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Schatten class composition operators on the Hardy space

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Suppose $2 < p < \infty$ and φ is a holomorphic self-map of the open unit disk $\mathbb{D}$. We show the following assertions:

(1) If φ has bounded valence and

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p/2} \frac{\mathrm{d}A(z)}{(1-|z|^2)^2} < \infty, \tag{0.1}$$

then C_φ is in the Schatten p-class of the Hardy space H^2.

(2) There exists a holomorphic self-map φ (which is, of course, not of bounded valence) such that the inequality (0.1) holds and $C_\varphi : H^2 \to H^2$ does not belong to the Schatten p-class.

Keywords: composition operators; Hardy spaces; Schatten p-class

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1. Introduction and main results

1.1. Backgrounds and motivations

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk of the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of holomorphic functions on $\mathbb{D}$ and let φ be a holomorphic function on $\mathbb{D}$ with $\varphi(\mathbb{D}) \subset \mathbb{D}$. For $f \in H(\mathbb{D})$, the composition operator C_φ is a linear operator defined by $C_\varphi(f) = f \circ \varphi$.

Recall that a positive T on a separable Hilbert space H is in the trace class if

$$\mathrm{tr}(T) = \sum_{n=0}^{\infty} \langle Te_n, e_n\rangle_H < +\infty$$

for some (or all) orthonormal basis $\{e_n\}$ of H. For any $0 < p < \infty$, the Schatten p-class $\mathcal{S}_p(H)$ of H consists of bounded linear operators $T : H \to H$ such that $(T^*T)^{p/2}$ belongs to the trace class. In particular, $\mathcal{S}_1(H)$ is the trace class of H, and $\mathcal{S}_2(H)$ is called the Hilbert–Schmidt class. It is easy to check that $T \in \mathcal{S}_p(H)$

if and only if $T^* \in \mathcal{S}_p(H)$. For more details about Schatten p-class operators, we refer the readers to Zhu [16].

The Hardy space H^2 is a Hilbert space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|_{H^2}^2 = \sup_{0<r<1} \int_0^{2\pi} |f(re^{i\theta})|^2 \frac{\mathrm{d}\theta}{2\pi} < \infty.$$

For $\alpha > -1$, the weighted Bergman space A_α^2 consists of holomorphic functions f on $\mathbb{D}$ satisfying

$$\|f\|_{A_\alpha^2}^2 = \int_{\mathbb{D}} |f(z)|^2 \mathrm{d}A_\alpha(z) < \infty,$$

where $\mathrm{d}A_\alpha(z) = (\alpha+1)(1-|z|^2)^\alpha \mathrm{d}A(z)$ and $\mathrm{d}A(z)$ is the normalized area measure on $\mathbb{D}$. When $\alpha = 0$, the space A_0^2 is usually denoted by A^2. Properties of composition operator on A_α^2 and H^2 has been widely investigated for decades, see e.g. [3, 8, 16]. In particular, conditions for C_φ that belong to $\mathcal{S}_p(A_\alpha^2)$ and $\mathcal{S}_p(H^2)$ are also characterized, see [1, 2, 4–7, 9, 10, 12, 14].

It is well known (see e.g. Zhu [15]) that H^2 can be viewed as the limit case of A_α^2 as $\alpha \to -1^+$ in some sense. It is also known that for $0 < p < \infty$, $C_\varphi \in \mathcal{S}_p(H^2)$ if and only if

$$\int_{\mathbb{D}} \left(\frac{N_\varphi(z)}{\log \frac{1}{|z|}} \right)^{p/2} \mathrm{d}\lambda(z) < \infty,$$

where

$$\mathrm{d}\lambda(z) = (1-|z|^2)^{-2} \mathrm{d}A(z)$$

is the Möbius invariant measure on $\mathbb{D}$, and

$$N_\varphi(z) = \sum_{w \in \varphi^{-1}(z)} \log \frac{1}{|w|}$$

is the Nevanlinna counting function of φ. Similarly, $C_\varphi \in \mathcal{S}_p(A_\alpha^2)$ if and only if

$$\int_{\mathbb{D}} \left(\frac{N_{\varphi,\alpha+2}(z)}{(\log \frac{1}{|z|})^{\alpha+2}} \right)^{p/2} \mathrm{d}\lambda(z) < \infty,$$

where $N_{\varphi,\alpha+2}(z)$ is a generalized Nevanlinna counting function of φ given by

$$N_{\varphi,\alpha+2}(z) = \sum_{w \in \varphi^{-1}(z)} \left(\log \frac{1}{|w|} \right)^{\alpha+2}.$$

See Luecking-Zhu [5].

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1.2. Main results

A holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ is of bounded valence if there is a positive integer N such that for each $z \in \mathbb{D}$, the set $\varphi^{-1}(z)$ contains at most N points. Zhu [14] shows that if $\alpha > -1$, $2 \leqslant p < \infty$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is an analytic function of bounded valence, then C_φ is in the Schatten class $\mathcal{S}_p$ of A^2_α if and only if

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p(\alpha+2)/2} \mathrm{d}\lambda(z) < \infty.$$

Meanwhile, Zhu [16, Exercise 11.6.7] says that if $p > 2$ and $C_\varphi \in \mathcal{S}_p(H^2)$, then

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p/2} \mathrm{d}\lambda(z) < \infty.$$

These observations hint us to give the following result.

THEOREM 1.1. *If* $2 < p < \infty$, φ *has bounded valence and*

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p/2} \mathrm{d}\lambda(z) < \infty, \tag{1.1}$$

then $C_\varphi \in \mathcal{S}_p(H^2)$.

For $p > 2$, Xia [10] constructs a holomorphic map $\varphi : \mathbb{D} \to \mathbb{D}$ such that

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p} \mathrm{d}\lambda(z) < \infty$$

and such that $C_\varphi : A^2 \to A^2$ does not belong to the Schatten class $\mathcal{S}_p(A^2)$. Motivated by Xia [10], we prove the following theorem:

THEOREM 1.2. *For any* $2 < p < \infty$, *there exists a holomorphic function* $\varphi : \mathbb{D} \to \mathbb{D}$ *such that*

$$\int_{\mathbb{D}} \left(\frac{1 - |z|^2}{1 - |\varphi(z)|^2} \right)^{p/2} \mathrm{d}\lambda(z) < \infty, \tag{1.2}$$

but $C_\varphi : H^2 \to H^2$ *does not belong to the Schatten class* $\mathcal{S}_p(H^2)$.

The proof of theorem 1.1 is based on Wirths-Xiao [9] and Zhu [14]. The proof of theorem 1.2 is modified from Xia [10]. Although the idea of the proof of theorem 1.2 is coming from [10], there are several technical barriers we need to overcome. Thus, we need to adapt Xia's construction for our situation.

Notation. Throughout this paper, we only write $U \lesssim V$ (or $V \gtrsim U$) for $U \leqslant cV$ for a positive constant c, and moreover $U \approx V$ for both $U \lesssim V$ and $V \lesssim U$. $\square$

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2. Preliminaries

For $\alpha > -1$, the Dirichlet-type space is a space of holomorphic functions f on $\mathbb{D}$ for which

$$\|f\|_\alpha^2 = |f(0)|^2 + \|f'\|_{A_\alpha^2}^2 < \infty.$$

It is easy to check that $A_\alpha^2 = \mathcal{D}_{\alpha+2}$ and $H^2 = \mathcal{D}_1$ with equivalent norms.

The following lemma is contained in [9, Theorem 3.2].

LEMMA 2.1. *Let $\alpha > -1$ and $0 < p < \infty$. Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then $C_\varphi \in \mathcal{S}_p(\mathcal{D}_\alpha)$ if and only if*

$$\int_{\mathbb{D}} \left(\int_{\mathbb{D}} \left(\frac{(1-|w|^2)^\varepsilon}{|1-\bar{w}\varphi(z)|^{1+\varepsilon}} \right)^{2+\alpha} |\varphi'(z)|^2 (1-|z|^2)^\alpha \mathrm{d}A(z) \right)^{p/2} \mathrm{d}\lambda(w) < \infty \tag{2.1}$$

for some (any) $\varepsilon > \max\{1/(2+\alpha),\, 2/(2p+p\alpha)\}$.

For fixed $\alpha > 0$, $f,\, g \in \mathcal{D}_\alpha$ with

$$f(z) = \sum_{n=0}^\infty a_n z^n \quad \text{and} \quad g(z) = \sum_{n=0}^\infty b^n z^n,$$

let

$$\langle f, g\rangle_{\mathcal{D}_\alpha} = \sum_{n=0}^\infty \frac{n!\Gamma(\alpha)}{\Gamma(n+\alpha)} a_n \overline{b_n}.$$

Then the reproducing kernel of $\mathcal{D}_\alpha$ associated with the inner product $\langle \cdot, \cdot\rangle_{\mathcal{D}_\alpha}$ is given by

$$K_{\alpha,w}(z) = K_\alpha(z,w) = \frac{1}{(1-\bar{w}z)^\alpha}, \quad z, w \in \mathbb{D}.$$

This means that for each $f \in \mathcal{D}_\alpha$,

$$f(w) = \langle f, K_{\alpha,w}\rangle_{\mathcal{D}_\alpha} \quad w \in \mathbb{D}.$$

Meanwhile, if we write

$$J_{\alpha,w}(z) = J_\alpha(z,w) = \frac{\partial}{\partial \bar{w}} K_\alpha(z,w) = \frac{\alpha z}{(1-\bar{w}z)^{\alpha+1}},$$

then

$$f'(w) = \langle f, J_{\alpha,w}\rangle_{\mathcal{D}_\alpha}. \tag{2.2}$$

Let

$$\|f\|_{\mathcal{D}_\alpha}^2 = \langle f, f\rangle_{\mathcal{D}_\alpha}.$$

Then

$$\|K_{\alpha,w}\|_{\mathcal{D}_\alpha}^2 = \frac{1}{(1-|w|^2)^\alpha}$$

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and

$$\|J_{\alpha,w}\|_{\mathcal{D}_\alpha}^2 = \langle J_{\alpha,w}, J_{\alpha,w}\rangle_{\mathcal{D}_\alpha} = J'_{\alpha,w}(w) = \frac{\alpha(1+\alpha|w|^2)}{(1-|w|^2)^{\alpha+2}} \approx \frac{1}{(1-|w|^2)^{\alpha+2}}. \tag{2.3}$$

Let

$$k_{\alpha,w}(z) = \frac{K_{\alpha,w}(z)}{\|K_{\alpha,w}\|_{\mathcal{D}_\alpha}} \quad \text{and} \quad j_{\alpha,w}(z) = \frac{J_{\alpha,w}(z)}{\|J_{\alpha,w}\|_{\mathcal{D}_\alpha}}.$$

The following lemma comes from [**11**, Lemma 10].

LEMMA 2.2. *Suppose $\alpha > 0$ and $T : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is a positive operator. Let*

$$\widehat{T}^{\alpha,t}(w) = \langle T j_{\alpha,w}, j_{\alpha,w}\rangle_{\mathcal{D}_\alpha}, \quad w \in \mathbb{D}.$$

(1) *Let $0 < p \leqslant 1$. If $\widehat{T}^{\alpha,t} \in L^p(\mathbb{D}, \mathrm{d}\lambda)$, then T is in $\mathcal{S}_p(\mathcal{D}_\alpha)$.*

(2) *Let $1 \leqslant p < \infty$. If T is in $\mathcal{S}_p(\mathcal{D}_\alpha)$, then $\widehat{T}^{\alpha,t} \in L^p(\mathbb{D}, \mathrm{d}\lambda)$.*

Immediately, we have the following theorem.

THEOREM 2.3. *Suppose $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function.*

(1) *If $0 < p \leqslant 2$ and*

$$\int_{\mathbb{D}} \left(\frac{(1-|z|^2)^{\alpha+2}|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}} \right)^{p/2} \mathrm{d}\lambda(z) < \infty, \tag{2.4}$$

then C_φ is in $\mathcal{S}_p$ of $\mathcal{D}_\alpha$.

(2) *If $2 \leqslant p < \infty$ and C_φ is in $\mathcal{S}_p$ of $\mathcal{D}_\alpha$, then (2.4) holds.*

Proof. Write $S = C_\varphi C_\varphi^*$, then $S : \mathcal{D}_\alpha \to \mathcal{D}_\alpha$ is a positive operator. We have

$$\begin{aligned} \widehat{S}^{\alpha,t}(w) = \langle S j_{\alpha,w}, j_{\alpha,w}\rangle_{\mathcal{D}_\alpha} &= \langle C_\varphi^* j_{\alpha,w}, C_\varphi^* j_{\alpha,w}\rangle_{\mathcal{D}_\alpha} \\ &= \frac{\langle C_\varphi^* J_{\alpha,w}, C_\varphi^* J_{\alpha,w}\rangle_{\mathcal{D}_\alpha}}{\|J_{\alpha,w}\|_{\mathcal{D}_\alpha}^2} = \frac{\|C_\varphi^* J_{\alpha,w}\|_{\mathcal{D}_\alpha}^2}{\|J_{\alpha,w}\|_{\mathcal{D}_\alpha}^2}. \end{aligned}$$

For each $f \in \mathcal{D}_\alpha$, (2.2) implies that

$$\begin{aligned} \langle f, C_\varphi^* J_{\alpha,w}\rangle_{\mathcal{D}_\alpha} &= \langle C_\varphi f, J_{\alpha,w}\rangle_{\mathcal{D}_\alpha} = f'(\varphi(w))\varphi'(w) \\ &= \varphi'(w)\langle f, J_{\alpha,\varphi(w)}\rangle_{\mathcal{D}_\alpha} = \langle f, \overline{\varphi'(w)} J_{\alpha,\varphi(w)}\rangle_{\mathcal{D}_\alpha}. \end{aligned}$$

Thus,

$$C_\varphi^* J_{\alpha,w} = \overline{\varphi'(w)} J_{\alpha,\varphi(w)}.$$

Then (2.3) implies that

$$\|C_\varphi^* J_{\alpha,w}\|_{\mathcal{D}_\alpha}^2 \approx \frac{|\varphi'(w)|^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.$$

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This gives that

$$\langle C_\varphi C_\varphi^* j_{\alpha,w}, j_{\alpha,w}\rangle_{\mathcal{D}_\alpha} = \frac{\langle C_\varphi^* J_{\alpha,w}, C_\varphi^* J_{\alpha,w}\rangle_{\mathcal{D}_\alpha}}{\|J_{\alpha,w}\|^2_{\mathcal{D}_\alpha}} \approx \frac{(1-|w|^2)^{2+\alpha}|\varphi'(w)|^2}{(1-|\varphi(w)|^2)^{2+\alpha}}.$$

An application of lemma 2.2 gives the desired assertions. □

By letting $p = 2$ in theorem 2.3, we have the following corollary.

COROLLARY 2.4. *Suppose $\alpha > 0$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is a holomorphic function. Then C_φ is in the Hilbert–Schmidt class of $\mathcal{D}_\alpha$ if and only if*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+2}}\, \mathrm{d}A(z) < \infty.$$

There are several well-known characterizations of the Hilbert–Schmidt compositions on H^2 and A^2_α, see e.g. [**3**, **13**, **16**]. Combine these characterizations with corollary 2.4, we have the following corollaries.

COROLLARY 2.5. *Suppose $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then the following statements are equivalent:*

(1) $C_\varphi \in \mathcal{S}_2(H^2)$.

(2) *The following inequality holds:*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^3}\, \mathrm{d}A(z) < \infty.$$

(3) *The following inequality holds:*

$$\int_{\mathbb{D}} \frac{N_\varphi(z)}{\log \frac{1}{|z|}}\, \mathrm{d}\lambda(z) < \infty.$$

(4) *The following inequality holds:*

$$\int_0^{2\pi} \frac{\mathrm{d}\theta}{(1-|\varphi(\mathrm{e}^{\mathrm{i}\theta})|^2)} < \infty.$$

COROLLARY 2.6. *Suppose $\alpha > -1$ and $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Then the following statements are equivalent*:

(1) $C_\varphi \in \mathcal{S}_2(A^2_\alpha)$.

(2) *The following inequality holds*:

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^{\alpha+2}|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+4}}\, \mathrm{d}A(z) < \infty.$$

(3) *The following inequality holds*:

$$\int_{\mathbb{D}} \frac{N_{\varphi,\alpha+2}(z)}{(\log \frac{1}{|z|})^{\alpha+2}}\, \mathrm{d}\lambda(z) < \infty.$$

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(4) *The following inequality holds*:

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^\alpha}{(1-|\varphi(z)|^2)^{2+\alpha}}\,\mathrm{d}A(z) < \infty.$$

3. Proof of theorem 1.1

Theorem 1.1 is just the case $\alpha = 1$ of the following proposition.

Proposition 3.1. *Suppose* $\alpha > 0$, $2 \leqslant p < \infty$ *and* $p\alpha > 2$. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *is a holomorphic function which has bounded valence and*

$$\int_{\mathbb{D}} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2}\right)^{p\alpha/2} \mathrm{d}\lambda(z) < \infty, \tag{3.1}$$

then C_φ *is in the Schatten class* $\mathcal{S}_p$ *of* $\mathcal{D}_\alpha$.

The condition $p\alpha > 2$ in the above proposition is necessary. Indeed, if $0 < p\alpha \leqslant 2$, then the involved integral is trivially divergent.

Proof. When $p = 2$, the condition $p\alpha > 2$ implies that $\alpha > 1$. Notice that in this case $\mathcal{D}_\alpha = A^2_{\alpha-2}$. According to [**14**], the condition (3.1) implies that $C_\varphi \in \mathcal{S}_p(A^2_{\alpha-2})$.

Now we suppose $2 < p < \infty$. According to lemma 2.1, if we can check the inequality (2.1) for some $\varepsilon > \max\{1/(2+\alpha), 2/(2p+p\alpha)\}$, then we have $C_\varphi \in \mathcal{S}_p(\mathcal{D}_\alpha)$. Write $q = p/2$, then $q > 1$. Let

$$F(w) = \int_{\mathbb{D}} \frac{(1-|w|^2)^{(2+\alpha)\varepsilon}}{|1-\bar{w}\varphi(z)|^{(2+\alpha)(1+\varepsilon)}} |\varphi'(z)|^2 (1-|z|^2)^\alpha \mathrm{d}A(z).$$

Then it is sufficient to check that $F \in L^q(\mathbb{D}, \mathrm{d}\lambda)$.

Let

$$H(w,z) = \frac{(1-|w|^2)^{(\alpha+2)\varepsilon}(1-|\varphi(z)|^2)^\alpha(1-|z|^2)^2|\varphi'(z)|^2}{|1-\bar{w}\varphi(z)|^{(2+\alpha)(1+\varepsilon)}}$$

and

$$h(z) = \left(\frac{(1-|z|^2)}{(1-|\varphi(z)|^2)}\right)^\alpha.$$

Then,

$$F(w) = \int_{\mathbb{D}} H(w,z)h(z)\mathrm{d}\lambda(z).$$

Recall that $\varphi : \mathbb{D} \to \mathbb{D}$ is holomorphic. Schwarz's lemma implies that

$$\frac{(1-|z|^2)^2|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^2} \leqslant 1. \tag{3.2}$$

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Then, for each $\varepsilon > 1/(2+\alpha)$, Forelli–Rudin's estimate implies that

$$\begin{aligned}\int_{\mathbb{D}} H(w,z)\mathrm{d}\lambda(w) &= (1-|\varphi(z)|^2)^{\alpha}(1-|z|^2)^2|\varphi'(z)|^2 \int_{\mathbb{D}} \frac{(1-|w|^2)^{(\alpha+2)\varepsilon-2}\mathrm{d}A(w)}{|1-\bar{w}\varphi(z)|^{(2+\alpha)(1+\varepsilon)}} \\ &\lesssim \frac{(1-|\varphi(z)|^2)^{\alpha}(1-|z|^2)^2|\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{2+\alpha}} \\ &\leqslant 1. \end{aligned} \tag{3.3}$$

Meanwhile, recall that φ is of bounded valence. Let $n_{\varphi}(z)$ be the number of points in $\varphi^{-1}(z)$. Then,

$$\sup_{z\in\mathbb{D}} n_{\varphi}(z) < \infty$$

and

$$\begin{aligned}\int_{\mathbb{D}} H(w,z)\mathrm{d}\lambda(z) &= \int_{\mathbb{D}} \frac{(1-|w|^2)^{(\alpha+2)\varepsilon}(1-|\varphi(z)|^2)^{\alpha}|\varphi'(z)|^2}{|1-\bar{w}\varphi(z)|^{(2+\alpha)(1+\varepsilon)}}\,\mathrm{d}A(z) \\ &= (1-|w|^2)^{(\alpha+2)\varepsilon}\int_{\mathbb{D}} \frac{n_{\varphi}(z)(1-|z|^2)^{\alpha}}{|1-\bar{w}z|^{(2+\alpha)(1+\varepsilon)}}\,\mathrm{d}A(z) \\ &\lesssim 1. \end{aligned} \tag{3.4}$$

Put (3.3) and (3.4) together. Application of Schur's test tells us that the integral operator with kernel $H(w, z)$ is bounded on $L^q(\mathbb{D}, \mathrm{d}\lambda)$. Recall that condition (3.1) implies that $h \in L^q(\mathbb{D}, \mathrm{d}\lambda)$. This gives that $F \in L^q(\mathbb{D}, \mathrm{d}\lambda)$ as desired. □

4. Proof of theorem 1.2

4.1. Construction of φ

The construction is modified from Xia [10]. We adapt some parameters for our argument. For $n = 1, 2, \ldots$, let

$$T_n = \left(2^{-(n+1)}, 2^{-n}\right] \quad \text{and} \quad S_n = \left((4/3)2^{-(n+1)}, (5/3)2^{-(n+1)}\right].$$

That is, S_n is the middle third of T_n. Let $t_n = (4/3)2^{-(n+1)}$ be the left end-point of S_n.

For fixed $p \in (2, \infty)$, let ε be a fixed rational number such that

$$0 < \varepsilon < \frac{2}{p} < 1.$$

We can choose a strictly increasing sequence $k(1) < \cdots < k(n) < \ldots$ of positive integers such that

$$2^{-(\frac{2}{p}+\varepsilon)k(n)} \cdot 2 \cdot 2^{\varepsilon k(n)} = 2^{-\frac{2}{p}k(n)+1} \leqslant (1/3)2^{-(n+1)} = |S_n|$$

for all n and such that every $\varepsilon k(n)$ is an integer.

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For integers $n \geqslant 1$ and $1 \leqslant j \leqslant 2^{\varepsilon k(n)}$, recall that t_n is the left end-point of S_n. Define the intervals

$$J_{n,j} = (a_{n,j}, c_{n,j}) = \left(t_n + 2^{-(\frac{2}{p}+\varepsilon)k(n)} \cdot 2 \cdot (j-1), t_n + 2^{-(\frac{2}{p}+\varepsilon)k(n)} \cdot 2 \cdot j\right)$$

and

$$I_{n,j} = (a_{n,j}, b_{n,j}) = \left(t_n + 2^{-(\frac{2}{p}+\varepsilon)k(n)} \cdot 2 \cdot (j-1), t_n + 2^{-(\frac{2}{p}+\varepsilon)k(n)} \cdot (2j-1)\right).$$

It is easy to check that $I_{n,j}$ is the left half of $J_{n,j}$, $J_{n,j}$'s are pairwise disjoint,

$$\bigcup_{j=1}^{2^{\varepsilon k(n)}} J_{n,j} \subset S_n,$$

and the length of the interval $I_{n,j}$ is denoted by ρ_n, that is

$$\rho_n = |I_{n,j}| = b_{n,j} - a_{n,j} = 2^{-(\frac{2}{p}+\varepsilon)k(n)}. \tag{4.1}$$

We now define a measurable function u on the unit circle $\mathbb{T} = \{w \in \mathbb{C} : |w| = 1\}$ as follows:

$$u(\mathrm{e}^{\mathrm{i}t}) = 2^{-k(n)} \quad \text{if } t \in \bigcup_{j=1}^{2^{\varepsilon k(n)}} I_{n,j}, n \geqslant 1,$$

$$u(\mathrm{e}^{\mathrm{i}t}) = 1 \quad \text{if } t \in (-\pi, \pi] \setminus \left(\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{\varepsilon k(n)}} I_{n,j}\right).$$

The harmonic extension of u to $\mathbb{D}$ is also denoted by u. Let

$$h(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\mathrm{e}^{\mathrm{i}t} + z}{\mathrm{e}^{\mathrm{i}t} - z} u(\mathrm{e}^{\mathrm{i}t})\, \mathrm{d}t$$

and

$$\varphi(z) = \exp(-h(z)) \tag{4.2}$$

for all $z \in \mathbb{D}$. Then, $\mathrm{Re}(h(z)) = u(z) > 0$ for each $z \in \mathbb{D}$, and thus,

$$|\varphi(z)| = \mathrm{e}^{\mathrm{Re}(h(z))} = \mathrm{e}^{-u(z)} < 1.$$

This implies $\varphi(\mathbb{D}) \subset \mathbb{D}$. We will need the fact that $\varphi \in H^2$ with

$$\|\varphi\|_{H^2}^2 = \sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\varphi(r\mathrm{e}^{\mathrm{i}\theta})\right|^2 \mathrm{d}\theta = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^2 \mathrm{d}\theta. \tag{4.3}$$

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4.2. Estimates

For $z \in \mathbb{D}$ and $\mathrm{e}^{\mathrm{i}t} \in \mathbb{T}$, let

$$P(z, \mathrm{e}^{\mathrm{i}t}) = \frac{1-|z|^2}{|\mathrm{e}^{\mathrm{i}t}-z|^2}$$

be the Poisson kernel. It is shown in [10, p. 2508] that if $1/2 \leqslant r < 1$ and $|\theta - t| \leqslant 5$, then there exist constants $0 < \alpha < \beta < \infty$ such that

$$\frac{\alpha(1-r)}{(1-r)^2+(\theta-t)^2} \leqslant \frac{1}{2\pi} P(r\mathrm{e}^{\mathrm{i}\theta}, \mathrm{e}^{\mathrm{i}t}) \leqslant \frac{\beta(1-r)}{(1-r)^2+(\theta-t)^2}. \tag{4.4}$$

We have the following lemma modified from [10, Lemma 4].

LEMMA 4.1. *For any positive integer n and $1 \leqslant j \leqslant 2^{\varepsilon k(n)}$, let $G_{n,j}$ be the Carleson box based on $I_{n,j}$, i.e.*

$$G_{n,j} = \{r\mathrm{e}^{\mathrm{i}\theta} : \theta \in I_{n,j}, 0 < 1-r \leqslant \rho_n\}. \tag{4.5}$$

Then there is a constant C_1 independent of n, j such that

$$\int_{G_{n,j}} \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{p/2} \mathrm{d}\lambda(z) \leqslant C_1 2^{-\frac{p\varepsilon}{2}k(n)}. \tag{4.6}$$

Proof. Given such a pair of n, j, we write

$$G_{n,j} = \bigcup_{\nu=0}^{k(n)} G_{n,j}^{\nu},$$

where

$$G_{n,j}^{0} = \left\{r\mathrm{e}^{\mathrm{i}\theta} : \theta \in I_{n,j}, 0 < 1-r \leqslant \rho_n \cdot 2^{-k(n)}\right\},$$

and

$$G_{n,j}^{\nu} = \left\{r\mathrm{e}^{\mathrm{i}\theta} : \theta \in I_{n,j}, \rho_n \cdot 2^{-k(n)} \cdot 2^{\nu-1} < 1-r \leqslant \rho_n \cdot 2^{-k(n)} \cdot 2^{\nu}\right\},$$

for $1 \leqslant \nu \leqslant k(n)$.

It is shown in [10, p. 2509] that there is a constant $0 < c < 1$ independent of n, j such that

$$1-|\varphi(z)| = 1-\mathrm{e}^{-u(z)} \geqslant 1-\exp(-c2^{-k(n)+\nu})$$

if $z \in G_{n,j}^{\nu}$ and $0 \leqslant \nu \leqslant k(n)$. Let $\delta = \inf_{0<x\leqslant 1} x^{-1}(1-\mathrm{e}^{-x})$. Then,

$$\inf_{z\in G_{n,j}^{\nu}} (1-|\varphi(z)|)^{p/2} \geqslant (\delta c)^{p/2} \cdot 2^{-p/2k(n)} \cdot 2^{p/2\nu}, \quad 0 \leqslant \nu \leqslant k(n). \tag{4.7}$$

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This implies that

$$
\begin{aligned}
&\int_{G_{n,j}} \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{p/2} \mathrm{d}\lambda(z) \\
&= \int_{G_{n,j}^0} \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{p/2} \mathrm{d}\lambda(z) + \sum_{\nu=1}^{k(n)} \int_{G_{n,j}^\nu} \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{p/2} \mathrm{d}\lambda(z) \\
&\leqslant \frac{2^{p/2k(n)}}{(\delta c)^{p/2}} \int_{G_{n,j}^0} (1-|z|^2)^{p/2-2} \mathrm{d}A(z) \\
&\quad + \sum_{\nu=1}^{k(n)} \frac{2^{p/2k(n)}}{(\delta c)^{p/2}\cdot 2^{p/2\nu}} \int_{G_{n,j}^\nu} (1-|z|)^{p/2-2} \mathrm{d}A(z).
\end{aligned} \tag{4.8}
$$

Notice that $p/2-2>-1$. Straightforward computation shows that

$$
\begin{aligned}
\int_{G_{n,j}^0} (1-|z|^2)^{p/2-2} \mathrm{d}A(z) &= \frac{1}{\pi} \int_{I_{n,j}} \mathrm{d}\theta \int_{1-\rho_n\cdot 2^{-k(n)}}^{1} (1-r^2)^{p/2-2} r\,\mathrm{d}r \\
&\leqslant C_2 \rho_n^{p/2} \cdot 2^{-(p/2-1)k(n)}
\end{aligned} \tag{4.9}
$$

for some $C_2>0$, and

$$
\begin{aligned}
\int_{G_{n,j}^\nu} (1-|z|)^{p/2-2} \mathrm{d}A(z) &= \frac{1}{\pi} \int_{I_{n,j}} \mathrm{d}\theta \int_{1-\rho_n\cdot 2^{-k(n)}\cdot 2^{\nu}}^{1-\rho_n\cdot 2^{-k(n)}\cdot 2^{\nu-1}} (1-r)^{p/2-2} r\,\mathrm{d}r \\
&\leqslant C_3 \rho_n^{p/2} \cdot 2^{-(p/2-1)k(n)} \cdot 2^{(p/2-1)\nu}
\end{aligned} \tag{4.10}
$$

for some $C_3>0$. Put (4.8), (4.9) and (4.10) together, we have

$$
\begin{aligned}
&\int_{G_{n,j}} \left(\frac{1-|z|}{1-|\varphi(z)|}\right)^{p/2} \mathrm{d}\lambda(z) \\
&\leqslant \frac{C_2\cdot 2^{k(n)}\cdot \rho_n^{p/2}}{(\delta c)^{p/2}} + \sum_{\nu=1}^{k(n)} \frac{2^{p/2k(n)}\cdot C_3\rho_n^{p/2}\cdot 2^{-(p/2-1)k(n)}\cdot 2^{(p/2-1)\nu}}{(\delta c)^{p/2}\cdot 2^{p/2\nu}} \\
&= 2^{k(n)}\cdot \rho_n^{p/2}\cdot \left(\frac{C_2}{(\delta c)^{p/2}} + \frac{C_3}{(\delta c)^{p/2}} \sum_{\nu=1}^{k(n)} 2^{-\nu}\right).
\end{aligned}
$$

Recall the inequality (4.1), we get the desired inequality (4.6) by letting

$$
C_1 = \frac{C_2}{(\delta c)^{p/2}} + \frac{C_3}{(\delta c)^{p/2}} \sum_{\nu=1}^{\infty} 2^{-\nu} = \frac{C_2+C_3}{(\delta c)^{p/2}}. \qquad \square
$$

The following lemma is quoted from [**10**, Lemma 7].

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LEMMA 4.2. *There is a $C_4 > 0$ such that*

$$u(z) \geqslant C_4 \quad \textit{for every} \quad z \in \mathbb{D} \setminus \left(\bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{\varepsilon k(n)}} G_{n,j} \right),$$

where $G_{n,j}$ is defined by (4.5).

4.3. Proof of theorem 1.2

Let φ be the holomorphic self-map of $\mathbb{D}$ given by (4.2). It is sufficient to check the inequality (1.2) for this φ, and $C_\varphi \notin \mathcal{S}_p(H^2)$.

Let

$$G = \bigcup_{n=1}^{\infty} \bigcup_{j=1}^{2^{\varepsilon k(n)}} G_{n,j},$$

where $G_{n,j}$ is given by (4.5). For $z \in \mathbb{D} \setminus G$, lemma 4.2 implies that

$$|\varphi(z)| = \mathrm{e}^{-\mathrm{Re}(h(z))} = \mathrm{e}^{-u(z)} \leqslant \mathrm{e}^{-C_4}.$$

Since $p/2 - 2 > -1$, we have

$$\begin{aligned} \int_{\mathbb{D}\setminus G} \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{p/2} \mathrm{d}\lambda(z) &\leqslant \frac{1}{(1-\mathrm{e}^{-C_4})^{p/2}} \int_{\mathbb{D}\setminus G} (1-|z|^2)^{p/2-2} \mathrm{d}A(z) \\ &\leqslant \frac{1}{(1-\mathrm{e}^{-C_4})^{p/2}} \int_{\mathbb{D}} (1-|z|^2)^{p/2-2} \mathrm{d}A(z) < \infty. \end{aligned} \tag{4.11}$$

Meanwhile, lemma 4.1 implies that

$$\begin{aligned} \int_G \left(\frac{1-|z|^2}{1-|\varphi(z)|^2} \right)^{p/2} \mathrm{d}\lambda(z) &\approx \int_G \frac{(1-|z|)^{p/2-2}}{(1-|\varphi(z)|)^{p/2}} \, \mathrm{d}A(z) \\ &= \sum_{n=1}^{\infty} \sum_{j=1}^{2^{\varepsilon k(n)}} \int_{G_{n,j}} \frac{(1-|z|)^{p/2-2}}{(1-|\varphi(z)|)^{p/2}} \, \mathrm{d}A(z) \\ &\leqslant C_1 \sum_{n=1}^{\infty} 2^{\varepsilon k(n)} \cdot 2^{-\frac{p\varepsilon}{2} k(n)} \leqslant C_1 \sum_{n=1}^{\infty} 2^{-(p/2-1)\varepsilon k(n)} < \infty, \end{aligned} \tag{4.12}$$

where the last inequality is following from the fact that $p/2 - 1 > 0$. Now (1.2) follows from (4.11) and (4.12) easily.

It remains to check that $C_\varphi \notin \mathcal{S}_p(H^2)$, or equivalently, $\mathrm{tr}((C_\varphi^* C_\varphi)^{\frac{p}{2}}) = \infty$. Let $e_\ell(z) = z^\ell$, $\ell = 0, 1, 2, \ldots$. It is well known that $\{e_\ell : \ell \geqslant 0\}$ is an orthonormal

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basis for H^2. Since $p/2 > 1$, we have

$$\left\langle \left(C_\varphi^* C_\varphi\right)^{p/2} e_\ell, e_\ell \right\rangle_{H^2} \geqslant \left(\left\langle C_\varphi^* C_\varphi e_\ell, e_\ell \right\rangle_{H^2}\right)^{p/2}$$
$$= \|C_\varphi e_\ell\|_{H^2}^p = \left\|\varphi^l\right\|_{H^2}^p = \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^{2\ell} \mathrm{d}\theta\right)^{p/2}.$$

Write

$$I_n = \bigcup_{j=1}^{2^{\varepsilon k(n)}} I_{n,j}.$$

Then,

$$|I_n| = 2^{\varepsilon k(n)} \rho_n = 2^{-\frac{2}{p}k(n)},$$

and

$$\left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right| = \exp(-u(\mathrm{e}^{\mathrm{i}\theta})) = \exp(-2^{-k(n)})$$

for almost every $\theta \in I_n$. Thus,

$$\int_{-\pi}^{\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^{2\ell} \mathrm{d}\theta \geqslant \sum_{n=1}^{\infty} \int_{I_n} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^{2\ell} \mathrm{d}\theta = \sum_{n=1}^{\infty} \mathrm{e}^{-2\ell \cdot 2^{-k(n)}} \cdot 2^{-\frac{2}{p}k(n)}.$$

Notice that

$$\left(\sum_n a_n\right)^s \geqslant \sum_n a_n^s$$

if $s \geqslant 1$ and $a_n \geqslant 0$. We get

$$\left(\int_{-\pi}^{\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^{2\ell} \mathrm{d}\theta\right)^{p/2} \geqslant \left(\sum_{n=1}^{\infty} \mathrm{e}^{-2\ell \cdot 2^{-k(n)}} \cdot 2^{-\frac{2}{p}k(n)}\right)^{p/2} \geqslant \sum_{n=1}^{\infty} \mathrm{e}^{-p\ell \cdot 2^{-k(n)}} \cdot 2^{-k(n)}.$$

This gives that

$$\operatorname{tr}\left((C_\varphi^* C_\varphi)^{p/2}\right) = \sum_{\ell=0}^{\infty} \left\langle (C_\varphi^* C_\varphi)^{p/2} e_\ell, e_\ell \right\rangle_{H^2} \geqslant \sum_{\ell=0}^{\infty} \left(\frac{1}{2\pi}\int_{-\pi}^{\pi} \left|\varphi(\mathrm{e}^{\mathrm{i}\theta})\right|^{2\ell} \mathrm{d}\theta\right)^{p/2}$$
$$\geqslant \frac{1}{(2\pi)^{p/2}} \sum_{\ell=0}^{\infty} \sum_{n=1}^{\infty} \mathrm{e}^{-p\ell \cdot 2^{-k(n)}} \cdot 2^{-k(n)}$$
$$= \frac{1}{(2\pi)^{p/2}} \sum_{n=1}^{\infty} \left(2^{-k(n)} \sum_{\ell=0}^{\infty} \mathrm{e}^{-p\ell \cdot 2^{-k(n)}}\right)$$
$$= \frac{1}{(2\pi)^{p/2}} \sum_{n=1}^{\infty} 2^{-k(n)} \cdot \frac{1}{1 - \mathrm{e}^{-p \cdot 2^{-k(n)}}}.$$

Since

$$\sup_{x>0} \frac{1-e^{-x}}{x} \leqslant 1.$$

We have

$$\frac{1}{1-e^{-p\cdot 2^{-k(n)}}} \geqslant \frac{1}{p\cdot 2^{-k(n)}}.$$

Then,

$$\sum_{n=1}^{\infty} 2^{-k(n)} \cdot \frac{1}{1-e^{-p\cdot 2^{-k(n)}}} \geqslant \sum_{n=1}^{\infty} 2^{-k(n)} \cdot \frac{1}{p\cdot 2^{-k(n)}} = \sum_{n=1}^{\infty} \frac{1}{p} = \infty.$$

This implies that $C_\varphi \notin \mathcal{S}_p(H^2)$ and the proof is complete.

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