ldots, k\}$. There are exactly k^{mn} of them. We shall show that at most $\binom{m}{p}\binom{n}{q}k^{mn-pq+1}$ of these arrays contain a $p\times q; 1$ sub-array.

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Given any $m \times n; k$ array A there are $\binom{m}{p}\binom{n}{q}$ possibilities of choosing its $p \times q$ sub-array. Moreover, this sub-array is a $p \times q; 1$ sub-array in exactly $k^{mn-pq+1}$ cases since there are k possibilities of choosing its entries as well as k possibilities of choosing each of the remaining $mn - pq$ entries of A. Now, by the hypothesis,

$$\binom{m}{p}\binom{n}{q} k^{mn-pq+1} < k^{mn} ,$$

so there exists an $m \times n; k$ array containing no $p \times q; 1$ sub-array, which is the desired result. In particular, if $m \leq (p!)^{1/p} k^{(p^2-1)/2p}$ then $\binom{m}{p} < \frac{m^p}{p!} \leq k^{(p^2-1)/2}$ or $\binom{m}{p}\binom{m}{p} < k^{p^2-1}$ so there exists an $m \times m; k$ array containing no $p \times p; 1$ sub-array.

Proof of Theorem 6. Let A be a $(kp - k + 1) \times (k\binom{kp-k+1}{p}(q - 1)+1);k$ array. Given any column of A there exists (by the pigeon-hole principle) a p-tuple of rows such that their intersection with the column forms a $p \times 1; 1$ array. There are exactly $\binom{kp-k+1}{p}$ possibilities of choosing such a p-tuple; hence, by the pigeon-hole principle again, $k(q - 1) + 1$ of the choices must be the same. Finally, q of these $k(q - 1) + 1$ choices must correspond to the same element of $\{1, 2, \ldots, k\}$. In other words, A contains a $p \times q; 1$ sub-array.

To prove the second part, we observe that there exists a set $\mathcal{G}$ of $kp - k + 1 \times 1; k$ arrays with entries from the set $\{1, 2, \ldots, k\}$ provided that

(i) if $A \in \mathcal{G}$, then A contains exactly one $p \times 1; 1$ sub-array;

(ii) $\mathcal{G}$ has exactly $k\binom{kp-k+1}{p}$ elements;

(iii) a $(kp - k + 1) \times k\binom{kp-k+1}{p}; k$ array whose columns are all elements of $\mathcal{G}$ contains no $p \times 2; 1$ sub-array.

Hence, there exists a $(kp - k + 1) \times k(q - 1)\binom{kp-k+1}{p}; k$ array containing no $p \times q; 1$ sub-array, an array whose columns are elements of $\mathcal{G}$, each of them being used exactly $(q - 1)$-times.

Proof of Theorem 7. Let $A = (a_{ij})$ be a $(4p - 3) \times (\binom{2p-1}{p} + 2);2$ array with elements from the set $\{1, 2\}$. Evidently, there is a set

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$S \subset \{1, 2, \ldots, 4p - 3\}$, $|S| = 2p - 1$, such that $a_{i1} = a_{k1}$ whenever $i, k \in S$, say $a_{i1} = a_{k1} = 1$. Now, given the j-th column of A, there exist a set $T = T(j) \subset S$, $|T| = p$ and an element $r = r(j)$ of $\{1, 2\}$ such that $a_{ij} = r(j)$ whenever $i \in T(j)$. If $r(j) = 2$ for every $j = 2, 3, \ldots, \binom{2p-1}{p} + 2$ then there are integers j_1, j_2 such that $2 \leq j_1 < j_2 \leq \binom{2p-1}{p} + 2$, $T(j_1) = T(j_2) = T$ and A contains a $p \times 2; 1$ sub-array (a_{ij}), $i \in T$, $j \in \{j_1, j_2\}$. If there exists an integer j_0, $2 \leq j_0 \leq \binom{2p-1}{p} + 2$ such that $r(j_0) = 1$ then A contains a $p \times 2; 1$ sub-array (a_{ij}) $i \in T(j_0)$, $j \in \{1, j_0\}$.

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