Proceedings of the Royal Society of Edinburgh, page 1 of 51
DOI:10.1017/prm.2024.6

On the Sobolev stability threshold for shear flows near Couette in 2D MHD equations

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(Received 16 June 2023; accepted 14 January 2024)

In this work, we study the Sobolev stability of shear flows near Couette in the 2D incompressible magnetohydrodynamics (MHD) equations with background magnetic field $(\alpha, 0)^\top$ on $\mathbb{T}\times\mathbb{R}$. More precisely, for sufficiently large α, we show that when the initial datum of the shear flow satisfies $\|U(y)-y\|_{H^{N+6}} \ll 1$, with $N>1$, and the initial perturbations $u_{\rm in}$ and $b_{\rm in}$ satisfy $\|(u_{\rm in}, b_{\rm in})\|_{H^{N+1}} = \epsilon \ll \nu^{\frac{5}{6}+\tilde{\delta}}$ for any fixed $\tilde{\delta}>0$, then the solution of the 2D MHD equations remains $\nu^{-(\frac{1}{3}+\frac{\tilde{\delta}}{2})}\epsilon$-close to $(e^{\nu t\partial_{yy}}U(y), 0)^\top$ for all $t>0$.

Keywords: MHD equations; Couette flow; stability

2020 *Mathematics Subject Classification:* 76E05; 76E25

1. Introduction

1.1. Problem statement and background

Consider the 2D incompressible MHD equations on $\mathbb{T}\times\mathbb{R}$:

$$\begin{cases} \partial_t\tilde{u} + \tilde{u}\cdot\nabla\tilde{u} - \tilde{b}\cdot\nabla\tilde{b} = \nu\Delta\tilde{u} - \nabla\tilde{p}, \\ \partial_t\tilde{b} + \tilde{u}\cdot\nabla\tilde{b} - \tilde{b}\cdot\nabla\tilde{u} = \mu\Delta\tilde{b}, \\ \mathrm{div}\tilde{u} = \mathrm{div}\tilde{b} = 0. \end{cases} \tag{1.1}$$

Here $\mathbb{T}$ is the periodized interval $[0,1]$, $\tilde{u} = \tilde{u}(t,x,y) : [0,\infty)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}^2$ denotes the velocity, $\tilde{b} = \tilde{b}(t,x,y) : [0,\infty)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}^2$ denotes magnetic field, $\tilde{p} = \tilde{p}(t,x,y) : [0,\infty)\times\mathbb{T}\times\mathbb{R}\to\mathbb{R}$ denotes pressure, ν denotes the inverse Reynolds number, and μ denotes the inverse magnetic Reynolds number. For a derivation of (1.1) and a general overview of MHD equations we refer the reader to [**17**, **30**].

Note that the shear profile $u_s = (e^{\nu t\partial_{yy}}U(y), 0)^\top$ is a solution of (1.1) in any uniform magnetic field $b_s = (\alpha, 0)^\top$, where $U(y)$ is a given smooth function, and

α is a constant in $\mathbb{R}$. A natural question is to study the long time stability of the equilibrium state (u_s, b_s). To this end, let us introduce the perturbations u and b by $\tilde{u} = u + u_s$ and $\tilde{b} = b + b_s$. Then (u, b) solves

$$\begin{cases} \partial_t u + \bar{U}\partial_x u + \begin{pmatrix} \bar{U}' u^2 \\ 0 \end{pmatrix} - \alpha\partial_x b - \nu\Delta u = -u\cdot\nabla u + b\cdot\nabla b - \nabla\tilde{p}, \\ \partial_t b + \bar{U}\partial_x b - \begin{pmatrix} \bar{U}' b^2 \\ 0 \end{pmatrix} - \alpha\partial_x u - \mu\Delta b = -u\cdot\nabla b + b\cdot\nabla u, \\ \nabla\cdot u = \nabla\cdot b = 0, \\ u(0) = u_{\rm in}, \quad b(0) = b_{\rm in}, \end{cases} \tag{1.2}$$

where we have used the notations $u = (u_1, u_2)^\top,\ b = (b_1, b_2)^\top$ and $\bar{U} = e^{\nu t\partial_{yy}} U(y)$. The corresponding perturbed vorticity and current density take the form of

$$\omega = \partial_x u^2 - \partial_y u^1, \quad j = \partial_x b^2 - \partial_y b^1.$$

Let ψ and ϕ be the steam functions such that

$$u = (-\partial_y\psi, \partial_x\psi)^\top = \nabla^\perp\psi, \quad b = (-\partial_y\phi, \partial_x\phi)^\top = \nabla^\perp\phi.$$

Then we infer from (1.2) that the system of (ω, j) is of the following form

$$\begin{cases} \partial_t\omega + \bar{U}\partial_x\omega - \alpha\partial_x j - \bar{U}''\partial_x\psi - \nu\Delta\omega = -(u\cdot\nabla)\omega + (b\cdot\nabla)j, \\ \partial_t j + \bar{U}\partial_x j - \alpha\partial_x\omega + \bar{U}''\partial_x\phi + 2\bar{U}'\partial_{xy}\phi - \mu\Delta j = -(u\cdot\nabla)j + (b\cdot\nabla)\omega \\ \qquad\qquad +2\partial_{xy}\psi(2\partial_{xx}\phi - j) - 2\partial_{xy}\phi(2\partial_{xx}\psi - \omega), \\ u = \nabla^\perp\psi, \quad b = \nabla^\perp\phi, \\ \Delta\psi = \omega, \quad \Delta\phi = j. \end{cases} \tag{1.3}$$

The mathematical theory of the stability of shear flows gained quite some attention in the last decade. The nonlinear stability of the 2D Couette flow on $\mathbb{T}\times\mathbb{R}$ in the Euler equation was first obtained by Bedrossian and Masmoudi in [**6**] when the initial perturbation is smoother than the Gevrey space of class 2. In particular, the *inviscid damping*, an important hydrodynamic phenomenon manifests itself as the algebraic decay of the velocity for the inviscid fluids, was rigorously justified at the nonlinear level in [**6**]. A generalization of the results in [**6**] to the finite channel $\mathbb{T}\times[0,1]$ for initial perturbation with compact support was given by Ionescu and Jia in [**22**]. The nonlinear inviscid damping for a class of monotone shear flows was proved in [**23**] and [**33**], independently. The stability results for the 2D Couette flow in inhomogeneous fluids can be found in [**16**, **51**]. The instability result for the 2D Couette flow on $\mathbb{T}\times\mathbb{R}$ was shown by Deng and Masmoudi in [**18**] when the initial perturbation is less smooth than the Gevrey space of class 2. For more general shear flows, the stability/instability results are more difficult to achieve due to the presence of nonlocal term. We refer to [**15**, **25**, **26**, **45**, **52**] for the linear inviscid damping results for general *monotone* shear flows, and to [**24**, **46**, **47**] for the linear stability results for non-monotone shear flows.

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If the viscosity is taken into consideration, consider the toy model

$$\partial_t g + y\partial_x g = \nu\Delta g, \quad (t, x, y) \in [0, \infty) \times \mathbb{T} \times \mathbb{R}$$

it is not difficult to show that the non-zero mode $g_{\neq}$ undergo *enhanced dissipation*, namely for some $c > 0$ there holds

$$\|g_{\neq}(t)\|_{L^2} \lesssim e^{-c\nu^{\frac{1}{3}}t}\|g(0)\|_{L^2}. \tag{1.4}$$

The inviscid damping mentioned above and the enhanced dissipation demonstrated in (1.4) are two main stability mechanisms which are closely related to the stability threshold problem. That is, *how small should initial perturbations be in terms of the viscous coefficient to ensure nonlinear stability*? Since the early experiments of Reynolds [**36**], the the linear stability or instability of shear flows *at high Reynolds number* has been a classical problem in applied fluid mechanics [**10**, **27**, **35**, **39**, **48**]. Significant progress has been made by Bedrossian, Germain and Masmoudi in [**3**–**5**] for the nonlinear stability of the Couette flow on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ by using the time dependent Fourier multiplier method. Then Wei and Zhang [**44**] proved the nonlinear stability of the Couette flow on $\mathbb{T} \times \mathbb{R} \times \mathbb{T}$ as long as the initial perturbation satisfying $\|u_{\rm in}\|_{H^2} \ll \mathrm{Re}^{-1}$ with Re the Reynolds number. For the 2D the domain $\mathbb{T} \times \mathbb{R}$, a series of results can be found in [**7**, **8**, **34**]. Recently, Li, Masmoudi and the first author of this paper [**29**] studied the relationship between the size and the regularity of the initial perturbation that ensures the nonlinear asymptotic stability. For the periodic finite channels, a delicate resolvent estimate method was developed in [**12**, **14**]. In particular, Chen, Wei and Zhang showed that the size of the initial perturbation in [**44**] still ensures the stability of the Couette flow in the presence of physical boundary in [**14**].

In the presence of magnetic field, the behaviours of the shear flows become more complicated. On the one hand, it is classically known that a strong background magnetic field can have a stability effect on a conducting fluid, see [**11**, **40**, **42**], for instance. On the other hand, the magnetic field may destabilize the system [**13**] even with shear flows (including Couette flow) that are asymptotically stable. Tataronis and Grossmam [**41**] predicted that the decaying of the vertical components of velocity and magnetic field due to the phase mixing. Ren and Zhao [**38**] gave a rigorous mathematical proof under the assumption that the magnetic field is positive and strictly monotone. In [**20**], Hirota, Tatsuno and Yoshida investigated the linearized behaviour of the ideal MHD equation around Couette flow $(k_f y, 0)$ and linear magnetic field $(k_m y, 0)$. They predicted that if $|k_f| < |k_m|$, then the magnetic island appears in the final state, namely the linear asymptotic stability fails, and if $|k_m| < |k_f|$, then linear damping holds and the magnetic island will be destructed. Recently, the generation of magnetic island was rigorously proved by Zhai, Zhang and Zhao in [**49**], and the rigorously mathematical proof for the destruction of magnetic fields was given by Ren, Wei and Zhang in [**37**]. We refer to [**28**, **32**] for more recent results for the linear stability results on shear flows in magnetic field. For the nonlinear stability result on this direction, Liss [**31**] proved that for strong and suitably oriented background fields $\alpha(\sigma, 0, 1)^\top$, the Couette flow $(y, 0, 0)^\top$ is asymptotically stable provided the initial perturbations u_{in} and b_{in} satisfy

$$\|u_{\mathrm{in}}\|_{H^N} + \|b_{\mathrm{in}}\|_{H^N} \ll \nu = \mu, \tag{1.5}$$

with N sufficiently large.

In this paper, under the same condition $\nu = \mu$ in [**31**], we study the nonlinear stability of the time dependent shear flow $u_s = (\bar{U}(t, y), 0)$ in a uniform magnetic field $b_s = (\alpha, 0)^\top$ on $\mathbb{T} \times \mathbb{R}$. More precisely, for $0 < \nu = \mu \ll 1$, given an initial norm X_i and a final norm X_f, our goal in this paper is determine a constant $\gamma = \gamma(X_i, X_f) \geqslant 0$ as small as possible, such that if the initial perturbations $\bar{u}_{\mathrm{in}}$ and $\bar{b}_{\mathrm{in}}$ satisfy

$$\|(u_{\mathrm{in}}, b_{\mathrm{in}})\|_{X_i} \leqslant c_0 \nu^\gamma, \tag{1.6}$$

for c_0 sufficiently small (independent of ν), then the solution of (1.1) is global in time and converges back to (u_s, b_s) as $t \to \infty$ in the sense that

$$\|(u, b)\|_{L^\infty X_f} \lesssim c_0, \quad \lim_{t \to \infty} \|(u(t), b(t))\|_{X_f} = 0. \tag{1.7}$$

1.2. The main result

Our main result is stated as follows.

THEOREM 1.1. *Let $N > 1$, $\mu = \nu \in (0, 1]$. There exist two sufficiently large constants $\alpha_0 > 0$ and $C \geqslant 1$ independent of ν, such that for all $|\alpha| \geqslant \alpha_0$, if the shear flow $U = U(y)$ satisfies*

$$\|U(y) - y\|_{H^{N+6}(\mathbb{R})} = \delta \leqslant C^{-1}, \tag{1.8}$$

with δ independent of ν, and the initial perturbation obeys

$$\|(\omega_{\mathrm{in}}, j_{\mathrm{in}})\|_{H^N} + \|(u_{\mathrm{in}}, b_{\mathrm{in}})\|_{H^N} = \epsilon \leqslant C^{-1} \nu^{\frac{5}{6} + \tilde{\delta}}, \tag{1.9}$$

for any fixed $\tilde{\delta} > 0$, then the global in time solution (ω, j) to (1.3) obeys

$$\left\|(\omega, j) \circ (x + t\bar{U}(t, y), y)\right\|_{L^\infty(0,\infty; H^N(\mathbb{T}\times\mathbb{R}))} \leqslant C\epsilon \nu^{-(\frac{1}{3} + \frac{\tilde{\delta}}{2})}, \tag{1.10}$$

and the enhanced dissipation estimate

$$\left\|(\omega_{\neq}, j_{\neq}) \circ (x + t\bar{U}(t, y), y)\right\|_{L^2(0,\infty; H^N(\mathbb{T}\times\mathbb{R}))} \leqslant C\epsilon \nu^{-\frac{1}{2} - \frac{\tilde{\delta}}{2}}. \tag{1.11}$$

REMARK 1.2. The bounds (1.10) and (1.11) follow from (4.3), (4.5) in theorem 4.1 and lemma A.1 immediately. So the rest part of this paper aims to prove theorem 4.1.

REMARK 1.3. It is not difficult to obtain the explicit enhanced dissipation decay $e^{-c\nu^{\frac{1}{3}} t}$ for $(\omega_{\neq}, j_{\neq})$ like (1.4). In fact, we just need to change the multiplier $\tilde{K}$

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(see (4.2)) slightly. Let $\mathbf{e}_{\neq}$ be a multiplier defined by

$$\widehat{\mathbf{e}_{\neq} f}(k,\eta) = e^{\mathbf{1}_{k\neq 0} c\nu^{\frac{1}{3}} t} \hat{f}(k,\eta).$$

Set

$$\mathsf{M} = \mathbf{e}_{\neq} \tilde{K}.$$

Replacing the multiplier $\tilde{K}$ in (4.10) by M, the proof of proposition 4.3 is still valid. In particular, one can use (3.21) to absorb the bad term $c\nu^{\frac{1}{3}} \left\| \mathsf{M} Z_{\neq}^{\pm} \right\|_{L^2}^2$ arising from the time evolution of $\mathbf{e}_{\neq}$ for sufficiently small c (see § 2.3 for the definition of $Z^{\pm}$). We refer to [**50**] for more details of the use of the multiplier $\mathbf{e}_{\neq}$ to get the explicit enhanced dissipation decay $e^{-c\nu^{\frac{1}{3}} t}$.

REMARK 1.4. At first glance, it looks as if the loss $\nu^{-\frac{1}{2}}$ on the right hand side of (1.11) might come from the diffusion $-\nu\Delta(\omega, j)$. As a matter of fact, roughly speaking, due to the linear growth $\langle t \rangle$ caused by the stretch $2\bar{U}'\partial_{xy}\phi$ (together with the oscillation stemming from the strong background magnetic field $(\alpha, 0)^{\top}$) and the enhanced dissipation $e^{-c\nu^{\frac{1}{3}} t}$ as mentioned in remark 1.3, $(\omega_{\neq}, j_{\neq})$ behaves like $\langle t \rangle e^{-c\nu^{\frac{1}{3}} t}$. Clearly, $\left\| \langle t \rangle e^{-c\nu^{\frac{1}{3}} t} \right\|_{L_t^2} \approx \nu^{-\frac{1}{2}}$. This explains the loss $\nu^{-\frac{1}{2}}$ on the right hand side of (1.11).

REMARK 1.5. Compared with the Couette flow case $\bar{U}(t,y) = U(y) = y$, the extra exponent $\tilde{\delta}$ in (1.9) is derived from the linear stretch term $2\bar{U}'\partial_{xy}\phi$ in (1.3), which is absent in the 2D Navier–Stokes equations around shear flows near Couette (see [**8**]). It is open whether one can remove the extra exponent $\tilde{\delta}$ in (1.9).

REMARK 1.6. The constant C in theorem 1.1 depends on $\frac{1}{|\alpha|}$. As a matter of fact, if $\alpha = 0$, the oscillations $-\alpha\partial_x j$ and $-\alpha\partial_x \omega$ disappear in (1.3). As a result, instead of the linear growth $\langle t \rangle$ in the case $|\alpha| \geqslant \alpha_0$, for $\alpha = 0$ the linear stretch term $2\bar{U}'\partial_{xy}\phi$ in (1.3) will lead to $\langle t \rangle^2$ growth even for $\bar{U}(t,y) = U(y) = y$, which costs more smallness of the initial perturbations in terms of some power of ν to ensure the stability.

1.3. Notations

(1) Throughout this paper, we use the standard notation

$$\langle x \rangle = \sqrt{1 + x^2}$$

we write $\langle \nabla \rangle^s$ for the operator with symbol

$$\langle \nabla \rangle^s = \left(1 + k^2 + \eta^2\right)^{\frac{s}{2}}.$$

(2) We use the notation $f \lesssim g$ to mean that there exist some constant $C > 0$ such that $f \leqslant Cg$. This constant C may depend on N and α, but not on ν.

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(3) For a function $f(x, y)$, We denote the projection of f onto the zero frequencies in x by

$$f_0(y) = \int_{\mathbb{T}} f(x, y)\mathrm{d}x.$$

Then we write

$$f_{\neq}(x, y) = f(x, y) - f_0(y)$$

for the projection onto the nonzero frequencies in x.

(4) The Fourier transform of function f is denoted by

$$\mathcal{F}(f) = \hat{f}(k, \eta) = \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{R}} e^{-i(kx+\eta y)} f(x, y)\mathrm{d}x\mathrm{d}y.$$

The Fourier multiplier with symbol $m(t, k, \eta)$ is given by

$$mf = \mathcal{F}^{-1}\left(m(t, k, \eta)\mathcal{F}f\right).$$

(5) For any $a \in \mathbb{R}$, we use the shorthand notation

$$O_a^t = e^{a\partial_x t} \tag{1.12}$$

to denote the multiplier with symbol e^{iakt}. We then write $\partial_t O_a^t$ to denote the Fourier multiplier with symbol $iake^{iakt}$.

(6) For $s \geqslant 0$, we define the Sobolev space H^s by using the norm

$$\|f\|_{H^s} := \|\langle\nabla\rangle^s f\|_{L^2}.$$

The notation $L^pL^q = L_t^pL_{x,y}^q$ is used for the Banach space $L^p([0, T]; L^q(\Omega))$ with norm

$$\|f(t, x)\|_{L^pL^q} = \big\|\|f(t, \cdot)\|_{L^q}\big\|_{L^p}.$$

(7) For two real functions f and g , we write the associated inner product as

$$\langle f, g\rangle = \int_{\mathbb{T}\times\mathbb{R}} fg\mathrm{d}x\mathrm{d}y,$$

and denote

$$\langle f, g\rangle_{H^s} = \langle\langle\nabla\rangle^s f, \langle\nabla\rangle^s g\rangle.$$

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2. Reformulations and key ideas of the proof

2.1. Elsässer variables under the restriction $\mu = \nu$

Since we focus on the case $\nu = \mu$ in this paper, the symmetry of the system (1.3) enables us to reformulate it by using the Elsässer variables

$$w^{\pm} = \omega \mp j.$$

Then $w^{\pm}$ solves

$$\begin{aligned}\partial_t w^{\pm} + \bar{U}\partial_x w^{\pm} \pm \alpha\partial_x w^{\pm} - \nu\Delta w^{\pm} \mp 2\bar{U}'\partial_{xy}\Delta^{-1}(w^- - w^+) - \bar{U}''\partial_x\Delta^{-1}w^{\mp} \\ = -\nabla^{\perp}\Delta^{-1}w^{\mp}\cdot\nabla w^{\pm} + \partial_{xy}\Delta^{-1}w^{\mp}(2\partial_{xx}\Delta^{-1}w^{\pm} - w^{\pm}) \\ - \partial_{xy}\Delta^{-1}w^{\pm}(2\partial_{xx}\Delta^{-1}w^{\mp} - w^{\mp}).\end{aligned}$$

Such kind of variables have played important roles in the study of large time behaviours of solutions to MHD equations in the absence of shear flows [**2**, **9**, **19**, **43**], and the nonlinear stability result [**31**] where the three dimensional Couette flow $(y, 0, 0)^{\top}$ is taken into consideration as mentioned in § 1.

Similar to the 3D case, the background magnetic field $(\alpha, 0)^{\top}$ introduces oscillations (see the $-\alpha\partial_x w^{\pm}$ terms in (2.1)) that may stabilize the system. Following [**31**], we define the profiles

$$z^{\pm} = O^t_{\pm\alpha} w^{\pm}$$

to hide the oscillations in the new unknowns $z^{\pm}$, where $O^t_{\pm\alpha}$ is defined in (1.12). Then $z^{\pm}$ solves

$$\begin{aligned}\partial_t z^{\pm} + \bar{U}\partial_x z^{\pm} - \nu\Delta z^{\pm} + \bar{U}'\partial_{xy}\Delta^{-1}\left(z^{\pm} - O^t_{\pm 2\alpha}z^{\mp}\right) - \bar{U}''\partial_x\Delta^{-1}O^t_{\pm 2\alpha}z^{\mp} \\ = -\nabla^{\perp}\Delta^{-1}O^t_{\pm 2\alpha}z^{\mp}\cdot\nabla z^{\pm} + \partial_{xy}\Delta^{-1}O^t_{\pm 2\alpha}z^{\mp}(2\partial_{xx}\Delta^{-1}z^{\pm} - z^{\pm}) \\ - \partial_{xy}\Delta^{-1}z^{\pm}(2\partial_{xx}\Delta^{-1}O^t_{\pm 2\alpha}z^{\mp} - O^t_{\pm 2\alpha}z^{\mp}).\end{aligned} \tag{2.1}$$

2.2. Change of coordinates

In this paper we use the coordinate transform introduced by Bedrossian, Vicol and Wang in [**8**] to unwind the decaying background shear flow $\bar{U}(t, y)$ in (2.1):

$$\begin{cases} X = x - t\bar{U}(t, y) \\ Y = \bar{U}(t, y). \end{cases} \tag{2.2}$$

Denote the spatial derivatives of the shear flow in the new coordinates as follows

$$\begin{cases} a(t, Y(t, y)) & = \partial_y\bar{U}(t, y), \\ b(t, Y(t, y)) & = \partial_y^2\bar{U}(t, y), \\ c(t, Y(t, y)) & = \partial_y^3\bar{U}(t, y), \\ d(t, Y(t, y)) & = \partial_y^4\bar{U}(t, y). \end{cases} \tag{2.3}$$

Note that by the chain rule, we have

$$b = a\partial_Y a. \tag{2.4}$$

For any function $\tilde{h}$ in the (x, y) coordinates, the corresponding function h in the (X, Y) coordinates is given by

$$h(t, X, Y) = \tilde{h}(t, x, y).$$

Then $\nabla \tilde{h}$ and $\Delta \tilde{h}$ can be rewritten in the new coordinate system (2.2) in terms of h as follows (the notations ∇_t and Δ_t are introduced naturally):

$$\nabla \tilde{h}(t, x, y) = (\partial_x \tilde{h}, \partial_y \tilde{h}) = (\partial_X h, a(\partial_Y - t\partial_X)h) = (\partial_X^t h, \partial_Y^t h) = \nabla_t h, \tag{2.5}$$

and

$$\begin{aligned} \Delta \tilde{h}(t, x, y) &= (\partial_{XX} + a^2 \partial_{YY}^L + b\partial_Y^L)h = \Delta_t h \\ &= \Delta_L h + \left((a^2 - 1)\partial_{YY}^L + b\partial_Y^L\right) h \\ &= \tilde{\Delta}_t h + b\partial_Y^L h, \end{aligned} \tag{2.6}$$

where we have used the notations

$$\partial_Y^L = \partial_Y - t\partial_X, \quad \partial_{YY}^L = (\partial_Y - t\partial_X)^2,$$

and the modified Laplace operator is given by

$$\tilde{\Delta}_t = \Delta_L + (a^2 - 1)\partial_{YY}^L, \quad \text{with} \quad \Delta_L = \partial_{XX} + \partial_{YY}^L. \tag{2.7}$$

Finally, using the fact $\partial_t \bar{U} = \nu \partial_y^2 \bar{U}$ and the definitions of a and b, we have

$$\partial_t \tilde{h} = \partial_t h + \partial_X h(-\bar{U} - t\partial_t \bar{U}) + \partial_Y h \partial_t \bar{U} = \partial_t h - Y\partial_X h + \nu b \partial_Y^L h. \tag{2.8}$$

In particular,

$$\partial_t \partial_y \bar{U} = \partial_t a + \nu b \partial_Y a, \quad \text{and} \quad \partial_t \partial_y^2 \bar{U} = \partial_t b + \nu b \partial_Y b,$$

which, together with the fact $\partial_t \bar{U} = \nu \partial_y^2 \bar{U}$ imply that

$$\partial_t a = \nu c - \nu b \partial_Y a, \quad \text{and} \quad \partial_t b = \nu d - \nu b \partial_Y b. \tag{2.9}$$

2.3. Definition of Δ_t^{-1} and system (2.1) under new coordinates

In this subsection, we first give the definition of the inverse of Δ_t in the spirit of Antonelli, Dolce, and Marcati [1]. For the sake of completeness, we sketch the definitions below.

To begin with, assume that $a^2 - 1 = (a^2 - 1)(t, Y)$ and $b = b(t, Y)$ are given functions in $L^\infty(\mathbb{R}^+; H^1(\mathbb{R}))$, then the fact that Δ_L^{-1} is well defined for $k \neq 0$ enables us to define an operator on $L^2(\mathbb{T} \times \mathbb{R})$:

$$\tilde{\Lambda} = \left((a^2 - 1)\partial_{YY}^L + b\partial_Y^L\right)(-\Delta_L)^{-1}, \tag{2.10}$$

with

$$\|\tilde{\Lambda}\|_{L^2 \to L^2} \leqslant C_* \left(\|a^2 - 1\|_{L^\infty H^1} + \|b\|_{L^\infty H^1}\right), \tag{2.11}$$

for some constant $C_* \geqslant 1$, see proposition 4.1 in [1] for more details.

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DEFINITION 2.1. *Assume that* $(\|a^2 - 1\|_{L^\infty H^1} + \|b\|_{L^\infty H^1}) \leqslant \delta$ *such that* $C_*\delta < 1$, *then for* $k \neq 0$, *let us define*

$$\Delta_t^{-1} := \Delta_L^{-1}\Lambda, \quad \text{i.e.} \quad \Lambda\Delta_t = \Delta_L, \tag{2.12}$$

where

$$\Lambda := (I - \tilde{\Lambda})^{-1} = \sum_{n=0}^{\infty} \tilde{\Lambda}^n.$$

REMARK 2.2. For the operator Λ, we also have the following useful identity

$$\Lambda = I + \tilde{\Lambda}\Lambda. \tag{2.13}$$

Now we are in a position to rewrite (2.1) under the coordinates defined by (2.2). To this end, let us denote

$$\begin{aligned} Z^{\pm}(t, X, Y) &= z^{\pm}(t, x, y), \\ U_0^1(t, Y) &= u_0^1(t, y), \\ B_0^1(t, Y) &= b_0^1(t, y). \end{aligned}$$

Note that

$$\partial_{XY}^L \Delta_t^{-1} = \partial_{XY}^L \Delta_L^{-1} \Delta_L \Delta_t^{-1} = S\Lambda, \quad \text{with} \quad S = \partial_{XY}^L \Delta_L^{-1}, \tag{2.14}$$

and

$$\nabla^\perp \Delta^{-1} O_{\pm 2\alpha}^t z_0^\mp \cdot \nabla z^\pm = -\partial_y \partial_{yy}^{-1} z_0^\mp \partial_x z^\pm = -\partial_y \partial_{yy}^{-1} w_0^\mp \partial_x z^\pm = \left(u_0^1 \pm b_0^1\right) \partial_x z_{/}^\pm = .$$

Then it follows from (2.1) that the equations of $Z^\pm$ take the form of

$$\partial_t Z^\pm - \nu\tilde{\Delta}_t Z^\pm + a^2 S\Lambda \left(Z^\pm - O_{\pm 2\alpha}^t Z^\mp\right) - b\partial_X \Delta_t^{-1} O_{\pm 2\alpha}^t Z^\mp = \mathrm{NL}^\pm, \tag{2.15}$$

or

$$\partial_t Z^\pm - \nu\Delta_L Z^\pm + S\Lambda \left(Z^\pm - O_{\pm 2\alpha}^t Z^\mp\right) = \mathrm{LP}^\pm + \mathrm{NL}^\pm, \tag{2.16}$$

where

$$\mathrm{NL}^\pm := \mathrm{NLT}^\pm + \mathrm{NLS1}^\pm + \mathrm{NLS2}^\pm, \tag{2.17}$$

with

$$\begin{aligned} \mathrm{NLT}^\pm &= -\left(U_0^1 \pm B_0^1\right) \partial_X Z_{/}^\pm = -\nabla_t^\perp \Delta_t^{-1} O_{\pm 2\alpha}^t Z_{/}^\mp = \cdot \nabla_t Z^\pm, \\ \mathrm{NLS1}^\pm &= aS\Lambda O_{\pm 2\alpha}^t Z_{/}^\mp = (2\partial_{XX} \Delta_t^{-1} Z_{/}^\pm = -Z^\pm), \\ \mathrm{NLS2}^\pm &= -aS\Lambda Z_{/}^\pm = (2\partial_{XX} \Delta_t^{-1} O_{\pm 2\alpha}^t Z_{/}^\mp = -O_{\pm 2\alpha}^t Z^\mp), \end{aligned}$$

and

$$\mathrm{LP}^\pm := -(a^2 - 1)S\Lambda \left(Z^\pm - O_{\pm 2\alpha}^t Z^\mp\right) + b\partial_X \Delta_t^{-1} O_{\pm 2\alpha}^t Z^\mp + \nu(a^2 - 1)\partial_{YY}^L Z^\pm. \tag{2.18}$$

Here 'NLT', 'NLS' and 'LP' stand for 'nonlinear transport', 'nonlinear stretch' and 'linear perturbation', respectively.

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2.4. Toy model and key ideas

Compared with the Navier–Stokes equations around 2D shear flows near Couette [8], and the MHD equations around 3D Couette flows [31], we will encounter new difficulties for the MHD equations around 2D shear flows near Couette. In order to track the difficulties precisely, let us consider the toy model

$$\partial_t f^{\pm} + S\Lambda f^{\pm} - \nu \Delta_L f^{\pm} = 0, \tag{2.19}$$

or equivalently in Fourier variables

$$\partial_t \widehat{f^{\pm}} + \frac{k(\eta - kt)}{k^2 + (\eta - kt)^2} \widehat{\Lambda f^{\pm}} - \nu \left(k^2 + (\eta - kt)^2\right) \widehat{f^{\pm}} = 0. \tag{2.20}$$

It is worth pointing out that if the shear flow is the Couette flow, i.e., $\bar{U}(t, y) = U(y) = y$, the operator Λ in (2.20) will not appear. This scenario is reminiscent of the toy model introduced by Bedrossian, Germain and Masmoudi in [3] for the 3D Navier–Stokes equations near Couette:

$$\partial_t \hat{g}(t, k, \eta, l) + \frac{2k(\eta - kt)}{k^2 + (\eta - kt)^2 + l^2} \hat{g}(t, k, \eta, l) \\ - \nu \left(k^2 + (\eta - kt)^2 + l^2\right) \hat{g}(t, k, \eta, l) = 0.$$

To balance the interaction between $\frac{2k(\eta-kt)}{k^2+(\eta-kt)^2+l^2}\hat{g}$ and $-\nu(k^2 + (\eta - kt)^2 + l^2)\hat{g}$, the authors in [3] constructed a multiplier $m(t, k, \eta, l)$ satisfying

$$\frac{\partial_t m}{m} = \frac{2k(\eta - kt)}{k^2 + (\eta - kt)^2 + l^2} \quad \text{for} \quad t \in \left[\frac{\eta}{k}, \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}\right].$$

More precisely, the two dimensional analogue of the multiplier $m(t, k, \eta, l)$ is given as follows:

(1) if $k = 0 : m(t, 0, \eta) = 1$;

(2) if $k \neq 0, \frac{\eta}{k} < -1000\nu^{-\frac{1}{3}} : m(t, k, \eta) = 1$;

(3) if $k \neq 0, -1000\nu^{-\frac{1}{3}} < \frac{\eta}{k} < 0$:
- $m(t, k, \eta) = \frac{k^2+\eta^2}{k^2+(\eta-kt)^2}$ if $0 < t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$,
- $m(t, k, \eta) = \frac{k^2+\eta^2}{k^2+(1000k\nu^{-\frac{1}{3}})^2}$ if $t > \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$;

(4) if $k \neq 0, \frac{\eta}{k} > 0$:
- $m(t, k, \eta) = 1$ if $t < \frac{\eta}{k}$,
- $m(t, k, \eta) = \frac{k^2}{k^2+(\eta-kt)^2}$ if $\frac{\eta}{k} < t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$,
- $m(t, k, \eta) = \frac{k^2}{k^2+(1000k\nu^{-\frac{1}{3}})^2}$ if $t > \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$.

Notice, in particular, that

$$\nu^{\frac{2}{3}} \lesssim m(t,k,\eta) \leqslant 1, \tag{2.21}$$

and

$$m(t,k,\eta) \gtrsim \frac{k^2}{k^2+(\eta-kt)^2}. \tag{2.22}$$

In our case, for the Couette flow, it suffices to use $m^{\frac{1}{2}}(t,k,\eta)$ instead of $m(t,k,\eta)$ to suppress the potential growth caused by the linear stretch term $\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{f^{\pm}}$. Nevertheless, for general shear flows the presence of the operator Λ may amplify the linear stretch effect since the norm of Λ may be larger than 1 even though the shear flow $\bar{U}$ is close to Couette. Roughly speaking, one can regard the linear stretch term in (2.20) as $(1+O(\tilde{\delta}))\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{f^{\pm}}$ with $\tilde{\delta}>0$. Then it is natural to modify the multiplier $m^{\frac{1}{2}}(t,k,\eta)$ so as to suppress the extra growth resulting from the operator Λ. To this end, our strategy is to replace $m^{\frac{1}{2}}(t,k,\eta)$ with $\tilde{m}^{\frac{1}{2}}(t,k,\eta)$, where $\tilde{m}(t,k,\eta)$ is defined by

$$\tilde{m}(t,k,\eta) = m^{1+\frac{3}{2}\tilde{\delta}}(t,k,\eta), \tag{2.23}$$

with arbitrary $\tilde{\delta}>0$. Note that

$$\frac{\partial_t \tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\eta)} = \left(1+\frac{3}{2}\tilde{\delta}\right)\frac{\partial_t m(t,k,\eta)}{m(t,k,\eta)}. \tag{2.24}$$

Then (2.21) and (2.22) reduce to

$$\nu^{\frac{2}{3}+\tilde{\delta}} \lesssim \tilde{m}(t,k,\eta) \leqslant 1, \tag{2.25}$$

and

$$\tilde{m}(t,k,\eta) \gtrsim \left(\frac{k^2}{k^2+(\eta-kt)^2}\right)^{1+\frac{3}{2}\tilde{\delta}}. \tag{2.26}$$

For $(t,k,\eta)\in[0,\infty)\times\mathbb{Z}\backslash\{0\}\times\mathbb{R}$, let us define the following three disjoint sets

$$\begin{aligned}
D_{dam} &= \left\{(t,k,\eta) : t < \frac{\eta}{k}\right\},\\
D_{dis} &= \left\{(t,k,\eta) : \frac{\eta}{k} < -1000\nu^{-\frac{1}{3}}\right\} \cup \left\{-1000\nu^{-\frac{1}{3}} < \frac{\eta}{k}, t \geqslant \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}\right\},\\
D_{mul} &= \left\{(t,k,\eta) : -1000\nu^{-\frac{1}{3}} < \frac{\eta}{k} \leqslant 0, t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}\right\}\\
&\quad \cup \left\{(t,k,\eta) : \frac{\eta}{k} > 0, \frac{\eta}{k} \leqslant t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}\right\}.
\end{aligned}$$

Then the effects of the linear stretch are summarized as follows:

(1) if $(t,k,\eta)\in D_{dam}$, then $k(\eta-kt)\geqslant 0$, and thus the linear stretch term behaves as a damping.

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(2) if $(t,k,\eta)\in D_{dis}$, we have

$$\left|t-\frac{\eta}{k}\right|\geqslant 1000\nu^{-\frac{1}{3}},$$

and thus

$$\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\leqslant\frac{1}{1000^3}\nu\left(k^2+(\eta-kt)^2\right),\tag{2.27}$$

which means that the linear stretch is dominated by the dissipation.

(3) by the definition of $\tilde{m}$, there holds

$$\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{mul}}(t,k,\eta)=\frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{-\partial_t(\tilde{m}^{1/2})(t,k,\eta)}{\tilde{m}^{1/2}(t,k,\eta)},\tag{2.28}$$

that is to say, the effect of the linear stretch is balanced by the evolution of $\tilde{m}^{\frac{1}{2}}$ on D_{mul}.

Remark 2.3. Clearly, the following decomposition of unity holds

$$1=\mathbf{1}_{D_{dam}}(t,k,\eta)+\mathbf{1}_{D_{dis}}(t,k,\eta)+\mathbf{1}_{D_{mul}}(t,k,\eta),\quad\text{for all}\quad (t,k,\eta)\in[0,\infty)\times\mathbb{Z}\backslash\{0\}\times\mathbb{R}.$$

Then it follows from (2.27) and (2.28) that

$$\begin{aligned}&\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\\&=\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dam}}(t,k,\eta)+\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dis}}(t,k,\eta)\\&\quad-\frac{1}{2}\frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{\dot{\tilde{m}}(t,k,\eta)}{\tilde{m}(t,k,\eta)}\\&\leqslant\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dam}}(t,k,\eta)+\frac{1}{1000^3}\nu\left(k^2+(\eta-kt)^2\right)\\&\quad-\frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{\partial_t(\tilde{m}^{1/2})(t,k,\eta)}{\tilde{m}^{1/2}(t,k,\eta)}.\end{aligned}\tag{2.29}$$

The treatment of the general shear flow $(\bar{U},0)^\top$ is much more complicated than that of the Couette flow $(y,0)^\top$. In fact, for the case $\bar{U}(t,y)=U(y)=y$, once the multiplier $m(t,k,\eta)$ is well defined as above, then it is straightforward to deal with the linear stretch term, see (3.24). In particular, the damping effect stemming from the linear stretch term when $(t,k,\eta)\in D_{dam}$ can be ignored. However, for the case that $U(y)$ is close to y, the appearance of the operator Λ makes the damping effect of the linear stretch $\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{\Lambda f^{\pm}}$ unclear even though $(t,k,\eta)\in D_{dam}$. Our strategy

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is to isolate the damping effect by using (2.13):

$$\mathbf{1}_{D_{dam}}\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{\Lambda f^{\pm}}=\mathbf{1}_{D_{dam}}\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{f^{\pm}}+\mathbf{1}_{D_{dam}}\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\widehat{\tilde{\Lambda}\Lambda f^{\pm}}. \tag{2.30}$$

The first term on the right hand side of (2.30) is the main part, and the second term is the perturbation. Some delicate commutator estimates will be performed to treat the perturbation. As a result, lots of errors appear. To close the estimates, the damping effect captured by the main part of the linear stretch in (2.30), together with the dissipation and other good terms, will be used to absorb the errors. See **Step I** of § 4.1 for more details.

In addition, the presence of the operator Λ leads us to estimate the composition $B\circ\Lambda$ for some multiplier B involved in this paper. The continuity of $B\circ\Lambda$ depends on the commutator estimates of B. That's why we give a collection of commutator estimates in Appendix B. In particular, some extra terms appear in the commutator estimate of $\sqrt{-\frac{\partial_t(\tilde{m}^{\frac{1}{2}})}{\tilde{m}^{\frac{1}{2}}}}\tilde{K}$, see lemma B.4. This motivates us to establish a composition inequality for the multipliers whose commutator estimates have *extra errors with better commutator estimates*, see lemma C.1 for the details.

On the other hand, it is worth pointing out that the above toy model (2.19) ignored the oscillation terms $O^t_{\pm 2\alpha}S\Lambda f^{\mp}$, which should be taken into consideration as well in the energy estimates. In order to take advantage of the oscillator O^t_α, noting that

$$\widehat{O^t_\alpha}=\frac{1}{i\alpha k}\partial_t\widehat{O^t_\alpha}\quad\text{for}\quad k\neq 0, \tag{2.31}$$

integrating by parts with respect to the time variable will be exploited as that in [**31**]. In this way, we will inevitably encounter the time derivative of the operator Λ. To achieve this, using again (2.13), we obtain an important relation

$$\partial_t\Lambda=\Lambda\partial_t\tilde{\Lambda}\Lambda, \tag{2.32}$$

where $\partial_t\tilde{\Lambda}$ is derived from (2.10)

$$\partial_t\tilde{\Lambda}=\Big[2(a^2-1)S+b\partial_X\Delta_L^{-1}+2\tilde{\Lambda}S\Big]+\left(2a\partial_t a\partial^L_{YY}+\partial_t b\partial^L_Y\right)(-\Delta_L)^{-1}=:\tilde{\Lambda}^1_t+\tilde{\Lambda}^2_t. \tag{2.33}$$

See the estimates of OLS_5 in **Step II** of § 4.1 for more details.

Finally, we would like to remark that, unlike [**1**], the coefficients a and b hidden in the definition of Δ_t^{-1} are time dependent (see (2.10) and (2.12)), and $\partial_t a$ and $\partial_t b$ are involved when performing integrating by parts in time (see (2.33)). Accordingly, recalling (2.3) and (2.9), we find that some higher derivatives of $\bar{U}$ are actually involved. This partially explains the extra regularities required in (1.8).

3. Stability of the Couette flow

For comparison, we discuss in this section the case that the shear flow is the Couette flow. In fact, under the condition $\bar{U}(t,y)=U(y)=y$, the change of coordinates (2.2)

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reduces to

$$X = x - ty, \quad Y = y. \tag{3.1}$$

Without causing confusion, we continue to use the unknowns and notations introduced in § 2. Then it is easy to see that the system (2.15) now reads

$$\begin{aligned} &\partial_t Z^{\pm} - \nu \Delta_L Z^{\pm} + S\left(Z^{\pm} - O^t_{\pm 2\alpha} Z^{\mp}\right) \\ &\quad = -\nabla_L^{\perp} \Delta_L^{-1} O^t_{\pm 2\alpha} Z^{\mp} \cdot \nabla_L Z^{\pm} + S O^t_{\pm 2\alpha} Z^{\mp} (2\partial_{XX} \Delta_L^{-1} Z^{\pm} - Z^{\pm}) \\ &\qquad - S Z^{\pm} (2\partial_{XX} \Delta_L^{-1} O^t_{\pm 2\alpha} Z^{\mp} - O^t_{\pm 2\alpha} Z^{\mp}). \end{aligned} \tag{3.2}$$

The purpose of this section is to establish the following theorem.

THEOREM 3.1. *Let $\mu = \nu \in (0, 1], N > 1$. There exist a universal constant $\alpha_0 > 0$, and a positive constant δ depending only on N and α, such that if $|\alpha| \geqslant \alpha_0$ and*

$$\|(\omega_{\text{in}}, j_{\text{in}})\|_{H^N} + \|(u_{\text{in}}, b_{\text{in}})\|_{H^N} = \epsilon \leqslant \delta \nu^{\frac{5}{6}}, \tag{3.3}$$

then the following estimates hold

$$\|Z^{\pm}\|_{L^\infty H^N} + \nu^{\frac{1}{2}} \|\nabla_L Z^{\pm}\|_{L^2 H^N} \lesssim \epsilon \nu^{-\frac{1}{3}}, \tag{3.4}$$

$$\|(u_0^1 \mp b_0^1)\|_{L^\infty H^N} + \nu^{\frac{1}{2}} \|\partial_y (u_0^1 \mp b_0^1)\|_{L^2 H^N} \lesssim \epsilon. \tag{3.5}$$

and

$$\|Z^{\pm}_{\neq}\|_{L^2 H^N} \lesssim \epsilon \nu^{-\frac{1}{2}}. \tag{3.6}$$

Theorem 3.1 will be proved by using the Fourier multiplier method. In other words, the norms involved in the proof are defined based on special, time-dependent Fourier multipliers. Apart from the multiplier $m(t, k, \eta)$ defined in § 2, we also need two extra multipliers that are modified from the ones introduced in the study of the stability of the three dimensional Couette flow in [**3**]. More precisely, define

$$-\frac{\dot{M}_1}{M_1} = \frac{k^2}{k^2 + |\eta - kt|^2} \quad \text{and} \quad M_1(0, k, \eta) = 1; \tag{3.7}$$

$$-\frac{\dot{M}_2}{M_2} = \frac{\nu^{\frac{1}{3}}}{\left(\nu^{\frac{1}{3}} \left|t - \frac{\eta}{k}\right|\right)^2 + 1} \quad \text{and} \quad M_2(0, k, \eta) = 1. \tag{3.8}$$

The multiplier M_1 is used to capture the inviscid damping effect in terms of the L^2 time integrability, and M_2 is designed to show the enhanced dissipation effect. Clearly, M_1 and M_2 can be given explicitly, and then one deduces that

$$M_i \approx 1, \quad \text{for} \quad i = 1, 2. \tag{3.9}$$

For more properties of multiplier M_1 and M_2, one can refer to lemma 4.1 of [**8**].

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Let us denote

$$M := M_1 M_2, \quad K := \langle\nabla\rangle^N m^{1/2} M. \tag{3.10}$$

Then from (2.21) and (3.9), we find that for any $f = f(X, Y) \in H^N$,

$$\nu^{\frac{1}{3}} \|f\|_{H^N} \lesssim \|m^{\frac{1}{2}} f\|_{H^N} \approx \|Kf\|_{L^2} \lesssim \|f\|_{H^N}. \tag{3.11}$$

In addition, we need to estimate the interactions between the non-zero modes in the treatment of the nonlinear terms, so the following lemma is introduced.

LEMMA 3.2. *Let $N > 1$. Then for all $f \in L^2 H^N$ and g such that $\nabla_L g \in L^2 H^N$, there holds*

$$\begin{aligned} &\|\nabla_L^\perp \Delta_L^{-1} f_{\neq} \cdot \nabla_L g_{\neq}\|_{L^1 H^N} \\ &\quad \leqslant C\left(\|Kf_{\neq}\|_{L^2L^2} + \nu^{-\frac{1}{3}} \left\|\sqrt{-\frac{\dot{M}}{M}} K f_{\neq}\right\|_{L^2L^2}\right) \|\nabla_L K g_{\neq}\|_{L^2L^2}. \end{aligned} \tag{3.12}$$

Proof. We first write

$$\nabla_L^\perp \Delta_L^{-1} f_{\neq} \cdot \nabla_L g_{\neq} = -\partial_Y^L \Delta_L^{-1} f_{\neq} \partial_X g_{\neq} + \partial_X \Delta_L^{-1} f_{\neq} \partial_Y^L g_{\neq}.$$

Thanks to (2.22), there hold

$$|k| \lesssim \sqrt{k^2 + (\eta - kt)^2} m^{1/2}, \quad \text{i.e.} \quad |\partial_X| \lesssim |\nabla_L| m^{1/2}, \tag{3.13}$$

and

$$|\partial_Y^L \Delta_L^{-1}| \leqslant \frac{|\eta - kt|}{k^2 + (\eta - kt)^2} \leqslant \frac{1}{\sqrt{k^2 + (\eta - kt)^2}} \lesssim m^{1/2}, \quad k \neq 0. \tag{3.14}$$

Combining these two estimates with (3.11) yields

$$\begin{aligned} &\|\partial_Y^L \Delta_L^{-1} f_{\neq} \partial_X g_{\neq}\|_{L^1 H^N} \\ &\quad \lesssim \|m^{1/2} f_{\neq}\|_{L^2 H^N} \|\nabla_L m^{1/2} g_{\neq}\|_{L^2 H^N} \approx \|Kf_{\neq}\|_{L^2L^2} \|\nabla_L K g_{\neq}\|_{L^2L^2}. \end{aligned} \tag{3.15}$$

To estimate $\partial_X \Delta_L^{-1} f_{\neq} \partial_Y^L g_{\neq}$, in view of (2.21), (2.22) and the definition of M, we arrive at

$$|\partial_X \Delta_L^{-1}| \leqslant \frac{|k|}{k^2 + (\eta - kt)^2} \leqslant \min\left\{-\frac{\dot{M}_1}{M_1}, \sqrt{-\frac{\dot{M}_1}{M_1}}, \sqrt{-\frac{\dot{M}}{M}} m^{1/2}\right\}, \tag{3.16}$$

and

$$|\partial_Y^L| \lesssim \nu^{-1/3} m^{1/2} |\nabla_L|. \tag{3.17}$$

Accordingly,

$$\|\partial_X \Delta_L^{-1} f_{\neq} \partial_Y^L g_{\neq}\|_{L^1 H^N} \lesssim \nu^{-\frac{1}{3}} \left\| \sqrt{-\frac{\dot{M}}{M}} m^{1/2} f_{\neq} \right\|_{L^2 H^N} \|m^{1/2} \nabla_L g_{\neq}\|_{L^2 H^N}$$

$$\approx \nu^{-\frac{1}{3}} \left\| \sqrt{-\frac{\dot{M}}{M}} K f_{\neq} \right\|_{L^2 L^2} \|\nabla_L K g_{\neq}\|_{L^2 L^2}. \tag{3.18}$$

It follows from (3.15) and (3.18) that (3.12) holds. This completes the proof of lemma 3.2. □

3.1. Proof of theorem 3.1

To prove theorem 3.1, it suffices to establish the following *a priori* estimates:

$$\|KZ^{\pm}\|_{L^\infty L^2} + \nu^{\frac{1}{2}} \|\nabla_L K Z^{\pm}\|_{L^2 L^2} + \left\| \sqrt{-\frac{\dot{M}}{M}} K Z^{\pm} \right\|_{L^2 L^2} \leqslant 8\epsilon, \tag{3.19}$$

and

$$\left\|u_0^1 \mp b_0^1\right\|_{L^\infty H^N} + \nu^{\frac{1}{2}} \left\|\partial_y (u_0^1 \mp b_0^1)\right\|_{L^2 H^N} \leqslant 8\epsilon. \tag{3.20}$$

Indeed, (3.4) is a direct consequence of (3.11) and (3.19), and (3.5) is nothing but (3.20). To prove (3.6), by the definition of M_2, we have

$$1 \lesssim \nu^{-\frac{1}{6}} \left(\sqrt{-\frac{\dot{M}_2}{M_2}(t,k,\eta)} + \nu^{\frac{1}{2}} |k, \eta - kt| \right). \tag{3.21}$$

It follows that

$$\|KZ_{\neq}\|_{L^2 L^2} \lesssim \nu^{-\frac{1}{6}} \left(\nu^{\frac{1}{2}} \|\nabla_L K Z_{\neq}\|_{L^2 L^2} + \left\| \sqrt{-\frac{\dot{M}}{M}} K Z_{\neq} \right\|_{L^2 L^2} \right). \tag{3.22}$$

Thus, (3.6) follows from (3.11), (3.19) and (3.22) immediately.

Next we will prove (3.19) and (3.20) by using the standard continuity method. First of all, the local well-posedness of the 2D MHD equations in H^N ensures that there exists a $T_0 > 0$, such that

$$\|KZ^{\pm}\|_{L^\infty(0,T_0;L^2)} + \nu^{\frac{1}{2}} \|\nabla_L K Z^{\pm}\|_{L^2(0,T_0;L^2)} + \left\| \sqrt{-\frac{\dot{M}}{M}} K Z^{\pm} \right\|_{L^2(0,T_0;L^2)} \leqslant 2\epsilon,$$

and

$$\left\|u_0^1 \mp b_0^1\right\|_{L^\infty(0,T_0;H^N)} + \nu^{\frac{1}{2}} \left\|\partial_y (u_0^1 \mp b_0^1)\right\|_{L^2(0,T_0;H^N)} \leqslant 2\epsilon.$$

Then define $T^* \leqslant \infty$ to be the maximum of all time T such that (3.19) and (3.20) hold on $[0, T]$. By the continuity, $T^* > T_0$.

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We are left to prove that the constant 8 on the right of (3.19) and (3.20) can be replaced by 4, which implies that $T^* = \infty$. In fact, we have the following propositon.

PROPOSITION 3.3. *Let $\mu = \nu \in (0, 1], N > 1$. Assume that (3.19) and (3.20) hold on $[0, T^*]$. There exist a universal constant $\alpha_0 > 0$ and a positive constant δ depending only on N and α, such that if $|\alpha| \geqslant \alpha_0$ and (3.3) holds, then the same estimates in (3.19) and (3.20) hold with the occurrences of 8 on the right-hand side replaced by 4.*

The proof of proposition 3.3 will be achieved in the following two subsections.

3.2. Improvement of (3.19)

Recalling the definition of the multiplier K in (3.10), from (3.2), we derive the following energy identity:

$$\begin{aligned}
&\frac{1}{2}\|KZ^+(t)\|_{L^2}^2 + \nu\|\nabla_L KZ^+\|_{L^2L^2}^2 \\
&\quad + \left\|\sqrt{-\frac{\dot{M}}{M}}KZ^+\right\|_{L^2L^2}^2 + \left\|\sqrt{-\frac{\partial_t(m^{1/2})}{m^{1/2}}}KZ^+\right\|_{L^2L^2}^2 \\
&= \frac{1}{2}\|KZ^+(0)\|_{L^2}^2 - \int_0^t \left\langle SKZ^+, KZ^+\right\rangle \mathrm{d}t' + \int_0^t \left\langle SO_{2\alpha}^t KZ^-, KZ^+\right\rangle \mathrm{d}t' \\
&\quad + \int_0^t \left\langle KZ^+, K\left(\partial_t Z^+\right)_{\mathcal{NL}}\right\rangle \mathrm{d}t' \\
&= \frac{1}{2}\|KZ^+(0)\|_{L^2}^2 + \mathrm{LS} + \mathrm{OLS} + \mathcal{NL}.
\end{aligned} \tag{3.23}$$

By the definition of m, we have

$$\mathrm{LS} \leqslant \left\|\sqrt{-\frac{\partial_t(m^{1/2})}{m^{1/2}}}KZ^+\right\|_{L^2L^2}^2 + \frac{\nu}{2}\|\nabla_L KZ^+\|_{L^2L^2}^2. \tag{3.24}$$

Thanks to the fact (2.31), one can estimate OLS by integrating by parts in time:

$$\mathrm{OLS} = \sum_{i=1}^{5} \mathrm{OLS}_i, \tag{3.25}$$

where

$$\begin{aligned}
\mathrm{OLS}_1 &= \frac{1}{2\alpha}\left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{/}^- = (t), KZ^+(t)\right\rangle, \\
\mathrm{OLS}_2 &= -\frac{1}{2\alpha}\left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{/}^- = (0), KZ^+(0)\right\rangle,
\end{aligned}$$

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$$\mathrm{OLS}_3 = -\frac{1}{2\alpha}\int_0^t \left\langle \left(S\frac{2\dot{K}}{K} + \dot{S} \right) \partial_X^{-1} O_{2\alpha}^t K Z^{-}_{/} =, KZ^{+} \right\rangle \mathrm{d}t',$$

$$\mathrm{OLS}_4 = -\frac{1}{2\alpha}\int_0^t \left\langle S \partial_X^{-1} O_{2\alpha}^t K Z^{-}_{/} =, K\partial_t Z^{+} \right\rangle \mathrm{d}t',$$

$$\mathrm{OLS}_5 = -\frac{1}{2\alpha}\int_0^t \left\langle S \partial_X^{-1} O_{2\alpha}^t K \partial_t Z^{-}_{/} =, KZ^{+} \right\rangle \mathrm{d}t'.$$

Clearly,

$$\mathrm{OLS}_1 + \mathrm{OLS}_2 \leqslant \frac{1}{2|\alpha|}\left(\|KZ^{-}_{\neq}(t)\|_{L^2} \|KZ^{+}_{\neq}(t)\|_{L^2} + \|KZ^{-}_{\neq}(0)\|_{L^2} \|KZ^{+}_{\neq}(0)\|_{L^2} \right). \tag{3.26}$$

To bound OLS_3, in Fourier variables, we find that

$$\frac{\dot{K}}{K} = \frac{\partial_t(m^{1/2})}{m^{1/2}} + \frac{\dot{M}}{M} = \frac{1}{2}\frac{\dot{m}}{m} + \frac{\dot{M}}{M}. \tag{3.27}$$

On the support of $\dot{m}$, there holds

$$\frac{\dot{m}}{m} = \frac{2k(\eta - kt)}{k^2 + (\eta - kt)^2} = 2S(t, k, \eta). \tag{3.28}$$

Thus,

$$\left| \frac{\partial_t(m^{1/2})}{m^{1/2}} \right| + |S| \leqslant \frac{|k(\eta - kt)|}{k^2 + (\eta - kt)^2} \leqslant \min\left\{ \frac{1}{2}, \sqrt{-\frac{\dot{M}_1}{M_1}} \right\}. \tag{3.29}$$

Moreover,

$$|\dot{S}| \leqslant \frac{k^2}{k^2 + (\eta - kt)^2} = -\frac{\dot{M}_1}{M_1} \leqslant -\frac{\dot{M}}{M}. \tag{3.30}$$

It follows that

$$\mathrm{OLS}_3 \leqslant \frac{2}{|\alpha|} \left\| \sqrt{-\frac{\dot{M}}{M}} KZ^{-}_{\neq} \right\|_{L^2L^2} \left\| \sqrt{-\frac{\dot{M}}{M}} KZ^{+} \right\|_{L^2L^2}. \tag{3.31}$$

As for OLS_4, in view of (3.2), we have

$$\mathrm{OLS}_4 = \sum_{i=1}^{4} \mathrm{OLS}_4^{(i)}, \tag{3.32}$$

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with

$$\mathrm{OLS}_4^{(1)} = -\frac{\nu}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, K\Delta_L Z^+\right\rangle \mathrm{d}t',$$

$$\mathrm{OLS}_4^{(2)} = \frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, KSZ^+\right\rangle \mathrm{d}t',$$

$$\mathrm{OLS}_4^{(3)} = -\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, KO_{2\alpha}^t Z_{\neq}^-\right\rangle \mathrm{d}t',$$

$$\mathrm{OLS}_4^{(4)} = -\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, K\left(\partial_t Z^+\right)_{\mathcal{NL}}\right\rangle \mathrm{d}t'.$$

Integrating by parts, and using the fact $|S| \leqslant \frac{1}{2}$, we have

$$\begin{aligned}\mathrm{OLS}_4^{(1)} &= \frac{\nu}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t \nabla_L KZ_{\neq}^-, \nabla_L KZ^+\right\rangle \mathrm{d}t' \\ &\leqslant \frac{\nu}{4|\alpha|}\|\nabla_L KZ_{\neq}^-\|_{L^2L^2}\|\nabla_L KZ_{\neq}^+\|_{L^2L^2}.\end{aligned}$$

Thanks to the fact $|S| \leqslant \sqrt{-\frac{\dot{M}_1}{M_1}}$, one deduces that

$$\mathrm{OLS}_4^{(2)} \leqslant \frac{1}{2|\alpha|}\left\|\sqrt{-\frac{\dot{M}}{M}}KZ_{\neq}^+\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}KZ_{\neq}^-\right\|_{L^2L^2}.$$

Owing to the periodicity in X variable, we find that

$$\mathrm{OLS}_4^{(3)} = -\frac{1}{4\alpha}\int_0^t\int_{\mathbb{T}\times\mathbb{R}} \partial_X\left(S\partial_X^{-1}O_{2\alpha}^t KZ_{/}^-=\right)^2 dXdYdt' = 0. \tag{3.33}$$

To bound $\mathrm{OLS}_4^{(4)}$, by (3.2), we get

$$\begin{aligned}\mathrm{OLS}_4^{(4)} = &\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, K\left(\nabla_L^\perp\Delta_L^{-1}O_{2\alpha}^t Z^-\cdot\nabla_L Z^+\right)\right\rangle \mathrm{d}t' \\ &-\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, K\left(SO_{2\alpha}^t Z^-(2\partial_{XX}\Delta_L^{-1}Z^+ - Z^+)\right)\right\rangle \mathrm{d}t' \\ &+\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{\neq}^-, K\left(SZ^+(2\partial_{XX}\Delta_L^{-1}O_{2\alpha}^t Z^- - O_{2\alpha}^t Z^-)\right)\right\rangle \mathrm{d}t' \\ = &\,\mathcal{NLT} + \mathcal{NLS}1 + \mathcal{NLS}2,\end{aligned} \tag{3.34}$$

with

$$\begin{aligned}\mathcal{NLT} = &\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{/}^-=, K\left(\nabla_L^\perp\Delta_L^{-1}O_{2\alpha}^t Z_{/}^-=\cdot\nabla_L Z_{/}^+=\right)\right\rangle \mathrm{d}t' \\ &+\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t KZ_{/}^-=, K\left(\nabla_L^\perp\Delta_L^{-1}O_{2\alpha}^t Z_0^-\cdot\nabla_L Z_{/}^+=\right)\right\rangle \mathrm{d}t'\end{aligned}$$

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$$+\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O^t_{2\alpha}KZ^-_{/} =, K\left(\nabla_L^{\perp}\Delta_L^{-1}O^t_{2\alpha}Z^-_{/} = \cdot\nabla_L Z_0^+\right)\right\rangle \mathrm{d}t'$$
$$=\mathcal{NLT}(\neq,\neq)+\mathcal{NLT}(0,\neq)+\mathcal{NLT}(\neq,0).$$

By using lemma 3.2, we are led to

$$\begin{aligned}
&\mathcal{NLT}(\neq,\neq)\\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^\infty L^2}\|\nabla_L^{\perp}\Delta_L^{-1}O^t_{2\alpha}Z^-_{\neq}\cdot\nabla_L Z^+_{\neq}\|_{L^1H^N}\\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^\infty L^2}\left(\|KZ^-_{\neq}\|_{L^2L^2}+\nu^{-\frac{1}{3}}\left\|\sqrt{-\frac{\dot{M}}{M}}KZ^-_{\neq}\right\|_{L^2L^2}\right)\|\nabla_L KZ^+_{\neq}\|_{L^2L^2}\\
&\lesssim \nu^{-\frac{5}{6}}\epsilon^3.
\end{aligned}\tag{3.35}$$

In view of (3.16), one deduces that

$$\begin{aligned}
\mathcal{NLT}(\neq,0)&=\frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}KZ^-_{\neq}, K\left(\partial_X\Delta_L^{-1}O^t_{2\alpha}Z^-_{\neq}\partial_Y Z_0^+\right)\right\rangle \mathrm{d}t'\\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^\infty L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}KZ^-_{\neq}\right\|_{L^2L^2}\|\partial_Y Z_0^+\|_{L^2H^N}\\
&\lesssim \nu^{-\frac{1}{2}}\epsilon^3.
\end{aligned}\tag{3.36}$$

Recalling that $z^{\pm}=O^t_{\pm\alpha}w^{\pm}$, we then have $z_0^{\pm}=w_0^{\pm}=\omega_0\mp j_0$. Consequently,

$$\partial_y z_0^{\pm}=\partial_y\omega_0\mp\partial_y j_0=-\left(\partial_{yy}u_0^1\mp\partial_{yy}b_0^1\right),$$

and hence

$$\partial_y\partial_{yy}^{-1}z_0^{\pm}=-\left(u_0^1\mp b_0^1\right).\tag{3.37}$$

Using (3.13), we arrive at

$$\begin{aligned}
\mathcal{NLT}(0,\neq)&=-\frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}KZ^-_{\neq}, K\left(\partial_Y\partial_{YY}^{-1}O^t_{2\alpha}Z_0^-\partial_X Z^+_{\neq}\right)\right\rangle \mathrm{d}t'\\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^2L^2}\|\partial_Y\partial_{YY}^{-1}Z_0^-\|_{L^\infty H^N}\|\partial_X Z^+_{\neq}\|_{L^2H^N}\\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^2L^2}\|u_0^1+b_0^1\|_{L^\infty H^N}\|\nabla_L KZ^+_{\neq}\|_{L^2L^2}\\
&\lesssim \nu^{-\frac{2}{3}}\epsilon^3.
\end{aligned}\tag{3.38}$$

$\mathcal{NLS}1$ and $\mathcal{NLS}2$ can be treated in the same way, we only estimate $\mathcal{NLS}2$ now. To this end, we infer from (2.22) that

$$|S|\leqslant\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\leqslant\frac{|k|}{\sqrt{k^2+(\eta-kt)^2}}\lesssim m^{1/2}.\tag{3.39}$$

This, together with (2.21) and the obvious fact $|\partial_{XX}\Delta_L^{-1}| \leqslant 1$, implies that

$$
\begin{aligned}
\mathcal{NLS}2 &\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^\infty L^2}\|SZ^+(2\partial_{XX}\Delta_L^{-1}O^t_{2\alpha}Z^- - O^t_{2\alpha}Z^-)\|_{L^1H^N} \\
&\lesssim \frac{1}{|\alpha|}\|KZ^-_{\neq}\|_{L^2L^2}\|m^{1/2}Z^+\|_{L^2H^N}\left(\nu^{-\frac{1}{3}}\|m^{1/2}Z^-\|_{L^\infty H^N}\right) \\
&\lesssim \nu^{-\frac{2}{3}}\epsilon^3.
\end{aligned} \tag{3.40}
$$

Note that OLS_5 can be treated in the same manner as OLS_4, and $\mathcal{NL}$ in (3.23) can be treated in the same way as $\mathrm{OLS}_4^{(4)}$. On the other hand, one can obtain an energy identity for KZ^- similar to (3.23), and estimate the corresponding right hand side terms analogously. Putting the energy estimates for KZ^+ and KZ^- together, we find that for sufficiently large $|\alpha|$, LS, OLS_1, OLS_3, $\mathrm{OLS}_4^{(1)}$, $\mathrm{OLS}_4^{(2)}$ can be absorbed by the left hand. In conclusion, there exist a universal constant $\alpha_0 > 0$, and a positive constant C depending only on N and α, such that if $|\alpha| > \alpha_0$, we have

$$
\|KZ^\pm(t)\|^2_{L^2} + \nu\|\nabla_L KZ^\pm\|^2_{L^2L^2} + \left\|\sqrt{-\frac{\dot{M}}{M}}KZ^\pm\right\|^2_{L^2L^2} \leqslant 2\|KZ^\pm(0)\|^2_{L^2} + C\nu^{-\frac{5}{6}}\epsilon^3, \tag{3.41}
$$

which suffices to improve (3.19) as long as $\epsilon \ll \nu^{\frac{5}{6}}$.

3.3. Improvement of (3.20)

Since $u_0^2 = b_0^2 = 0$, we derive from (1.2) for $\bar{U}(t,y) = U(y) = y$ that (u_0^1, b_0^1) solves

$$
\begin{cases}
\partial_t u_0^1 - \nu\partial_{yy}u_0^1 = -\left(u\cdot\nabla u^1 - b\cdot\nabla b^1\right)_0, \\
\partial_t b_0^1 - \nu\partial_{yy}b_0^1 = -\left(u\cdot\nabla b^1 - b\cdot\nabla u^1\right)_0.
\end{cases}
$$

Accordingly, in the coordinate system (3.1), we have

$$
\partial_t\left(U_0^1 \mp B_0^1\right) - \nu\partial_{YY}\left(U_0^1 \mp B_0^1\right) = -\left((U_{\neq} \pm B_{\neq})\cdot\nabla_L(U^1_{\neq} \mp B^1_{\neq})\right)_0. \tag{3.42}
$$

Then

$$
\begin{aligned}
&\frac{1}{2}\left\|(U_0^1 \mp B_0^1)(t)\right\|^2_{H^N} + \nu\|\partial_Y(U_0^1 \mp B_0^1)\|^2_{L^2H^N} \\
&\quad = \frac{1}{2}\left\|(U_0^1 \mp B_0^1)(0)\right\|^2_{H^N} \\
&\qquad - \int_0^t \left\langle U_0^1 \mp B_0^1, \left((U_{\neq} \pm B_{\neq})\cdot\nabla_L(U^1_{\neq} \mp B^1_{\neq})\right)_0\right\rangle_{H^N} \mathrm{d}t'.
\end{aligned} \tag{3.43}
$$

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Thanks to the divergence free condition, we have

$$
\begin{aligned}
&-\int_0^t \left\langle U_0^1 \mp B_0^1, \big((U_{\neq} \pm B_{\neq}) \cdot \nabla_L (U_{\neq}^1 \mp B_{\neq}^1)\big)_0 \right\rangle_{H^N} \mathrm{d}t' \\
&\quad = \int_0^t \left\langle \partial_Y (U_0^1 \mp B_0^1), \big((U_{\neq}^2 \pm B_{\neq}^2)(U_{\neq}^1 \mp B_{\neq}^1)\big)_0 \right\rangle_{H^N} \mathrm{d}t' \\
&\quad \lesssim \left\| \partial_Y (U_0^1 \mp B_0^1) \right\|_{L^2 H^N} \left\| U_{\neq}^2 \pm B_{\neq}^2 \right\|_{L^2 H^N} \left\| U_{\neq}^1 \mp B_{\neq}^1 \right\|_{L^\infty H^N}. \qquad (3.44)
\end{aligned}
$$

Note that

$$
u_{\neq}^2 \pm b_{\neq}^2 = \partial_x \Delta^{-1} (\omega \pm j) = \partial_x \Delta^{-1} w^{\mp} = \partial_x \Delta^{-1} O_\alpha^t z^{\mp},
$$

and

$$
u_{\neq}^1 \mp b_{\neq}^1 = -\partial_y \Delta^{-1} (\omega_{\neq} \mp j_{\neq}) = -\partial_y \Delta^{-1} w_{\neq}^{\pm} = -\partial_y \Delta^{-1} O_{-\alpha}^t z_{\neq}^{\pm}.
$$

Then in view of (3.14) and (3.16), we find that

$$
\left\| U_{\neq}^2 \pm B_{\neq}^2 \right\|_{L^2 H^N} = \left\| \partial_X \Delta_L^{-1} O_\alpha^t Z^{\mp} \right\|_{L^2 H^N} \lesssim \left\| \sqrt{-\frac{\dot{M}}{M}} K Z^{\mp} \right\|_{L^2 L^2}, \qquad (3.45)
$$

and

$$
\left\| U_{\neq}^1 \mp B_{\neq}^1 \right\|_{L^\infty H^N} = \left\| \partial_Y^L \Delta_L^{-1} O_{-\alpha}^t Z_{\neq}^{\pm} \right\|_{L^\infty H^N} \lesssim \left\| K Z_{\neq}^{\pm} \right\|_{L^\infty L^2}. \qquad (3.46)
$$

Substituting (3.44)–(3.46) into (3.43), noting that the coordinate system (3.1) is the same as the original system in y variable, and using the hypotheses (3.19) and (3.20), we are led to

$$
\begin{aligned}
&\frac{1}{2} \left\| (u_0^1 \mp b_0^1)(t) \right\|_{H^N}^2 + \nu \left\| \partial_y (u_0^1 \mp b_0^1) \right\|_{L^2 H^N}^2 \\
&\quad \leqslant \frac{1}{2} \left\| (u_0^1 \mp b_0^1)(0) \right\|_{H^N}^2 + C \left\| \partial_y (u_0^1 \mp b_0^1) \right\|_{L^2 H^N} \left\| \sqrt{-\frac{\dot{M}}{M}} K Z_{\neq}^{-} \right\|_{L^2 L^2} \left\| K Z_{\neq}^{+} \right\|_{L^\infty L^2} \\
&\quad \leqslant \frac{1}{2} \| (u_0^1 \mp b_0^1)(0) \|_{H^N}^2 + C \nu^{-\frac{1}{2}} \epsilon^3, \qquad (3.47)
\end{aligned}
$$

which is sufficient to improve (3.20) provided $\epsilon \ll \nu^{\frac{1}{2}}$. Combining (3.41) with (3.47), we complete the proof of proposition 3.3 and hence of theorem 3.1.

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4. Stability of the shear flow close to Couette

In this section, we study the stability of the shear flow $(\bar{U}(t,y),0)^\top = (e^{\nu t\partial_{yy}}U(y),0)^\top$, with $U(y)$ satisfying

$$\|U(y)-y\|_{H^{N+6}} \leqslant \delta. \tag{4.1}$$

The multiplier that will be used in this section is given by

$$\tilde{K} := \langle\nabla\rangle^N \tilde{m}^{1/2} M, \tag{4.2}$$

where $\tilde{m}$ and M are given in (2.23) and (3.10), respectively. To simplify the presentation, let us denote

$$|S|_d =: \left|\partial_{XY}^L \Delta_L^{-1}\right|_d = \left|\partial_{XY}^L \Delta_L^{-1}\right| \mathbf{1}_{D_{dam}}, \text{ with symbol } \frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dam}}(t,k,\eta).$$

The aim of this section is to establish the following theorem.

THEOREM 4.1. *Let $N>1$. Assume that the shear flow $(U(y),0)$ satisfies (4.1), and $\|(\omega_{\text{in}},j_{\text{in}})\|_{H^N} + \|(u_{\text{in}},b_{\text{in}})\|_{H^N} = \epsilon \leqslant \delta\nu^{\frac{5}{6}+\tilde{\delta}}$. Then there exist two positive constants α_0 and δ_0 independent of ν, such that for all $|\alpha| \geqslant \alpha_0$ and $\delta \leqslant \delta_0$, the solution to (1.2) and the profile $Z^\pm$ satisfy the global in time estimates*

$$\left\|Z^\pm\right\|_{L^\infty H^N} + \nu^{\frac{1}{2}}\left\|\nabla_L Z^\pm\right\|_{L^2H^N} \lesssim \nu^{-(\frac{1}{3}+\frac{\tilde{\delta}}{2})}\epsilon, \tag{4.3}$$

$$\left\|U_0^1 \mp B_0^1\right\|_{L^\infty H^N} + \nu^{\frac{1}{2}}\left\|\partial_y(U_0^1 \mp B_0^1)\right\|_{L^2H^N} \lesssim \epsilon, \tag{4.4}$$

and

$$\left\|Z_{\neq}^\pm\right\|_{L^2H^N} \lesssim \nu^{-\frac{1}{2}-\frac{\tilde{\delta}}{2}}\epsilon. \tag{4.5}$$

Similar to lemma 3.2, we give the following lemma to treat the non-zero frequency interactions of the nonlinear term.

LEMMA 4.2. *Let $N>1$. Assume that (4.1) holds with δ sufficiently small. Then for all $f \in L^2H^N$ and g such that $\nabla_L g \in L^2H^N$, there holds*

$$\begin{aligned}&\|\nabla_t^\perp \Delta_t^{-1} f_{\neq} \cdot \nabla_t g_{\neq}\|_{L^1H^N} \\ &\quad\leqslant \frac{C\nu^{-\tilde{\delta}}}{1-C\delta}\left(\|\tilde{K}f_{\neq}\|_{L^2L^2} + \nu^{-\frac{1}{3}}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}f_{\neq}\right\|_{L^2L^2}\right)\left\|\nabla_L\tilde{K}g_{\neq}\right\|_{L^2L^2}.\end{aligned}$$

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Proof. In view of (2.25) and (2.26), there hold

$$|k| \lesssim \nu^{-\tilde{\delta}/2}\sqrt{k^2+(\eta-kt)^2}\tilde{m}^{1/2}, \quad \text{i.e.} \quad |\partial_X| \lesssim \nu^{-\tilde{\delta}/2}|\nabla_L|\tilde{m}^{1/2}, \tag{4.6}$$

$$|\partial_Y^L\Delta_L^{-1}| \leqslant \frac{|\eta-kt|}{k^2+(\eta-kt)^2} \leqslant \frac{1}{\sqrt{k^2+(\eta-kt)^2}} \lesssim \nu^{-\tilde{\delta}/2}\tilde{m}^{1/2}, \quad \text{for} \quad k\neq 0, \tag{4.7}$$

$$|\partial_X\Delta_L^{-1}| \leqslant \frac{|k|}{k^2+(\eta-kt)^2} \leqslant \nu^{-\tilde{\delta}/2}\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{m}^{1/2}, \tag{4.8}$$

and

$$|\partial_Y^L| \lesssim \nu^{-(\frac{1}{3}+\frac{\tilde{\delta}}{2})}\tilde{m}^{1/2}|\nabla_L|. \tag{4.9}$$

Note that the condition (4.1) ensures that (A.3) holds. This, together with the commutator estimates (B.1), (B.9), enables us to use lemma C.1 and the fact

$$\nabla_t^\perp\Delta_t^{-1}f_{\neq}\cdot\nabla_t g_{\neq} = -a\partial_Y^L\Delta_L^{-1}\Lambda f_{\neq}\partial_X g_{\neq} + a\partial_X\Delta_L^{-1}\Lambda f_{\neq}\partial_Y^L g_{\neq},$$

to obtain

$$\begin{aligned}
&\|\nabla_t^\perp\Delta_t^{-1}f_{\neq}\cdot\nabla_t g_{\neq}\|_{L^1H^N}\\
&\quad\leqslant \|a\partial_Y^L\Delta_L^{-1}\Lambda f_{\neq}\partial_X g_{\neq}\|_{L^1H^N} + \|a\partial_X\Delta_L^{-1}\Lambda f_{\neq}\partial_Y^L g_{\neq}\|_{L^1H^N}\\
&\quad\leqslant C\nu^{-\tilde{\delta}}\left(1+\|a-1\|_{L^\infty H^N}\right)\left(\|\tilde{m}^{1/2}\Lambda f_{\neq}\|_{L^2H^N} + \nu^{-\frac{1}{3}}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{m}^{1/2}\Lambda f_{\neq}\right\|_{L^2H^N}\right)\\
&\qquad\times\|\nabla_L\tilde{m}^{1/2}g_{\neq}\|_{L^2H^N}\\
&\quad\leqslant \frac{C\nu^{-\tilde{\delta}}}{1-C\delta}\left(\|\tilde{m}^{1/2}f_{\neq}\|_{L^2H^N} + \nu^{-\frac{1}{3}}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{m}^{1/2}f_{\neq}\right\|_{L^2H^N}\right)\|\nabla_L\tilde{m}^{1/2}g_{\neq}\|_{L^2H^N}.
\end{aligned}$$

Then (4.2) follows immediately. □

4.1. Proof of theorem 4.1

The proof of theorem 4.1 is similar to that of theorem 3.1. By the definition of M_2, (3.22) still holds with K replaced by $\tilde{K}$, it suffices to establish the following *a priori* estimates:

$$\|\tilde{K}Z^\pm\|_{L^\infty L^2} + \nu^{\frac{1}{2}}\|\nabla_L\tilde{K}Z^\pm\|_{L^2L^2} + \left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^\pm\right\|_{L^2L^2} \leqslant 8\epsilon, \tag{4.10}$$

and

$$\left\|U_0^1 \mp B_0^1\right\|_{L^\infty H^N} + \nu^{\frac{1}{2}}\left\|\partial_y(U_0^1\mp B_0^1)\right\|_{L^2H^N} \leqslant 8\epsilon. \tag{4.11}$$

Let us define T^* to be the end point of the largest interval $[0,T]$ such that (4.10) and (4.11) hold for all $0\leqslant t\leqslant T$. We are left to establish the following proposition.

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PROPOSITION 4.3. *Assume that the conditions in theorem 4.1 hold, and that (4.10) and (4.11) hold on $[0, T^*]$. Then there exist two positive constants α_0 and δ_0 independent of ν, such that for all $|\alpha| \geqslant \alpha_0$ and $\delta \leqslant \delta_0$, the same estimates in (4.10) and (4.11) hold with the occurrences of 8 on the right-hand side replaced by 4.*

Proof. We first improve (4.10). Similar to (3.23), from (2.16) and the definition of $\tilde{K}$ in (4.2), we have the following energy identity:

$$\begin{aligned}
&\frac{1}{2}\|\tilde{K}Z^+(t)\|_{L^2}^2 + \nu\|\nabla_L \tilde{K}Z^+\|_{L^2L^2}^2 + \left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^+\right\|_{L^2L^2}^2 \\
&\quad + \left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+\right\|_{L^2L^2}^2 \\
&= \frac{1}{2}\|\tilde{K}Z^+(0)\|_{L^2}^2 - \int_0^t \left\langle \tilde{K}S\Lambda Z^+, \tilde{K}Z^+\right\rangle \mathrm{d}t' + \int_0^t \left\langle \tilde{K}O_{2\alpha}^t S\Lambda Z^-, \tilde{K}Z^+\right\rangle \mathrm{d}t' \\
&\quad + \int_0^t \left\langle \tilde{K}Z^+, \tilde{K}\mathrm{LP}^+\right\rangle \mathrm{d}t' + \int_0^t \left\langle \tilde{K}Z^+, \tilde{K}\mathrm{NL}^+\right\rangle \mathrm{d}t' \\
&= \frac{1}{2}\|\tilde{K}Z^+(0)\|_{L^2}^2 + \mathrm{LS} + \mathrm{OLS} + \mathcal{LP} + \mathcal{NL}. \qquad (4.12)
\end{aligned}$$

The improvement of (4.10) will be achieved by the following four steps.

Step I: estimates of LS**.** We first split LS into two parts:

$$\begin{aligned}
\mathrm{LS} &= -\frac{1}{4\pi^2}\sum_{k\neq 0}\iint \tilde{K}(k,\eta)\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}\left(\mathbf{1}_{D_{dam}}(t,k,\eta) + \mathbf{1}_{D_{dam}^c}(t,k,\eta)\right) \\
&\quad \times \widehat{\Lambda Z_{\neq}^+}(k,\eta)\tilde{K}(k,\eta)\bar{\hat{Z}}_{\neq}^+(k,\eta)d\eta \mathrm{d}t \\
&= \mathrm{LS}^{dam} + \mathrm{LS}^*. \qquad (4.13)
\end{aligned}$$

Thanks to (2.13), one can split LS^{dam} into two parts

$$\begin{aligned}
\mathrm{LS}^{dam} &= -\left\|\sqrt{|S|_d}\tilde{K}Z_{\neq}^+\right\|_{L^2L^2}^2 \\
&\quad - \frac{1}{4\pi^2}\sum_{k\neq 0}\iint \tilde{K}(k,\eta)\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dam}} \\
&\quad \times \widehat{\tilde{\Lambda}\Lambda Z_{\neq}^+}(k,\eta)\tilde{K}(k,\eta)\bar{\hat{Z}}_{\neq}^+(k,\eta)d\eta \mathrm{d}t \\
&=: \mathrm{LS}_1^{dam} + \mathrm{LS}_2^{dam}.
\end{aligned}$$

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By the definition of $\tilde{\Lambda}$ in (2.10), there holds

$$
\begin{aligned}
&\left|\widehat{\tilde{\Lambda}\Lambda Z^+_{\neq}}\right|(k,\eta)\\
&\leqslant \int_\xi \left(|\widehat{a^2-1}|(\eta-\xi)\frac{(\xi-kt)^2}{k^2+(\xi-kt)^2}+|\hat{b}|(\eta-\xi)\frac{|\xi-kt|}{k^2+(\xi-kt)^2}\right)|\widehat{\Lambda Z^+_{\neq}}|(k,\xi)d\xi\\
&\leqslant \int_\xi \left(|\widehat{a^2-1}|(\eta-\xi)+|\hat{b}|(\eta-\xi)\right)|\widehat{\Lambda Z^+_{\neq}}|(k,\xi)d\xi.
\end{aligned}
$$

Combining this with (B.8), (B.11), and (A.3) yields

$$
\begin{aligned}
&\mathrm{LS}_2^{dam}\\
&\leqslant C\sum_{k\neq 0}\iiint \langle\eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})+\frac{3}{2}}\left(|\widehat{a^2-1}|(\eta-\xi)+|\hat{b}|(\eta-\xi)\right)|\widehat{\tilde{K}\Lambda Z^+_{\neq}}|(k,\xi)\\
&\quad\times\left(\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}+\sqrt{\frac{|k|}{k^2+(\xi-kt)^2}}\right)\\
&\quad\times\sqrt{\frac{k(\eta-kt)}{k^2+(\eta-kt)^2}}\mathbf{1}_{D_{dam}}|\widehat{\tilde{K}Z^+_{\neq}}|(k,\eta)d\xi d\eta \mathrm{d}t\\
&\leqslant C\delta\left(\left\|\sqrt{|S|_d}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}\right)\left\|\sqrt{|S|_d}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}.
\end{aligned}
\tag{4.14}
$$

Using (B.8), (B.9), (B.11), lemma C.1 and (2.29), we find that

$$
\begin{aligned}
&\left\|\sqrt{|S|}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}\\
&\leqslant \frac{1}{1-C\delta}\left\|\sqrt{|S|}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}+\frac{C\delta}{1-C\delta}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}\\
&\leqslant \frac{1}{1-C\delta}\Bigg(\left\|\sqrt{|S|_d}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}+\sqrt{\frac{1}{1000^3}}\nu^{\frac{1}{2}}\left\|\nabla_L\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\\
&\quad+\frac{1}{\sqrt{1+\frac{3}{2}\tilde{\delta}}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\Bigg)+\frac{C\delta}{(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2},
\end{aligned}
\tag{4.15}
$$

where $|S|$ denotes the multiplier with symbol $\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}$. Substituting this into (4.14), and using (B.8), (B.9) and lemma C.1 again to bound the second term on

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the right hand side of (4.14), one deduces that

$$\begin{aligned}
\mathrm{LS}_2^{dam} \leqslant{} & C\delta\Bigg\{\frac{1}{1-C\delta}\Bigg(\left\|\sqrt{|S|_d}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}+\sqrt{\frac{1}{1000^3}}\nu^{\frac{1}{2}}\left\|\nabla_L\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2} \\
&+\frac{1}{\sqrt{1+\frac{3}{2}\tilde{\delta}}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}\Bigg)+\frac{C\delta}{(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2} \\
&+\frac{1}{1-C\delta}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}\Bigg\}\left\|\sqrt{|S|_d}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2} \\
\leqslant{} & \left(\frac{2C\delta}{1-C\delta}+\frac{C\delta}{2(1-C\delta)^2}\right)\left\|\sqrt{|S|_d}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}^2 \\
&+\frac{C\delta}{2(1-C\delta)}\Bigg(\frac{1}{1000^3}\nu\left\|\nabla_L\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}^2 \\
&+\frac{1}{1+\frac{3}{2}\tilde{\delta}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}^2\Bigg) \\
&+\frac{C\delta}{2(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}^2. \qquad (4.16)
\end{aligned}$$

Next we turn to bound LS^*. From (2.27) and (2.28), we infer that

$$\begin{aligned}
\mathrm{LS}^* \leqslant{} & \frac{1}{4\pi^2}\sum_{k\neq 0}\iint \tilde{K}(k,\eta)\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{dam}^c}|\widehat{\Lambda Z_{\neq}^{+}}|(k,\eta)\tilde{K}(k,\eta)|\hat{Z}_{\neq}^{+}|(k,\eta)d\eta \mathrm{d}t \\
\leqslant{} & \frac{\nu}{1000^3}\sum_{k\neq 0}\iint \tilde{K}(k,\eta)\left(k^2+(\eta-kt)^2\right)|\widehat{\Lambda Z_{\neq}^{+}}|(k,\eta)\tilde{K}(k,\eta)|\hat{Z}_{\neq}^{+}|(k,\eta)d\eta \mathrm{d}t \\
&+\frac{1}{1+\frac{3}{2}\tilde{\delta}}\sum_{k\neq 0}\iint \tilde{K}(k,\eta)\frac{-\partial_t(\tilde{m}^{1/2})(t,k,\eta)}{\tilde{m}^{1/2}(t,k,\eta)}|\widehat{\Lambda Z_{\neq}^{+}}|(k,\eta)\tilde{K}(k,\eta)|\hat{Z}_{\neq}^{+}|(k,\eta)d\eta \mathrm{d}t \\
\leqslant{} & \frac{\nu}{1000^3}\|\nabla_L\tilde{K}\Lambda Z_{\neq}^{+}\|_{L^2L^2}\|\nabla_L\tilde{K}Z_{\neq}^{+}\|_{L^2L^2} \\
&+\frac{1}{1+\frac{3}{2}\tilde{\delta}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}\Lambda Z_{\neq}^{+}\right\|_{L^2L^2}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}. \qquad (4.17)
\end{aligned}$$

By virtue of (B.8) and lemma C.1, we have

$$\|\nabla_L\tilde{K}\Lambda Z_{\neq}^{+}\|_{L^2L^2} \leqslant \frac{1}{1-C\delta}\|\nabla_L\tilde{K}Z_{\neq}^{+}\|_{L^2L^2}. \qquad (4.18)$$

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Now we are left to bound $\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}$. In fact, thanks to (B.8), (B.10), and lemma C.1, we are led to

$$\begin{aligned}
&\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2} \\
&\quad\leqslant \frac{1}{1-C\delta}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} + \sqrt{\frac{1+\frac{3}{2}\tilde{\delta}}{1000^3}}\frac{C\delta}{(1-C\delta)^2}\nu^{\frac{1}{2}}\left\|\nabla_L\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} \\
&\qquad + \frac{C\delta\sqrt{1+\frac{3}{2}\tilde{\delta}}}{(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} + \frac{C\delta\sqrt{1+\frac{3}{2}\tilde{\delta}}}{1-C\delta}\left\|\sqrt{|S|}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}, \quad (4.19)
\end{aligned}$$

where we have used (4.18) to bound $\|\nabla_L\tilde{K}\Lambda Z^+_{\neq}\|_{L^2L^2}$, and used (B.8), (B.9) and lemma C.1 to bound $\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2}$, respectively. It follows from (4.15) and (4.19) that

$$\begin{aligned}
&\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}\Lambda Z^+_{\neq}\right\|_{L^2L^2} \\
&\quad\leqslant \frac{1}{(1-C\delta)^2}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} \\
&\qquad + \sqrt{\frac{1+\frac{3}{2}\tilde{\delta}}{1000^3}}\frac{2C\delta}{(1-C\delta)^2}\nu^{\frac{1}{2}}\left\|\nabla_L\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} \\
&\qquad + \frac{C\delta\sqrt{1+\frac{3}{2}\tilde{\delta}}}{(1-C\delta)^2}\left\|\sqrt{|S|_d}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2} + \frac{C\delta\sqrt{1+\frac{3}{2}\tilde{\delta}}}{(1-C\delta)^3}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}.
\end{aligned} \quad (4.20)$$

Substituting (4.18) and (4.20) into (4.17), and using Cauchy–Schwarz inequality, we arrive at

$$\begin{aligned}
\mathrm{LS}^* \leqslant{}& \frac{1}{1-C\delta}\frac{1}{1000^3}\nu\|\nabla_L\tilde{K}Z^+_{\neq}\|^2_{L^2L^2} + \frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{1}{(1-C\delta)^2}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2} \\
&+ \frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{C\delta}{(1-C\delta)^2}\left(\frac{1+\frac{3}{2}\tilde{\delta}}{1000^3}\nu\left\|\nabla_L\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2} + \left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}\right) \\
&+ \frac{1}{2}\frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{C\delta}{(1-C\delta)^2}\left(\left(1+\frac{3}{2}\tilde{\delta}\right)\left\|\sqrt{|S|_d}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}\right.
\end{aligned}$$

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$$
\begin{aligned}
&+\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}\Bigg) \\
&+\frac{1}{2}\frac{1}{1+\frac{3}{2}\tilde{\delta}}\frac{C\delta}{(1-C\delta)^3}\left(\left(1+\frac{3}{2}\tilde{\delta}\right)\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}\right. \\
&\left.+\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}\right) \\
=&\frac{C_\delta}{1+\frac{3}{2}\tilde{\delta}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}+\frac{1}{(1-C\delta)^2}\frac{1}{1000^3}\nu\|\nabla_L\tilde{K}Z^+_{\neq}\|^2_{L^2L^2} \\
&+\frac{C\delta}{2(1-C\delta)^2}\left\|\sqrt{|S|_d}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2}+\frac{C\delta}{2(1-C\delta)^3}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|^2_{L^2L^2},
\end{aligned}
$$

where

$$
C_\delta := \frac{1}{(1-C\delta)^2}+\frac{3C\delta}{2(1-C\delta)^2}+\frac{C\delta}{2(1-C\delta)^3}, \tag{4.21}
$$

which is increasing in $\delta \in (0, \frac{1}{3C}]$, and $C_\delta \to 1$ as $\delta \to 0+$.

Step II: estimates of OLS**.** Similar to (3.25), we write

$$
\mathrm{OLS} = \sum_{i=1}^{5} \mathrm{OLS}_i, \tag{4.22}
$$

where

$$
\begin{aligned}
\mathrm{OLS}_1 &= \frac{1}{2\alpha}\left\langle \tilde{K}\partial_X^{-1}O^t_{2\alpha}S\Lambda Z^-_{\neq}(t), \tilde{K}Z^+(t)\right\rangle, \\
\mathrm{OLS}_2 &= -\frac{1}{2\alpha}\left\langle \tilde{K}\partial_X^{-1}O^t_{2\alpha}S\Lambda Z^-_{\neq}(0), \tilde{K}Z^+(0)\right\rangle, \\
\mathrm{OLS}_3 &= -\frac{1}{2\alpha}\int_0^t \left\langle \partial_X^{-1}O^t_{2\alpha}\left(S\frac{2\dot{\tilde{K}}}{\tilde{K}}+\dot{S}\right)\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}Z^+\right\rangle \mathrm{d}t', \\
\mathrm{OLS}_4 &= -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}S\partial_X^{-1}O^t_{2\alpha}\Lambda Z^-_{\neq}, \tilde{K}\partial_t Z^+\right\rangle \mathrm{d}t', \\
\mathrm{OLS}_5 &= -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}S\partial_X^{-1}O^t_{2\alpha}\partial_t(\Lambda Z^-_{\neq}), \tilde{K}Z^+\right\rangle \mathrm{d}t'.
\end{aligned}
$$

We postpone the treatment of the nonlinear terms in OLS_4 and OLS_5 to **Step IV**, and focus on the linear terms here. The estimates for OLS_1 and OLS_2 are the same

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as (3.26), and thus omitted. To estimate OLS_3, similar to (3.27)–(3.29), we have

$$\frac{\dot{\tilde{K}}}{\tilde{K}} = \frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}} + \frac{\dot{M}}{M} = \frac{1}{2}\frac{\dot{\tilde{m}}}{\tilde{m}} + \frac{\dot{M}}{M}, -\frac{\dot{\tilde{m}}}{\tilde{m}}$$
$$= (1+\frac{3}{2}\tilde{\delta})\frac{2k(\eta-kt)}{k^2+(\eta-kt)^2}\mathbf{1}_{D_{mul}}(t,k,\eta) = 2(1+\frac{3}{2}\tilde{\delta})S(t,k,\eta)\mathbf{1}_{D_{mul}}(t,k,\eta),$$

and

$$\left|\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}\right| \leqslant (1+\frac{3}{2}\tilde{\delta})|S| \leqslant \frac{(1+\frac{3}{2}\tilde{\delta})|k(\eta-kt)|}{k^2+(\eta-kt)^2} \leqslant (1+\frac{3}{2}\tilde{\delta})\min\left\{\frac{1}{2}, \sqrt{-\frac{\dot{M}_1}{M_1}}\right\}.$$

Combining these calculations with (3.30), and using (B.8), (B.9) and lemma C.1, we are led to

$$\begin{aligned}\mathrm{OLS}_3 &\leqslant \frac{C}{|\alpha|}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}\Lambda Z^-_{\neq}\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^+\right\|_{L^2L^2}\\ &\leqslant \frac{C}{|\alpha|}\frac{1}{1-C\delta}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^+\right\|_{L^2L^2}.\end{aligned} \tag{4.23}$$

To bound OLS_4, by (2.16), we have

$$\mathrm{OLS}_4 = \sum_{i=1}^{5}\mathrm{OLS}_4^{(i)}, \tag{4.24}$$

where

$$\begin{aligned}\mathrm{OLS}_4^{(1)} &= -\frac{\nu}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}\Delta_L Z^+\right\rangle \mathrm{d}t',\\ \mathrm{OLS}_4^{(2)} &= \frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}S\Lambda Z^+\right\rangle \mathrm{d}t',\\ \mathrm{OLS}_4^{(3)} &= -\frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}O^t_{2\alpha}Z^-_{\neq}\right\rangle \mathrm{d}t',\\ \mathrm{OLS}_4^{(4)} &= -\frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}\mathrm{LP}^+\right\rangle \mathrm{d}t',\\ \mathrm{OLS}_4^{(5)} &= -\frac{1}{2\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq}, \tilde{K}\mathrm{NL}^+\right\rangle \mathrm{d}t'.\end{aligned}$$

Integrating by parts, using the fact $|S| \leqslant \frac{1}{2}$ and (4.18), we have

$$\begin{aligned}\mathrm{OLS}_4^{(1)} &= \frac{\nu}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t \nabla_L \tilde{K}\Lambda Z_{\neq}^-, \nabla_L \tilde{K} Z^+ \right\rangle \mathrm{d}t' \\ &\leqslant \frac{\nu}{4|\alpha|}\|\nabla_L \tilde{K}\Lambda Z_{\neq}^-\|_{L^2L^2}\|\nabla_L \tilde{K} Z_{\neq}^+\|_{L^2L^2} \\ &\leqslant \frac{\nu}{4|\alpha|(1-C\delta)}\|\nabla_L \tilde{K} Z_{\neq}^-\|_{L^2L^2}\|\nabla_L \tilde{K} Z_{\neq}^+\|_{L^2L^2}. \end{aligned} \tag{4.25}$$

Thanks to the fact $|S| \leqslant \sqrt{-\frac{\dot{M}_1}{M_1}}$, one deduces that

$$\begin{aligned}\mathrm{OLS}_4^{(2)} &\leqslant \frac{1}{2|\alpha|}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}\Lambda Z^+\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}\Lambda Z^-\right\|_{L^2L^2} \\ &\leqslant \frac{1}{|\alpha|(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K} Z^+\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K} Z^-\right\|_{L^2L^2}. \end{aligned} \tag{4.26}$$

Owing to the periodicity in X variable, we find that

$$\mathrm{OLS}_4^{(3)} = \frac{1}{4\alpha}\int_0^t \int_{\mathbb{T}\times\mathbb{R}} \partial_X \left(S\partial_X^{-1}O_{2\alpha}^t\tilde{K}\Lambda Z_{\neq}^-\right)^2 dX dY dt' = 0. \tag{4.27}$$

By (2.18), we deduce that

$$\mathrm{OLS}_4^{(4)} = \sum_{i=1}^{3}\mathrm{OLS}_4^{(4,i)}, \tag{4.28}$$

with

$$\begin{aligned}\mathrm{OLS}_4^{(4,1)} &= \frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t\tilde{K}\Lambda Z_{\neq}^-, \tilde{K}\left((a^2-1)S\Lambda\left(Z_{\neq}^+ - O_{2\alpha}^t Z_{\neq}^-\right)\right)\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_4^{(4,2)} &= -\frac{1}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t\tilde{K}\Lambda Z_{\neq}^-, \tilde{K}\left(b\partial_X\Delta_t^{-1}O_{2\alpha}^t Z_{\neq}^-\right)\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_4^{(4,3)} &= -\frac{\nu}{2\alpha}\int_0^t \left\langle S\partial_X^{-1}O_{2\alpha}^t\tilde{K}\Lambda Z_{\neq}^-, \tilde{K}\left((a^2-1)\partial_{YY}^L Z_{\neq}^+\right)\right\rangle \mathrm{d}t'. \end{aligned}$$

Using $|S| \leqslant \sqrt{-\frac{\dot{M}_1}{M_1}}$, (A.3) and lemma C.1, we have

$$\begin{aligned}\mathrm{OLS}_4^{(4,1)} &\leqslant \frac{1}{2|\alpha|}\left\|\tilde{K}S\Lambda Z_{\neq}^-\right\|_{L^2L^2}\left\|\tilde{K}\left((a^2-1)S\Lambda\left(Z_{\neq}^+ - O_{2\alpha}^t Z_{\neq}^-\right)\right)\right\|_{L^2L^2} \\ &\leqslant \frac{C\delta}{|\alpha|}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}\Lambda Z_{\neq}^-\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}\Lambda\left(Z_{\neq}^+ - O_{2\alpha}^t Z_{\neq}^-\right)\right\|_{L^2L^2}\end{aligned}$$

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$$\leqslant \frac{C\delta}{|\alpha|(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2} \times\left(\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\right). \tag{4.29}$$

Noting that

$$\partial_X\Delta_t^{-1}=\partial_X\Delta_L^{-1}\Lambda,$$

and

$$\begin{aligned}\left|\mathcal{F}\left(b\partial_X\Delta_t^{-1}O^t_{2\alpha}Z^-_{\neq}\right)\right|(t,k,\eta)&\leqslant\int_\xi|\hat{b}|(\eta-\xi)\frac{|k|}{k^2+(\xi-kt)^2}|\widehat{\Lambda O^t_{2\alpha}Z^-_{\neq}}|(k,\xi)d\xi\\&\leqslant\left(|\hat{b}|*\left(\sqrt{-\frac{\dot{M}_1}{M_1}}|\widehat{\Lambda O^t_{2\alpha}Z^-_{\neq}}|\right)\right)(t,k,\eta),\end{aligned} \tag{4.30}$$

then we have

$$\begin{aligned}\mathrm{OLS}_4^{(4,2)}&\leqslant\frac{C\delta}{|\alpha|}\left\|\tilde{K}S\Lambda Z^-_{\neq}\right\|_{L^2L^2}\left\|\tilde{K}\sqrt{-\frac{\dot{M}_1}{M_1}}\Lambda O^t_{2\alpha}Z^-_{\neq}\right\|_{L^2L^2}\\&\leqslant\frac{C\delta}{|\alpha|(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|^2_{L^2L^2}.\end{aligned} \tag{4.31}$$

Integrating by parts and using (2.4), we find that

$$\begin{aligned}\mathrm{OLS}_4^{(4,3)}&=\frac{\nu}{\alpha}\int_0^t\left\langle S\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq},\tilde{K}\left(b\partial^L_YZ^+_{\neq}\right)\right\rangle \mathrm{d}t'\\&\quad+\frac{\nu}{2\alpha}\int_0^t\left\langle \partial_Y^LS\partial_X^{-1}O^t_{2\alpha}\tilde{K}\Lambda Z^-_{\neq},\tilde{K}\left((a^2-1)\partial^L_YZ^+_{\neq}\right)\right\rangle \mathrm{d}t'\\&\leqslant\frac{C\delta\nu}{|\alpha|(1-C\delta)}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\left\|\partial^L_Y\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\\&\quad+\frac{C\delta\nu}{4|\alpha|}\left\|\partial^L_Y\tilde{K}\Lambda Z^-_{\neq}\right\|_{L^2L^2}\left\|\partial^L_Y\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\\&\leqslant\frac{C\delta\nu}{|\alpha|(1-C\delta)}\left\|\nabla_L\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\left\|\nabla_L\tilde{K}Z^+_{\neq}\right\|_{L^2L^2},\end{aligned} \tag{4.32}$$

where we have used (A.3), (A.8) to bound $\left\|\tilde{K}(b\partial^L_Y)\right\|_{L^2L^2}$ and $\left\|\tilde{K}((a^2-1)\partial^L_Y)\right\|_{L^2L^2}$. The estimates of $\mathrm{OLS}_4^{(5)}$ will be postponed in **Step IV**.

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Now we rewrite OLS_5 as follows:

$$\begin{aligned}\mathrm{OLS}_5 &= -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}S\partial_X^{-1}O_{2\alpha}^t\Lambda\partial_t Z_{\neq}^-, \tilde{K}Z_{\neq}^+\right\rangle \mathrm{d}t' \\ &\quad -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}S\partial_X^{-1}O_{2\alpha}^t\partial_t\Lambda Z_{\neq}^-, \tilde{K}Z_{\neq}^+\right\rangle \mathrm{d}t' \\ &= \mathrm{OLS}_{5,1}+\mathrm{OLS}_{5,2}.\end{aligned}$$

To estimate $\mathrm{OLS}_{5,1}$, instead of using (2.16), up to the nonlinear terms and some linear errors, we write $\partial_t Z^-$ in terms of $\nu\Delta_t Z^-$ by virtue of (2.6) and (2.15)

$$\partial_t Z^- = \nu\Delta_t Z^- - \nu b\partial_Y^L Z^- - a^2 S\Lambda\left(Z_{\neq}^- - O_{-2\alpha}^t Z_{\neq}^+\right) + b\partial_X\Delta_t^{-1}O_{-2\alpha}^t Z_{\neq}^+ + \mathrm{NL}^-.$$

Then thanks to the relation (2.12), we have

$$\begin{aligned}\Lambda\partial_t Z_{\neq}^- &= \nu\Delta_L Z_{\neq}^- - \nu\Lambda\left(b\partial_Y^L Z_{\neq}^-\right) - \Lambda\left(a^2 S\Lambda\left(Z_{\neq}^- - O_{-2\alpha}^t Z_{\neq}^+\right)\right) \\ &\quad + \Lambda\left(b\partial_X\Delta_t^{-1}O_{-2\alpha}^t Z_{\neq}^+\right) + \Lambda\mathrm{NL}_{\neq}^-.\end{aligned}$$

Thus,

$$\mathrm{OLS}_{5,1} = \sum_{i=1}^{5}\mathrm{OLS}_{5,1}^{(i)}, \tag{4.33}$$

where

$$\begin{aligned}\mathrm{OLS}_{5,1}^{(1)} &= -\frac{\nu}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^t\Delta_L Z_{\neq}^-, \tilde{K}SZ_{\neq}^+\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_{5,1}^{(2)} &= \frac{\nu}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^t\Lambda\left(b\partial_Y^L Z_{\neq}^-\right), \tilde{K}SZ_{\neq}^+\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_{5,1}^{(3)} &= \frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^t\Lambda\left(a^2 S\Lambda\left(Z_{\neq}^- - O_{-2\alpha}^t Z_{\neq}^+\right)\right), \tilde{K}SZ_{\neq}^+\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_{5,1}^{(4)} &= -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^t\Lambda\left(b\partial_X\Delta_t^{-1}O_{-2\alpha}^t Z_{\neq}^+\right), \tilde{K}SZ_{\neq}^+\right\rangle \mathrm{d}t', \\ \mathrm{OLS}_{5,1}^{(5)} &= -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^t\Lambda\mathrm{NL}_{\neq}^-, \tilde{K}SZ_{\neq}^+\right\rangle \mathrm{d}t'.\end{aligned}$$

Similar to (4.32), we obtain

$$\left|\mathrm{OLS}_{5,1}^{(1)}\right| + \left|\mathrm{OLS}_{5,1}^{(2)}\right| \leqslant \left(\frac{\nu}{4|\alpha|} + \frac{C\delta\nu}{|\alpha|(1-C\delta)}\right)\left\|\nabla_L\tilde{K}Z_{\neq}^-\right\|_{L^2L^2}\left\|\nabla_L\tilde{K}Z_{\neq}^+\right\|_{L^2L^2}. \tag{4.34}$$

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Similar to (4.26) and (4.29), we arrive at

$$\begin{aligned}\mathrm{OLS}_{5,1}^{(3)} \leqslant & \frac{1}{2|\alpha|}\left(\left\|\tilde{K}\Lambda\left(S\Lambda\left(Z_{\neq}^{-}-O_{-2\alpha}^{t}Z_{\neq}^{+}\right)\right)\right\|_{L^2L^2}\right.\\ & \left.+\left\|\tilde{K}\Lambda\left((a^2-1)S\Lambda\left(Z_{\neq}^{-}-O_{-2\alpha}^{t}Z_{\neq}^{+}\right)\right)\right\|_{L^2L^2}\right)\|\tilde{K}SZ_{\neq}^{+}\|_{L^2L^2}\\ \leqslant & \left(\frac{1}{|\alpha|(1-C\delta)^2}+\frac{C\delta}{|\alpha|(1-C\delta)^2}\right)\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}\\ & \times\left(\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{-}\right\|_{L^2L^2}\right). \end{aligned} \tag{4.35}$$

Clearly, $\mathrm{OLS}_{5,1}^{(4)}$ can be bounded in the same way as (4.31)

$$\left|\mathrm{OLS}_{5,1}^{(4)}\right| \leqslant \frac{C\delta}{|\alpha|(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}^2. \tag{4.36}$$

The estimates of $\mathrm{OLS}_{5,1}^{(5)}$ will be postponed in **Step IV**.

To estimate $\mathrm{OLS}_{5,2}$, we split it into two parts according to (2.32) and (2.33):

$$\mathrm{OLS}_{5,2} = -\frac{1}{2\alpha}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^{t}\partial_t\Lambda Z_{\neq}^{-}, \tilde{K}SZ^{+}\right\rangle \mathrm{d}t' = \mathrm{OLS}_{5,2}^{(1)}+\mathrm{OLS}_{5,2}^{(2)},$$

where

$$\begin{aligned}\mathrm{OLS}_{5,2}^{(1)} &= -\frac{1}{2|\alpha|}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^{t}\Lambda\tilde{\Lambda}_t^{1}\Lambda Z_{\neq}^{-}, \tilde{K}SZ_{\neq}^{+}\right\rangle \mathrm{d}t',\\ \mathrm{OLS}_{5,2}^{(2)} &= -\frac{1}{2|\alpha|}\int_0^t \left\langle \tilde{K}\partial_X^{-1}O_{2\alpha}^{t}\Lambda\tilde{\Lambda}_t^{2}\Lambda Z_{\neq}^{-}, \tilde{K}SZ_{\neq}^{+}\right\rangle \mathrm{d}t'.\end{aligned}$$

From (A.8) and (3.29), we obtain

$$\begin{aligned}\mathrm{OLS}_{5,2}^{(1)} &\leqslant \frac{1}{|\alpha|}\left\|\tilde{K}\Lambda\tilde{\Lambda}_t^{1}\Lambda Z_{\neq}^{-}\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}\\ &\leqslant \frac{1}{|\alpha|}\frac{C\delta}{(1-C\delta)^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{-}\right\|_{L^2L^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^{+}\right\|_{L^2L^2}.\end{aligned}$$

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The rest part $\mathrm{OLS}_{5,2}^{(2)}$ can be bounded as follows. Using the definition of $\tilde{\Lambda}_t^2$ in (2.33), lemmas A.2 and C.1, we are led to

$$\begin{aligned}
&\left\|\tilde{K}\tilde{\Lambda}_t^2\Lambda Z_{\neq}^-\right\|_{L^2L^2}\\
&\leqslant C\left\|\left(\langle\cdot\rangle^{N+(1+\frac{3}{2}\tilde{\delta})}\left(|\widehat{a\partial_t a}|+|\widehat{\partial_t b}|\right)\right)*|\widehat{\tilde{K}\Lambda Z_{\neq}^-}|\right\|_{L_t^2L_{k,\eta}^2}\\
&\leqslant C\left\|\langle\cdot\rangle^{N+(1+\frac{3}{2}\tilde{\delta})}\left(|\widehat{a\partial_t a}|+|\widehat{\partial_t b}|\right)\right\|_{L_t^2L_\eta^1}\|\tilde{K}\Lambda Z_{\neq}^-\|_{L^\infty L^2}\\
&\leqslant \frac{C}{1-C\delta}\left(\|a\partial_t a\|_{L^2H^{N+(1+\frac{3}{2}\tilde{\delta})+1}}+\|\partial_t b\|_{L^2H^{N+(1+\frac{3}{2}\tilde{\delta})+1}}\right)\|\tilde{K}Z_{\neq}^-\|_{L^\infty L^2}\\
&\leqslant \frac{C\delta\nu^{1/2}}{1-C\delta}\|\tilde{K}Z_{\neq}^-\|_{L^\infty L^2}.
\end{aligned}$$

Then it follows this and lemma C.1 that

$$\begin{aligned}
\mathrm{OLS}_{5,2}^{(2)}&\leqslant \frac{1}{|\alpha|(1-C\delta)}\left\|\tilde{K}\tilde{\Lambda}_t^2\Lambda Z_{\neq}^-\right\|_{L^2L^2}\left\|\tilde{K}SZ_{\neq}^+\right\|_{L^2L^2}\\
&\leqslant \frac{C\delta\nu^{1/2}}{|\alpha|(1-C\delta)^2}\left\|\tilde{K}Z_{\neq}^-\right\|_{L^\infty L^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}^+\right\|_{L^2L^2}.
\end{aligned}\tag{4.37}$$

Step III: estimates of $\mathcal{LP}$. By (2.18), we rewrite $\mathcal{LP}$ as

$$\mathcal{LP}=\int_0^t\left\langle\tilde{K}Z^+,\tilde{K}\mathrm{LP}^+\right\rangle\mathrm{d}t'=\mathcal{LP}_1+\mathcal{LP}_2+\mathcal{LP}_3,\tag{4.38}$$

where

$$\begin{aligned}
\mathcal{LP}_1&=-\int_0^t\left\langle\tilde{K}Z_{\neq}^+,\tilde{K}\left((a^2-1)S\Lambda(Z_{\neq}^+-O_{2\alpha}^tZ_{\neq}^-)\right)\right\rangle\mathrm{d}t',\\
\mathcal{LP}_2&=\int_0^t\left\langle\tilde{K}Z_{\neq}^+,\tilde{K}\left(b\partial_X\Delta_L^{-1}\Lambda O_{2\alpha}^tZ_{\neq}^-\right)\right\rangle\mathrm{d}t',\\
\mathcal{LP}_3&=\nu\int_0^t\left\langle\tilde{K}Z^+,\tilde{K}\left((a^2-1)\partial_{YY}^LZ^+\right)\right\rangle\mathrm{d}t'.
\end{aligned}$$

To estimate $\mathcal{LP}_1$, let us denote $Z_{\not=}:=Z_{\not}^+=-O_{2\alpha}^tZ_{\neq}^-$, and rewrite $\mathcal{LP}_1$ as

$$\begin{aligned}
\mathcal{LP}_1=&-\frac{1}{4\pi^2}\sum_{k\neq0}\int_0^t\int_\eta\int_\xi\tilde{K}(k,\eta)(\widehat{a^2-1})(\eta-\xi)\frac{k(\xi-kt)}{k^2+(\xi-kt)^2}\widehat{\Lambda Z_{\neq}}(k,\xi)\\
&\times\tilde{K}(k,\eta)\bar{\hat{Z}}_{\neq}^+(k,\eta)d\xi d\eta\mathrm{d}t'.
\end{aligned}$$

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Swapping the positions of η and ξ in (A.11), then using the resulting inequality, (A.3), lemma C.1, and (4.15), we have

$$\begin{aligned}
\mathcal{LP}_1 &\leqslant C\sum_{k\neq 0}\int_0^t\int_\eta\int_\xi \langle \eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})+\frac{3}{2}}|\widehat{a^2-1}|(\eta-\xi)\\
&\quad\times\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}|\widehat{\tilde{K}\Lambda Z_{\neq}}|(k,\xi)\\
&\quad\times\left(\sqrt{\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}}+\sqrt{\frac{|k|}{k^2+(\eta-kt)^2}}\right)|\widehat{\tilde{K}Z^+_{\neq}}|(k,\eta)d\xi d\eta \mathrm{d}t\\
&\leqslant C\delta\left\|\sqrt{|S|}\tilde{K}\Lambda Z_{\neq}\right\|_{L^2L^2}\left(\left\|\sqrt{|S|}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\right)\\
&\leqslant \frac{C\delta}{1-C\delta}\left(\left\|\sqrt{|S|}\tilde{K}Z_{\neq}\right\|_{L^2L^2}+\frac{C\delta}{1-C\delta}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z_{\neq}\right\|_{L^2L^2}\right)\\
&\quad\times\left(\left\|\sqrt{|S|}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^+_{\neq}\right\|_{L^2L^2}\right)\\
&\leqslant \frac{C\delta}{1-C\delta}\left(\left\|\sqrt{|S|_d}\tilde{K}Z^{\pm}_{\neq}\right\|^2_{L^2L^2}+\frac{1}{1000^3}\nu\left\|\nabla_L\tilde{K}Z^{\pm}_{\neq}\right\|^2_{L^2L^2}\right.\\
&\quad\left.+\frac{1}{1+\frac{3}{2}\tilde{\delta}}\left\|\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}\tilde{K}Z^{\pm}_{\neq}\right\|^2_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^{\pm}_{\neq}\right\|^2_{L^2L^2}\right). \qquad (4.39)
\end{aligned}$$

Now we turn to estimate $\mathcal{LP}_2$. Compared with (4.31), we need an extra commutator estimate of $\sqrt{\left|\partial_X\Delta_L^{-1}\right|}$. Indeed, using (A.8), (A.9) and lemma C.1, one easily deduces that

$$\begin{aligned}
\mathcal{LP}_2 &\leqslant C\sum_{k\neq 0}\int_0^t\int_\eta\int_\xi \tilde{K}(k,\eta)\left(|\hat{b}|(\eta-\xi)\frac{k^2}{k^2+(\xi-kt)^2}|\widehat{\Lambda Z^-_{\neq}}|(k,\xi)\right)\\
&\quad\times|\widehat{\tilde{K}Z^+_{\neq}}|(k,\eta)d\xi d\eta \mathrm{d}t'\\
&\leqslant C\sum_{k\neq 0}\int_0^t\int_\eta\int_\xi\langle\eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})+1}|\hat{b}|(\eta-\xi)\frac{|k|}{\sqrt{k^2+(\xi-kt)^2}}|\widehat{\tilde{K}\Lambda Z^-_{\neq}}|(k,\xi)\\
&\quad\times\frac{|k|}{\sqrt{k^2+(\eta-kt)^2}}|\widehat{\tilde{K}Z^+_{\neq}}|(k,\eta)d\xi d\eta \mathrm{d}t'
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\delta \left\| \sqrt{-\frac{\dot{M}_1}{M_1}} \tilde{K} \Lambda Z^-_{\neq} \right\|_{L^2L^2} \left\| \sqrt{-\frac{\dot{M}_1}{M_1}} \tilde{K} Z^+_{\neq} \right\|_{L^2L^2} \\
&\leqslant \frac{C\delta}{1 - C\delta} \left\| \sqrt{-\frac{\dot{M}_1}{M_1}} \tilde{K} Z^-_{\neq} \right\|_{L^2L^2} \left\| \sqrt{-\frac{\dot{M}_1}{M_1}} \tilde{K} Z^+_{\neq} \right\|_{L^2L^2}.
\end{aligned} \tag{4.40}
$$

To bound $\mathcal{LP}_3$, it is natural to divide it into two parts as follows

$$
\begin{aligned}
\mathcal{LP}_3 &= \nu \int_0^t \left\langle \tilde{K} Z^+_{\neq}, \tilde{K}\left((a^2-1)\partial^L_{YY} Z^+_{\neq}\right)\right\rangle \mathrm{d}t' \\
&\quad + \nu \int_0^t \left\langle \tilde{K} Z^+_0, \tilde{K}\left((a^2-1)\partial_{YY} Z^+_0\right)\right\rangle \mathrm{d}t' \\
&=: \mathcal{LP}_{3,\neq} + \mathcal{LP}_{3,0}.
\end{aligned}
$$

Similar to (4.32), we obtain

$$
\mathcal{LP}_{3,\neq} \leqslant C\delta\nu \|\nabla_L \tilde{K} Z^+_{/} = \|^2_{L^2L^2}. \tag{4.41}
$$

For $\mathcal{LP}_{3,0}$, integrating by parts and using (2.4) and (A.3) yields

$$
\begin{aligned}
\mathcal{LP}_{3,0} &\leqslant C\nu \|a^2-1\|_{L^\infty H^N} \|\partial_Y Z^+_0\|^2_{L^2H^N} + C\nu \|b\|_{L^2H^N} \|Z^+_0\|_{L^\infty H^N} \|\partial_Y Z^+_0\|_{L^2H^N} \\
&\leqslant C\delta \left(\nu \left\| \nabla_L \tilde{K} Z^+_0 \right\|^2_{L^2L^2} + \left\| \tilde{K} Z^+_0 \right\|^2_{L^\infty L^2} \right).
\end{aligned} \tag{4.42}
$$

Step IV: nonlinear estimates. We first collect all the above nonlinear terms to be estimated:

$$
\begin{aligned}
\mathrm{OLS}^{(5)}_4 &= -\frac{1}{2\alpha} \int_0^t \left\langle S \partial_X^{-1} O^t_{2\alpha} \tilde{K} \Lambda Z^-_{\neq}, \tilde{K} \mathrm{NL}^+_{\neq} \right\rangle \mathrm{d}t', \\
\mathrm{OLS}^{(5)}_{5,1} &= -\frac{1}{2\alpha} \int_0^t \left\langle \tilde{K} \partial_X^{-1} O^t_{2\alpha} \Lambda \mathrm{NL}^-_{\neq}, \tilde{K} S Z^+_{\neq} \right\rangle \mathrm{d}t', \\
\mathcal{NL} &= \int_0^t \left\langle \tilde{K} Z^+, \tilde{K} \mathrm{NL}^+ \right\rangle \mathrm{d}t'.
\end{aligned} \tag{4.43}
$$

Owing to lemma C.1 and the fact $|S| \leqslant \frac{1}{2}$, we observe that the bounds of the three quantities are essentially the same. To avoid unnecessary repetition, we only sketch the treatment of $\mathcal{NL}$ by modifying the nonlinear estimates in the Couette case, see (3.35)–(3.40). In fact, recalling the definition of $\mathrm{NL}^\pm$ (2.17), we write

$$
\mathcal{NL} = \int_0^t \left\langle \tilde{K} Z^+, \tilde{K} \mathrm{NLT}^+ \right\rangle \mathrm{d}t' + \int_0^t \left\langle \tilde{K} Z^+, \tilde{K} \mathrm{NLS}^+ \right\rangle \mathrm{d}t' = \mathcal{NLT} + \mathcal{NLS},
$$

here we use the shorthand notation $\mathrm{NLS}^\pm := \mathrm{NLS1}^\pm + \mathrm{NLS2}^\pm$. Noting that

$$
\left\langle \tilde{K} Z^+_0, \tilde{K}\left((U^1_0 + B^1_0)\partial_X Z^+\right)\right\rangle = 0,
$$

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by virtue of (4.6), (4.8), lemmas 4.2 and C.1, and the hypotheses (4.10) and (4.11), one deduces that

$$\begin{aligned}
\mathcal{NLT} &\leqslant \|\tilde{K}Z^+_{\neq}\|_{L^2L^2}\|U^1_0+B^1_0\|_{L^\infty H^N}\|\partial_X Z^+_{\neq}\|_{L^2H^N} \\
&\quad + \|\tilde{K}Z^+\|_{L^\infty L^2}\|\nabla^\perp_t\Delta^{-1}_t O^t_{2\alpha}Z^-_{\neq}\cdot\nabla_t Z^+_{\neq}\|_{L^1H^N} \\
&\quad + \|\tilde{K}Z^+\|_{L^\infty L^2}\left\|a\partial_X\Delta^{-1}_L\Lambda O^t_{2\alpha}Z^-_{\neq}\partial_Y Z^+_0\right\|_{L^1H^N} \\
&\leqslant C\nu^{-\frac{\tilde{\delta}}{2}}\|\tilde{K}Z^+_{\neq}\|_{L^2L^2}\|U^1_0+B^1_0\|_{L^\infty H^N}\|\nabla_L\tilde{K}Z^+_{\neq}\|_{L^2L^2} \\
&\quad + C\nu^{-\tilde{\delta}}\|\tilde{K}Z^+\|_{L^\infty L^2}\left(\|\tilde{K}Z^-_{\neq}\|_{L^2L^2}+\nu^{-\frac{1}{3}}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\right) \\
&\quad \times\|\nabla_L\tilde{K}Z^+_{\neq}\|_{L^2L^2} \\
&\quad + C\nu^{-\frac{\tilde{\delta}}{2}}\|\tilde{K}Z^+\|_{L^\infty L^2}\left\|\sqrt{-\frac{\dot{M}_1}{M_1}}\tilde{K}Z^-_{\neq}\right\|_{L^2L^2}\|\partial_Y Z^+_0\|_{L^2H^N} \\
&\leqslant C\nu^{-\frac{5}{6}-\tilde{\delta}}\epsilon^3.
\end{aligned} \tag{4.44}$$

Similar to (3.40), using (2.21), (2.23), (3.39), (A.3), (A.1), and lemma C.1, we arrive at

$$\begin{aligned}
\mathcal{NLS} &\leqslant C\|\tilde{K}Z^+\|_{L^\infty L^2}\left(\left\|aS\Lambda O^t_{2\alpha}Z^-_{\neq}(2\partial_{XX}\Delta^{-1}_L\Lambda Z^+_{\neq}-Z^+)\right\|_{L^1H^N}\right. \\
&\quad \left.+\left\|aS\Lambda Z^+_{\neq}(2\partial_{XX}\Delta^{-1}_L\Lambda O^t_{2\alpha}Z^-_{\neq}-O^t_{2\alpha}Z^-)\right\|_{L^1H^N}\right) \\
&\leqslant C\|\tilde{K}Z^+\|_{L^\infty L^2}\left(\nu^{-\frac{\tilde{\delta}}{2}}\|\tilde{m}^{1/2}Z^{\mp}_{\neq}\|_{L^2H^N}\right)\left(\nu^{-(\frac{1}{3}+\frac{\tilde{\delta}}{2})}\|\tilde{m}^{1/2}Z^{\pm}_{/}=\|_{L^2H^N}\right) \\
&\quad + C\|\tilde{K}Z^+_{\neq}\|_{L^2L^2}\left(\nu^{-\frac{\tilde{\delta}}{2}}\|\tilde{m}^{1/2}Z^{\mp}_{\neq}\|_{L^2H^N}\right)\left(\nu^{-(\frac{1}{3}+\frac{\tilde{\delta}}{2})}\|\tilde{m}^{1/2}Z^{\pm}_0\|_{L^\infty H^N}\right) \\
&\leqslant C\nu^{-\frac{2}{3}-\tilde{\delta}}\epsilon^3.
\end{aligned} \tag{4.45}$$

Now collecting the above estimates in **Step I–IV**, we conclude that there exist positive constant α_0 (sufficiently large) and δ_0 (sufficiently small) independent of ν, such that if $|\alpha|\geqslant\alpha_0$ and $\delta\leqslant\delta_0$, there holds

$$\begin{aligned}
&\|\tilde{K}Z^+(t)\|^2_{L^2}+\nu\|\nabla_L\tilde{K}Z^+\|^2_{L^2L^2}+\left\|\sqrt{-\frac{\dot{M}}{M}}\tilde{K}Z^+\right\|^2_{L^2L^2} \\
&\quad \leqslant 3\|\tilde{K}Z^+(0)\|^2_{L^2}+C\nu^{-\frac{5}{6}-\tilde{\delta}}\epsilon^3,
\end{aligned} \tag{4.46}$$

where the constant C depends only on N and α. It suffices to improve (4.10) as long as $\epsilon\ll\nu^{\frac{5}{6}+\tilde{\delta}}$.

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The improvement of (4.11) is similar to that of (3.20). Firstly, we write the equations of $U_0^1 \mp B_0^1$:

$$\partial_t \left(U_0^1 \mp B_0^1\right) - \nu\partial_{YY} \left(U_0^1 \mp B_0^1\right) = \nu(a^2-1)\partial_{YY} \left(U_0^1 \mp B_0^1\right) + \nu b\partial_Y \left(U_0^1 \mp B_0^1\right) \\ - \left((U_{\neq} \pm B_{\neq}) \cdot \nabla_t (U_{\neq}^1 \mp B_{\neq}^1)\right)_0 .$$

Then we have the energy identity

$$\begin{aligned} &\frac{1}{2}\left\|(U_0^1 \mp B_0^1)(t)\right\|_{H^N}^2 + \nu\|\partial_Y (U_0^1 \mp B_0^1)\|_{L^2H^N}^2 \\ &\quad= \frac{1}{2}\left\|(U_0^1 \mp B_0^1)(0)\right\|_{H^N}^2 \\ &\qquad - \int_0^t \left\langle U_0^1 \mp B_0^1, \left((U_{\neq} \pm B_{\neq}) \cdot \nabla_L (U_{\neq}^1 \mp B_{\neq}^1)\right)_0 \right\rangle_{H^N} \mathrm{d}t' \\ &\qquad + \nu \int_0^t \left\langle U_0^1 \mp B_0^1, (a^2-1)\partial_{YY} \left(U_0^1 \mp B_0^1\right)\right\rangle_{H^N} \mathrm{d}t' \\ &\qquad + \nu \int_0^t \left\langle U_0^1 \mp B_0^1, b\partial_Y \left(U_0^1 \mp B_0^1\right)\right\rangle_{H^N} \mathrm{d}t' \\ &\quad= \frac{1}{2}\left\|(U_0^1 \mp B_0^1)(0)\right\|_{H^N}^2 + \sum_{q=1}^{3} I_q. \end{aligned} \tag{4.47}$$

Using the divergence free condition $\nabla_t \cdot (U \pm B) = 0$, we find that

$$I_1 = \int_0^t \left\langle \partial_Y \left(U_0^1 \mp B_0^1\right), a \left((U_{\neq}^2 \pm B_{\neq}^2)(U_{\neq}^1 \mp B_{\neq}^1)\right)_0 \right\rangle_{H^N} \mathrm{d}t' \\ + \int_0^t \left\langle U_0^1 \mp B_0^1, \partial_Y a \left((U_{\neq}^2 \pm B_{\neq}^2)(U_{\neq}^1 \mp B_{\neq}^1)\right)_0 \right\rangle_{H^N} \mathrm{d}t'.$$

Recalling (2.4), it is easy to rewrite $\partial_Y a$ as follows

$$\partial_Y a = b + \left(\frac{1}{1-(1-a)} - 1\right) b = b + b\sum_{n=1}^{+\infty} a^n.$$

Combining this with (A.3) yields $\nu^{\frac{1}{2}}\|\partial_Y a\|_{L^2H^N} \lesssim \delta$. On the other hand, similar to (3.45) and (3.46), we have

$$\left\|U_{\neq}^2 \pm B_{\neq}^2\right\|_{L^2H^N} = \left\|\partial_X \Delta_L^{-1} \Lambda O_\alpha^t Z^{\mp}\right\|_{L^2H^N} \lesssim \nu^{-\frac{\tilde{\delta}}{2}} \left\|\sqrt{-\frac{\dot{M}}{M}} \tilde{K} Z^{\mp}\right\|_{L^2L^2},$$

and

$$\|U_{\neq}^1 \mp B_{\neq}^1\|_{L^\infty H^N} = \left\|a\partial_Y^L \Delta_L^{-1} \Lambda O_{-\alpha}^t Z_{\neq}^{\pm}\right\|_{L^\infty H^N} \lesssim \nu^{-\frac{\tilde{\delta}}{2}} \left\|\tilde{K} Z_{\neq}^{\pm}\right\|_{L^\infty L^2}.$$

It follows that

$$I_1 \lesssim \nu^{-\tilde{\delta}} \left(1 + \|a-1\|_{L^\infty H^N}\right) \left\|\partial_Y (U_0^1 \mp B_0^1)\right\|_{L^2 H^N} \times \left\| \sqrt{-\frac{\dot{M}}{M}} \tilde{K} Z^{\mp} \right\|_{L^2 L^2} \left\| \tilde{K} Z^{\pm}_{/} = \right\|_{L^\infty L^2} + \nu^{-\tilde{\delta}} \|\partial_t a\|_{L^2 H^N} \left\|U_0^1 \mp B_0^1\right\|_{L^\infty H^N} \left\| \sqrt{-\frac{\dot{M}}{M}} \tilde{K} Z^{\mp} \right\|_{L^2 L^2} \left\| \tilde{K} Z^{\pm}_{/} = \right\|_{L^\infty L^2}. \tag{4.48}$$

Integrating by parts, and using (A.3), it is easy to see that

$$I_2 + I_3 \lesssim \delta\nu \|\partial_Y (U_0^1 \mp B_0^1)\|^2_{L^2 H^N} + \delta\nu^{\frac{1}{2}} \|U_0^1 \mp B_0^1\|_{L^\infty H^N} \|\partial_Y (U_0^1 \mp B_0^1)\|_{L^2 H^N}. \tag{4.49}$$

Substituting (4.48) and (4.49) into (4.47), and using the hypotheses (4.10) and (4.11), we have

$$\|(U_0^1 \mp B_0^1)(t)\|^2_{H^N} + \nu \|\partial_Y (U_0^1 \mp B_0^1)\|^2_{L^2 H^N} \leqslant 2\|(U_0^1 \mp B_0^1)(0)\|^2_{H^N} + C\nu^{-\frac{1}{2}-\tilde{\delta}} \epsilon^3, \tag{4.50}$$

for some constant C independent of ν. Combining (4.46) and (4.50), one deduces proposition 4.3 under the hypotheses $\epsilon \ll \nu^{\frac{5}{6}+\tilde{\delta}}$ and hence of theorem 4.1. □

Appendix A. Estimates for the coefficients *a* and *b*

To begin with, we give a lemma to discuss the relation between the new coordinate system (2.2) and the original (x, y). Please refer to [**21**] and [**8**] for the proof.

Lemma A.1. *Let* $s' \geqslant 2, s' \geqslant s \geqslant 0, f \in H^s(\mathbb{R})$, *and* $g \in H^{s'}(\mathbb{R})$ *be such that* $\|g\|_{H^{s'}} \leqslant \delta$. *Then, there holds*

$$C_{s,s'}(\delta)^{-1} \|f \circ (I+g)\|_{H^s} \leqslant \|f\|_{H^s} \leqslant C_{s,s'}(\delta) \|f \circ (I+g)\|_{H^s}, \tag{A.1}$$

where the implicit constant obey $C_{s,s'}(\delta) \to 1$ *as* $\delta \to 0$.

From the properties of the heat equation and lemma A.1, we can deduce the energy estimates of the coefficients a and b.

Lemma A.2. *Let* $s \geqslant 0$. *Assume that* $U(y)$ *satisfies*

$$\|U(y) - y\|_{H^{s+2}} \leqslant \delta \leqslant 1, \tag{A.2}$$

then there holds

$$\|a-1\|_{L^\infty H^s} + \|b\|_{L^\infty H^s} + \nu^{\frac{1}{2}} \|b\|_{L^2 H^s} \lesssim \delta, \tag{A.3}$$

and

$$\|\partial_t a\|_{L^2 H^{s-1}} + \|\partial_t b\|_{L^2 H^{s-1}} \lesssim \delta \nu^{1/2}. \tag{A.4}$$

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Proof. Note that $\partial_y^l \bar{U}, l \geqslant 0$ solves

$$\partial_t \partial_y^l \bar{U} - \nu \partial_y^{l+2} \bar{U} = 0, \quad \partial_y^l \bar{U}|_{t=0} = \partial_y^l U.$$

Therefore, integrating by parts, we have

$$\nu \|\partial_y^2 \bar{U}\|_{L^2 H^s}^2 = -\int_{\mathbb{R}} \langle \partial_y \rangle^s \partial_t \left(\partial_y \bar{U} - 1\right) \langle \partial_y \rangle^s \left(\partial_y \bar{U} - 1\right) \mathrm{d}y = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_y \bar{U} - 1\|_{H^s}^2,$$

and

$$\nu \|\partial_y^{l+2} \bar{U}\|_{L^2 H^s}^2 = -\int_{\mathbb{R}} \langle \partial_y \rangle^s \partial_t \partial_y^{l+1} \bar{U} \langle \partial_y \rangle^s \partial_y^{l+1} \bar{U} \mathrm{d}y = -\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_y^{l+1} \bar{U}\|_{H^s}^2, \quad l \geqslant 1.$$

Consequently,

$$\frac{1}{2} \|\partial_y \bar{U} - 1\|_{L^\infty H^s}^2 + \nu \|\partial_y^2 \bar{U}\|_{L^2 H^s}^2 \leqslant \frac{1}{2} \|U' - 1\|_{H^s}^2,$$

and

$$\frac{1}{2} \|\partial_y^{l+1} \bar{U}\|_{L^\infty H^s}^2 + \nu \|\partial_y^{l+2} \bar{U}\|_{L^2 H^s}^2 \leqslant \frac{1}{2} \|\partial_y^{l+1} U\|_{H^s}^2, \quad l \geqslant 1.$$

Combining these two estimates with (2.3), (2.9) and lemma A.1, we find that

$$\begin{aligned}
\|a - 1\|_{L^\infty H^s} + \nu^{\frac{1}{2}} \|b\|_{L^2 H^s} &\lesssim \|\partial_y \bar{U} - 1\|_{L^\infty H^s}^2 + \nu^{\frac{1}{2}} \|\partial_y^2 \bar{U}\|_{L^2 H^s} \lesssim \|U' - 1\|_{H^s} \lesssim \delta, \\
\|b\|_{L^\infty H^s} &\lesssim \|\partial_y^2 \bar{U}\|_{L^\infty H^s} \lesssim \|U''\|_{H^s} \lesssim \delta, \|\partial_t a\|_{L^2 H^{s-1}} \\
&\leqslant C\nu \left(\|c\|_{L^2 H^{s-1}} + \|b\|_{L^2 H^{s-1}} \|\partial_Y a\|_{L^\infty H^{s-1}}\right) \\
&\leqslant C\nu \left(\|c\|_{L^2 H^{s-1}} + \|b\|_{L^2 H^{s-1}} \|a - 1\|_{L^\infty H^s}\right) \\
&\leqslant C\nu \left(\|\partial_y^3 \bar{U}\|_{L^2 H^{s-1}} + \|\partial_y^2 \bar{U}\|_{L^2 H^{s-1}} \|\partial_y \bar{U} - 1\|_{L^\infty H^s}\right) \\
&\leqslant C\nu^{1/2} \left(\|U''\|_{H^{s-1}} + \|U' - 1\|_{H^{s-1}} \|U' - 1\|_{H^s}\right) \\
&\lesssim \delta \nu^{\frac{1}{2}},
\end{aligned}$$

and

$$\begin{aligned}
\|\partial_t b\|_{L^2 H^{s-1}} &\leqslant C\nu \left(\|d\|_{L^2 H^{s-1}} + \|b\|_{L^\infty H^{s-1}} \|\partial_Y b\|_{L^2 H^{s-1}}\right) \\
&\leqslant C\nu \left(\|d\|_{L^2 H^{s-1}} + \|b\|_{L^\infty H^{s-1}} \|b\|_{L^2 H^s}\right) \\
&\leqslant C\nu \left(\|\partial_y^4 \bar{U}\|_{L^2 H^{s-1}} + \|\partial_y^2 \bar{U}\|_{L^\infty H^{s-1}} \|\partial_y^2 \bar{U}\|_{L^2 H^s}\right) \\
&\leqslant C\nu^{1/2} \left(\|U'''\|_{H^{s-1}} + \|U''\|_{H^{s-1}} \|U' - 1\|_{H^s}\right) \\
&\lesssim \delta \nu^{\frac{1}{2}}.
\end{aligned}$$

This completes the proof of lemma A.2. □

REMARK A.3. In this paper, we choose $s = N + 4$. Then (A.2) reduces to (4.1).

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Appendix B. Commutator estimates

To deal with the nonlinear terms, we need the following lemmas to exchange frequencies. The first one can be regarded as an analogue of Lemma A.1 in [3]. We give a proof below for the sake of completeness.

LEMMA B.1. *The multiplier $\tilde{m}$ satisfies*

$$\tilde{m}(t,k,\eta) \lesssim \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}\tilde{m}(t,k,\xi). \tag{B.1}$$

Proof. We only consider the case $k \neq 0$.

Case 1: $\frac{\eta}{k} < -1000\nu^{-\frac{1}{3}}, \tilde{m}(t,k,\eta) = 1$. It suffices to estimate $\tilde{m}^{-1}(t,k,\xi)$.

Case 1.1: $-1000\nu^{-\frac{1}{3}} < \frac{\xi}{k} < 0$. Now we have

$$\tilde{m}^{-1}(t,k,\xi) \leqslant \left(\frac{1+\left(1000\nu^{-\frac{1}{3}}\right)^2}{1+\left(\frac{\xi}{k}\right)^2}\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \left(\frac{1+\left(\frac{\eta}{k}\right)^2}{1+\left(\frac{\xi}{k}\right)^2}\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

Case 1.2: if $\frac{\xi}{k} > 0, \frac{\xi}{k} < t < \frac{\xi}{k} + 1000\nu^{-\frac{1}{3}}$.

$$\begin{aligned}\tilde{m}^{-1}(t,k,\xi) &= \left(\frac{k^2+(\xi-kt)^2}{k^2}\right)^{1+\frac{3}{2}\tilde{\delta}} = \left(1+\left(\frac{\xi}{k}-t\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}}\\ &\leqslant \left(1+\left(\frac{\eta}{k}-\frac{\xi}{k}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.\end{aligned}$$

Case 1.3: if $\frac{\xi}{k} > 0, t > \frac{\xi}{k} + 1000\nu^{-\frac{1}{3}}$.

$$\tilde{m}^{-1}(t,k,\xi) = \left(1+\left(1000\nu^{-\frac{1}{3}}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \left(1+\left(\frac{\eta}{k}-\frac{\xi}{k}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

Case 2: $-1000\nu^{-\frac{1}{3}} < \frac{\eta}{k} < 0$.

Case 2.1: $0 < t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$, and $\tilde{m}(t,k,\eta) = (\frac{k^2+\eta^2}{k^2+(\eta-kt)^2})^{1+\frac{3}{2}\tilde{\delta}}$.

- Case 2.1.1: $-1000\nu^{-\frac{1}{3}} < \frac{\xi}{k} < 0$.

$$\tilde{m}(t,k,\xi) \geqslant \left(\frac{k^2+\xi^2}{k^2+(\xi-kt)^2}\right)^{1+\frac{3}{2}\tilde{\delta}}, \quad \forall\, t \geqslant 0. \tag{B.2}$$

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Therefore,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(\frac{1+\left(\frac{\eta}{k}\right)^2}{1+\left(\frac{\xi}{k}\right)^2} \cdot \frac{1+\left(\frac{\xi}{k}-t\right)^2}{1+\left(\frac{\eta}{k}-t\right)^2} \right)^{1+\frac{3}{2}\tilde{\delta}}$$

$$\leqslant \begin{cases} \left(\frac{1+\left(\frac{\xi}{k}-t\right)^2}{1+\left(\frac{\eta}{k}-t\right)^2} \right)^{1+\frac{3}{2}\tilde{\delta}}, & \text{if } \left|\frac{\eta}{k}\right| \leqslant \left|\frac{\xi}{k}\right| \\ \left(\frac{1+\left(\frac{\eta}{k}\right)^2}{1+\left(\frac{\xi}{k}\right)^2} \right)^{1+\frac{3}{2}\tilde{\delta}}, & \text{if } \left|\frac{\eta}{k}\right| > \left|\frac{\xi}{k}\right| \end{cases}$$

$$\lesssim \langle \eta - \xi \rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

- Case 2.1.2: $\frac{\xi}{k} > 0, t > \frac{\xi}{k}$. In this case,

$$\tilde{m}(t,k,\xi) \geqslant \left(\frac{k^2}{k^2+(\xi-kt)^2} \right)^{1+\frac{3}{2}\tilde{\delta}}. \tag{B.3}$$

Therefore,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(\left(1+\left(\frac{\eta}{k}\right)^2\right) \cdot \frac{1+\left(\frac{\xi}{k}-t\right)^2}{1+\left(\frac{\eta}{k}-t\right)^2} \right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \left(1+\left(\frac{\eta}{k}-\frac{\xi}{k}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}}$$

$$\lesssim \langle \eta - \xi \rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

where we have used $\frac{\eta}{k} < 0 < \frac{\xi}{k} < t$.

Case 2.2: $t > \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$, and $\tilde{m}(t,k,\eta) = (\frac{k^2+\eta^2}{k^2+(1000k\nu^{-\frac{1}{3}})^2})^{1+\frac{3}{2}\tilde{\delta}}$.

- Case 2.2.1: $-1000\nu^{-\frac{1}{3}} < \frac{\xi}{k} < 0$. In this case,

$$\tilde{m}(t,k,\xi) \geqslant \left(\frac{k^2+\xi^2}{k^2+(1000k\nu^{-\frac{1}{3}})^2} \right)^{1+\frac{3}{2}\tilde{\delta}}, \tag{B.4}$$

Thus,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(\frac{k^2+\eta^2}{k^2+\xi^2} \right)^{1+\frac{3}{2}\tilde{\delta}} \lesssim \langle \eta - \xi \rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

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- Case 2.2.2: $\frac{\xi}{k} > 0, t > \frac{\xi}{k}$. In this case,

$$\tilde{m}(t,k,\xi) \geqslant \left(\frac{1}{1+(1000\nu^{-\frac{1}{3}})^2}\right)^{1+\frac{3}{2}\tilde{\delta}}, \tag{B.5}$$

Thus,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(1+\left(\frac{\eta}{k}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}} \leqslant \left(1+\left(\frac{\eta}{k}-\frac{\xi}{k}\right)^2\right)^{1+\frac{3}{2}\tilde{\delta}} \lesssim \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

Case 3: $\frac{\eta}{k} > 0$.

Case 3.1: $\frac{\eta}{k} < t < \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$, and $\tilde{m}(t,k,\eta) = (\frac{k^2}{k^2+(\eta-kt)^2})^{1+\frac{3}{2}\tilde{\delta}}$.

- Case 3.1.1: $-1000\nu^{-\frac{1}{3}} < \frac{\xi}{k} < 0$. In this case, (B.2) holds. Therefore,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(\frac{k^2}{k^2+\xi^2}\cdot\frac{k^2+(\xi-kt)^2}{k^2+(\eta-kt)^2}\right)^{1+\frac{3}{2}\tilde{\delta}} \lesssim \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

- Case 3.1.2: $\frac{\xi}{k} > 0$. Now (B.3) still holds. Thus,

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant \left(\frac{k^2+(\xi-kt)^2}{k^2+(\eta-kt)^2}\right)^{1+\frac{3}{2}\tilde{\delta}} \lesssim \langle \eta-\xi\rangle^{2(1+\frac{3}{2}\tilde{\delta})}.$$

Case 3.2: $t > \frac{\eta}{k} + 1000\nu^{-\frac{1}{3}}$, and $\tilde{m}(t,k,\eta) = (\frac{1}{1+(1000\nu^{-\frac{1}{3}})^2})^{1+\frac{3}{2}\tilde{\delta}}$.

- Case 3.2.1:$-1000\nu^{-\frac{1}{3}} < \frac{\xi}{k} < 0$. Using (B.4), we have

$$\frac{\tilde{m}(t,k,\eta)}{\tilde{m}(t,k,\xi)} \leqslant 1. \tag{B.6}$$

- Case 3.2.2: $\frac{\xi}{k} > 0$, In this case, In this case, (B.5) holds. Consequently, (B.6) holds as well.

The proof of lemma B.1 is completed. □

Recalling the definition of $\tilde{K}$ in (4.2), using (B.1) and the following fact

$$|k,\eta-kt| \lesssim \langle \eta-\xi\rangle |k,\xi-kt|, \quad \text{for} \quad k\neq 0. \tag{B.7}$$

we get the following corollary immediately.

Corollary B.2. *The multiplier $\tilde{K}$ satisfies*

$$\tilde{K}(t,k,\eta) \lesssim \langle \eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})}\tilde{K}(t,k,\xi). \tag{B.8}$$

In view of the definitions of M_1 and M_2 in (3.7) and (3.8), respectively, using (B.7) again, one easily derives the commutator estimates for $\sqrt{-\frac{\dot{M}_i}{M_i}}, i=1,2$.

LEMMA B.3. *For $k \neq 0$ there hold*

$$\sqrt{-\frac{\dot{M}_i(t,k,\eta)}{M_i(t,k,\eta)}} \lesssim \langle \eta-\xi\rangle \sqrt{-\frac{\dot{M}_i(t,k,\xi)}{M_i(t,k,\xi)}}, \quad \text{for} \quad i=1,2, \tag{B.9}$$

Finally, the estimates of $\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})}{\tilde{m}^{1/2}}}$ is given below.

LEMMA B.4. *For $k \neq 0$ there holds*

$$\begin{aligned}
&\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})(t,k,\eta)}{\tilde{m}^{1/2}(t,k,\eta)}}\tilde{K}(t,k,\eta) \\
&\lesssim \langle \eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})+1}\left(\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})(t,k,\xi)}{\tilde{m}^{1/2}(t,k,\xi)}}+\sqrt{\frac{1+\frac{3}{2}\tilde{\delta}}{1000^3}}\nu^{\frac{1}{2}}\left(k^2+(\xi-kt)^2\right)^{\frac{1}{2}}\right. \\
&\left.+\sqrt{1+\frac{3}{2}\tilde{\delta}}\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}\mathbf{1}_{D_{dam}}\right)\tilde{K}(t,k,\xi) \\
&+\sqrt{1+\frac{3}{2}\tilde{\delta}}\langle \eta-\xi\rangle^{N+(1+\frac{3}{2}\tilde{\delta})+\frac{3}{2}}\sqrt{-\frac{\dot{M}_1(t,k,\xi)}{M_1(t,k,\xi)}}\tilde{K}(t,k,\xi).
\end{aligned} \tag{B.10}$$

Proof. Using (B.7), it is not difficult to verify that

$$\sqrt{\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}} \lesssim \langle \eta-\xi\rangle^{\frac{3}{2}}\sqrt{\frac{|k|}{k^2+(\xi-kt)^2}}+\langle \eta-\xi\rangle\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}. \tag{B.11}$$

From the above inequality and (2.29) with η replaced by ξ, we are let to

$$\begin{aligned}
&\sqrt{\frac{|k(\eta-kt)|}{k^2+(\eta-kt)^2}} \\
&\lesssim \langle \eta-\xi\rangle\left(\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}\mathbf{1}_{D_{dam}}+\sqrt{\frac{1}{1000^3}}\nu^{\frac{1}{2}}\left(k^2+(\xi-kt)^2\right)^{\frac{1}{2}}\right. \\
&\left.+\frac{1}{\sqrt{1+\frac{3}{2}\tilde{\delta}}}\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})(t,k,\xi)}{\tilde{m}^{1/2}(t,k,\xi)}}\right)+\langle \eta-\xi\rangle^{\frac{3}{2}}\sqrt{-\frac{\dot{M}_1(t,k,\xi)}{M_1(t,k,\xi)}}.
\end{aligned}$$

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Combining this with (2.28) yields

$$\begin{aligned}
&\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})(t,k,\eta)}{\tilde{m}^{1/2}(t,k,\eta)}}\\
&\lesssim \langle \eta-\xi\rangle \left(\sqrt{-\frac{\partial_t(\tilde{m}^{1/2})(t,k,\xi)}{\tilde{m}^{1/2}(t,k,\xi)}}+\sqrt{\frac{1+\frac{3}{2}\tilde{\delta}}{1000^3}}\nu^{\frac{1}{2}}\left(k^2+(\xi-kt)^2\right)^{\frac{1}{2}}\right.\\
&\left.+\sqrt{1+\frac{3}{2}\tilde{\delta}}\sqrt{\frac{|k(\xi-kt)|}{k^2+(\xi-kt)^2}}\mathbf{1}_{D_{dam}}\right)+\sqrt{1+\frac{3}{2}\tilde{\delta}}\langle \eta-\xi\rangle^{\frac{3}{2}}\sqrt{-\frac{\dot{M}_1(t,k,\xi)}{M_1(t,k,\xi)}}.
\end{aligned} \tag{B.12}$$

Then (B.10) is a consequence of (B.8) and (B.12). We complete the proof of lemma B.4. □

Appendix C. Composition inequalities for the operator Λ in H^N

Since Λ is involved in the definition of Δ_t^{-1}, recalling the relation (2.14), it is inevitable to deal with the composition $B_1 \circ \Lambda$ for some multiplier B_1. Based on the commutator estimates of B, we establish the following composition inequality.

LEMMA C.1. *Let $l \in \mathbb{N}^+$, and $B_i, 1 \leqslant i \leqslant l$ be Fourier multipliers, satisfying*

$$|B_1(k,\eta)| \leqslant \sum_{i=1}^{l} C_i \langle \eta-\xi\rangle^{\beta_i}|B_i(k,\xi)|, \tag{C.1}$$

for some constants $C_i > 0$ and $\beta_i \geqslant 0, 1 \leqslant i \leqslant l$. If $\|a^2-1\|_{L^\infty H^{N+\beta+1}} + \|b\|_{L^\infty H^{N+\beta+1}} \leqslant \frac{\delta}{2}$ with $\beta = \max\{\beta_1,\cdots,\beta_l\}$, then we have

$$\|B_1\Lambda f_{\neq}\|_{H^N} \leqslant \frac{1}{1-C_1\delta}\|B_1 f_{\neq}\|_{H^N} + \frac{\delta}{1-C_1\delta}\sum_{i=2}^{l} C_i\|B_i\Lambda f_{\neq}\|_{H^N}. \tag{C.2}$$

Proof. We first write Δ_L in terms of Δ_t

$$\Delta_L = \Delta_t - (a^2-1)\partial_{YY}^L - b\partial_Y^L.$$

Then

$$\Lambda f_{\neq} = \Delta_L\Delta_t^{-1} f_{\neq} = f_{\neq} - (a^2-1)\partial_{YY}^L\Delta_t^{-1}f_{\neq} - b\partial_Y^L\Delta_t^{-1}f_{\neq}, \tag{C.3}$$

and thus

$$\begin{aligned}
&\|B_1\Lambda f_{\neq}\|_{H^N}\\
&\leqslant \|B_1 f_{\neq}\|_{H^N} + \left\|B_1\left((a^2-1)\partial_{YY}^L\Delta_t^{-1}f_{\neq}\right)\right\|_{H^N} + \left\|B_1\left(b\partial_Y^L\Delta_t^{-1}f_{\neq}\right)\right\|_{H^N}.
\end{aligned} \tag{C.4}$$

Thanks to (C.1), using Young's inequality for convolution, we find that

$$\begin{aligned}
&\left\|B_1\left((a^2-1)\partial_{YY}^L\Delta_t^{-1}f_{\neq}\right)\right\|_{H^N}\\
&=\left(\sum_{k\neq 0}\int_\eta\left|\langle k,\eta\rangle^N B_1(k,\eta)\int_\xi\widehat{(a^2-1)}(\eta-\xi)(\xi-kt)^2\widehat{\Delta_t^{-1}f_{\neq}}(k,\xi)d\xi\right|^2 d\eta\right)^{\frac{1}{2}}\\
&\leqslant\sum_{i=1}^{l}C_i\left(\sum_{k\neq 0}\int_\eta\Big(\int_\xi\langle\eta-\xi\rangle^{N+\beta}|\widehat{a^2-1}|(\eta-\xi)\right.\\
&\qquad\left.\times\langle k,\xi\rangle^N|B_i(k,\xi)|\left(k^2+(\xi-kt)^2\right)|\widehat{\Delta_t^{-1}f_{\neq}}|(k,\xi)d\xi\Big)^2 d\eta\right)^{\frac{1}{2}}\\
&\leqslant\sum_{i=1}^{l}C_i\|\langle\cdot\rangle^{N+\beta}\widehat{a^2-1}\|_{L^1_\eta}\|B_i\Lambda f_{\neq}\|_{H^N},
\end{aligned}\tag{C.5}$$

where we have used the fact $\Lambda=\Delta_L\Delta_t^{-1}$ in the last line above. Similarly, for $k\neq 0$, there holds

$$|\xi-kt|\leqslant|k(\xi-kt)|\leqslant\frac{1}{2}\left(k^2+(\xi-kt)^2\right).$$

Thus, we have

$$\begin{aligned}
&\left\|B_1\left(b\partial_Y^L\Delta_t^{-1}f_{\neq}\right)\right\|_{H^N}\\
&=\left(\sum_{k\neq 0}\int_\eta\left|\langle k,\eta\rangle^N B_1(k,\eta)\int_\xi\hat{b}(\eta-\xi)(\xi-kt)\widehat{\Delta_t^{-1}f_{\neq}}(k,\xi)d\xi\right|^2 d\eta\right)^{\frac{1}{2}}\\
&\leqslant\sum_{i=1}^{l}C_i\left(\sum_{k\neq 0}\int_\eta\Big(\int_\xi\langle\eta-\xi\rangle^{N+\beta}|\hat{b}|(\eta-\xi)\right.\\
&\qquad\left.\times\langle k,\xi\rangle^N|B_i(k,\xi)|\left(k^2+(\xi-kt)^2\right)|\widehat{\Delta_t^{-1}f_{\neq}}|(k,\xi)d\xi\Big)^2 d\eta\right)^{\frac{1}{2}}\\
&\leqslant\sum_{i=1}^{l}C_i\|\langle\cdot\rangle^{N+\beta}\hat{b}\|_{L^1_\eta}\|B_i\Lambda f_{\neq}\|_{H^N}.
\end{aligned}\tag{C.6}$$

Substituting (C.5) and (C.6) into (C.4), and using the restriction on a and b, we arrive at

$$\begin{aligned}
&\|B_1\Lambda f_{\neq}\|_{H^N}\\
&\leqslant\|B_1 f_{\neq}\|_{H^N}+\sum_{i=1}^{l}C_i\left(\|\langle\cdot\rangle^{N+\beta}\widehat{a^2-1}\|_{L^1_\eta}+\|\langle\cdot\rangle^{N+\beta}\hat{b}\|_{L^1_\eta}\right)\|B_i\Lambda f_{\neq}\|_{H^N}
\end{aligned}$$

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$$\leqslant \|B_1 f_{\neq}\|_{H^N} + \sqrt{\pi}\sum_{i=1}^{l} C_i \left(\|a^2-1\|_{L^\infty H^{N+\beta+1}} + \|b\|_{L^\infty H^{N+\beta+1}}\right)\|B_i\Lambda f_{\neq}\|_{H^N}$$

$$\leqslant \|B_1 f_{\neq}\|_{H^N} + \sum_{i=1}^{l} C_i\delta\|B_i\Lambda f_{\neq}\|_{H^N},$$

and hence (C.2) follows. This completes the proof of lemma C.1. □

COROLLARY C.2. *Let B be Fourier multiplier satisfying*

$$|B(k,\eta)| \leqslant C\langle \eta-\xi\rangle^{\beta}|B(k,\xi)|,$$

where C and β are two positive constants. If $\|a^2-1\|_{H^{N+\beta+2}} + \|b\|_{H^{N+\beta+2}} \leqslant \frac{\delta}{2}$, then we have

$$\|B\tilde{\Lambda} f_{\neq}\|_{H^N} \leqslant C\delta\|Bf_{\neq}\|_{H^N}, \tag{C.7}$$

and

$$\|B\Lambda\tilde{\Lambda}_t^1\Lambda f_{\neq}\|_{H^N} \leqslant \frac{C\delta}{(1-C\delta)^2}\left\|B\sqrt{-\frac{\dot{M}_1}{M_1}}f_{\neq}\right\|_{H^N}, \tag{C.8}$$

where $\tilde{\Lambda}$ and $\tilde{\Lambda}_t^1$ are given in (2.10) and (2.33), respectively.

Proof. Recalling the definition of $\tilde{\Lambda}$ in (2.10), following the proof of (C.5) and (C.6) with Δ_t^{-1} replaced by $(-\Delta_L)^{-1}$, we get (C.7) immediately. Now we turn to prove (C.8). In fact, using (2.33) and lemma C.1, we are led to

$$\begin{aligned}\|B\Lambda\tilde{\Lambda}_t^1\Lambda f_{\neq}\|_{H^N} &\leqslant \frac{1}{1-C\delta}\|B\tilde{\Lambda}_t^1\Lambda f_{\neq}\|_{H^N}\\ &\leqslant \frac{1}{1-C\delta}\Big(2\left\|B\left((a^2-1)S\Lambda f_{\neq}\right)\right\|_{H^N}\\ &\quad + \left\|B\left(b\partial_X\Delta_L^{-1}\Lambda f_{\neq}\right)\right\|_{H^N} + 2\left\|B\left(\tilde{\Lambda}S\Lambda f_{\neq}\right)\right\|_{H^N}\Big).\end{aligned} \tag{C.9}$$

In view of (3.16) and (3.29), similar to (C.5) and (C.6), and using the commutator estimate (B.9) and lemma C.1, we arrive at

$$\begin{aligned}&\left\|B\left((a^2-1)S\Lambda f_{\neq}\right)\right\|_{H^N} + \left\|B\left(b\partial_X\Delta_L^{-1}\Lambda f_{\neq}\right)\right\|_{H^N}\\ &\leqslant C\left\|\langle\eta-\xi\rangle^{N+\beta}\left(\widehat{(a^2-1)}+\hat{b}\right)\right\|_{L^1_\eta}\left\|B\sqrt{-\frac{\dot{M}_1}{M_1}}\Lambda f_{\neq}\right\|_{H^N}\\ &\leqslant \frac{C\delta}{1-C\delta}\left\|B\sqrt{-\frac{\dot{M}_1}{M_1}}f_{\neq}\right\|_{H^N}.\end{aligned}$$

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Moreover, it follows from (C.7), (3.29), and lemma C.1 that

$$\left\|B\left(\tilde{\Lambda}S\Lambda f_{\neq}\right)\right\|_{H^N} \leqslant C\delta\left\|B\sqrt{-\frac{\dot{M}_1}{M_1}}\Lambda f_{\neq}\right\|_{H^N} \leqslant \frac{C\delta}{1-C\delta}\left\|B\sqrt{-\frac{\dot{M}_1}{M_1}}f_{\neq}\right\|_{H^N}.$$

Substituting the above two inequalities into (C.9) yields (C.8). The proof is completed. □

Acknowledgements

R. Zi is partially supported by NSF of China under Grants 12222105.

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