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On the Extension of Bounded Holomorphic Maps from Gleason Parts of the Maximal Ideal Space of H^∞

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Abstract. Let H^∞ be the algebra of bounded holomorphic functions on the open unit disk, and let $\mathfrak{M}$ be its maximal ideal space. Let $\mathfrak{M}_a$ be the union of nontrivial Gleason parts (analytic disks) of $\mathfrak{M}$. In this paper, we study the problem of extensions of bounded Banach-valued holomorphic functions and holomorphic maps with values in Oka manifolds from Gleason parts of $\mathfrak{M}_a \backslash \mathbb{D}$. The resulting extensions satisfy the uniform boundedness principle in the sense that their norms are bounded by constants that do not depend on the choice of the Gleason part. The results extend fundamental results of D. Suárez on the characterization of the algebra of restrictions of Gelfand transforms of functions in H^∞ to Gleason parts of $\mathfrak{M}_a \backslash \mathbb{D}$. The proofs utilize our recent advances on $\bar\partial$-equations on quasi-interpolating sets and Runge-type approximations.

1 Formulation of main results

1.1

Recall that for a commutative unital complex Banach algebra A, the maximal ideal space $\mathfrak{M}(A) \subset A^*$ is the set of nonzero homomorphisms $A \to \mathbb{C}$ endowed with the Gelfand topology, the weak-$*$ topology of A^*. It is a compact Hausdorff space contained in the unit sphere of A^*. The *Gelfand transform* defined by $\hat{a}(\varphi) := \varphi(a)$ for $a \in A$ and $\varphi \in \mathfrak{M}(A)$ is a nonincreasing-norm morphism from A into the Banach algebra $C(\mathfrak{M}(A))$ of complex-valued continuous functions on $\mathfrak{M}(A)$.

Let H^∞ be the Banach algebra of bounded holomorphic functions on the open unit disk $\mathbb{D} \subset \mathbb{C}$ equipped with pointwise multiplication and supremum norm $\|\cdot\|_\infty$, and let $\mathfrak{M}$ be its maximal ideal space. Then the *Gelfand transform* $\hat{}: H^\infty \to C(\mathfrak{M})$ is an isometry and the map $\iota : \mathbb{D} \hookrightarrow \mathfrak{M}$ taking $z \in \mathbb{D}$ to the evaluation homomorphism $f \mapsto f(z)$, $f \in H^\infty$, is an embedding with dense image by the celebrated Carleson corona theorem [5]. In the sequel, we identify $\mathbb{D}$ with $\iota(\mathbb{D})$.

Let

$$\rho(z,w) := \left|\frac{z-w}{1-\bar{w}z}\right|, \qquad z,w \in \mathbb{D}, \tag{1.1}$$

be the pseudohyperbolic metric on $\mathbb{D}$.

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For $m_1, m_2 \in \mathfrak{M}$, the formula

$$\rho(m_1, m_2) := \sup\{|\hat{f}(m_2)| : f \in H^\infty,\ \hat{f}(m_1) = 0,\ \|f\|_{H^\infty} \le 1\} \tag{1.2}$$

gives an extension of ρ to $\mathfrak{M} \times \mathfrak{M}$. The extended function is lower semicontinuous on $\mathfrak{M} \times \mathfrak{M}$ (see [10, Theorem 6.2]) and determines a metric on $\mathfrak{M}$ with the property that any two open balls of radius 1 are either equal or disjoint. The *Gleason part* of $m \in \mathfrak{M}$ is then defined by $P(m) := \{m' \in \mathfrak{M} : \rho(m', m) < 1\}$. Hoffman's classification of Gleason parts [10] shows that there are only two cases: either $P(m) = \{m\}$ or $P(m)$ is an analytic disk. The latter case means that there is a *parameterization* of $P(m)$, i.e., a continuous one-to-one and onto map $L : \mathbb{D} \to P(m)$ such that $\hat{f} \circ L \in H^\infty$ for every $f \in H^\infty$. By $\mathfrak{M}_a$ we denote the union of all nontrivial (analytic disks) Gleason parts of $\mathfrak{M}$. It is known that $\mathfrak{M}_a \subset \mathfrak{M}$ is open and $P(m) \subset \mathfrak{M}_a$ if and only if m belongs to the closure of an interpolating sequence for H^∞.

For a Gleason part $P \subset \mathfrak{M}_a \backslash \mathbb{D}$, consider the closed ideal $I_P = \{f \in H^\infty : \hat{f} \equiv 0 \text{ on } P\}$. According to Gorkin [9], the quotient Banach algebra H^∞/I_P is semisimple with maximal ideal space $\bar{P}$ (- the closure of P in $\mathfrak{M}$). The corresponding Gelfand transform maps $f + I_P \in H^\infty/I_P$ to $\hat{f}|_{\bar{P}} \in C(\bar{P})$. In [15, 16], Suárez proved that the Gelfand transform maps H^∞/I_P isomorphically onto the closed subalgebra $\mathcal{O}(\bar{P})$ of functions $g \in C(\bar{P})$ such that $g \circ L \in H^\infty$, where L is a parameterization of P. Specifically, he proved the following interpolation result.

Theorem A *There is an absolute constant $C \ge 1$ such that for every $g \in \mathcal{O}(\bar{P})$, there exists $f \in H^\infty$ such that*

$$\hat{f}|_{\bar{P}} = g \quad \textit{and} \quad \|f\|_\infty \le C\|g\|_{C(\bar{P})}.$$

(Here and below for a normed space X, its norm is denoted by $\|\cdot\|_X$.)

In the framework of the Stein-like theory on $\mathfrak{M}$ developed in [2] analogous to the classical theory of complex functions on Stein spaces, and taking into account the results of Suárez, $\bar{P}$ can be viewed as an analog of a complex connected submanifold of $\mathfrak{M}$, and $\mathcal{O}(\bar{P})$ as an analog of the space of holomorphic functions on $\bar{P}$. Following this line, and using recent advances from the author's work [3, 4], in this paper we will prove analogs of Theorem A for such functions on $\bar{P}$ with values in complex Banach spaces and in some complex submanifolds of $\mathbb{C}^n$. Our results are related to some general interpolation results obtained previously in [2, Theorem 1.9] and [3, Theorem 1.11].

1.2

Our first result extends Theorem A to the case of Banach-valued functions.

Let X be a complex Banach space, and let $V \subset \mathbb{D}$ be an open subset. We denote by $H^\infty_{\text{comp}}(V, X)$ the Banach space of X-valued holomorphic functions f on V with relatively compact images equipped with norm $\|f\|_{H^\infty_{\text{comp}}(V,X)} := \sup_{z \in V} \|f(z)\|_X$. If $V = \mathbb{D} \cap \widehat{V}$ for an open set $\widehat{V} \subset \mathfrak{M}$, then every function $f \in H^\infty_{\text{comp}}(V, X)$ extends to a function with a relatively compact image $\hat{f} \in C(\widehat{V}, X)$ (see [14, Theorem 3.2] and [2, Proposition 1.3]). Let $P \subset \mathfrak{M}_a$ be a Gleason part with a parameterization $L : \mathbb{D} \to P$. We denote by $\mathcal{O}(\bar{P}, X) \subset C(\bar{P}, X)$ the Banach subspace of continuous X-valued functions g on $\bar{P}$ such that $g \circ L$ is a holomorphic X-valued function on $\mathbb{D}$ with norm $\|g\|_{C(\bar{P},X)} := \sup_{x\in\bar{P}} \|g(x)\|_X$.

Theorem 1.1 *There is an absolute constant $C \geq 1$ such that for every $g \in \mathcal{O}(\bar{P}, X)$, there exists $f \in H^\infty_{\rm comp}(\mathbb{D}, X)$ such that*

$$\hat{f}|_{\bar{P}} = g \quad \text{and} \quad \|f\|_{H^\infty_{\rm comp}(\mathbb{D},X)} \leq C\|g\|_{C(\bar{P},X)}.$$

It was established in [15, Theorem 4.1] that if the Gleason part P is a homeomorphic disk, i.e., if a parameterization $L : \mathbb{D} \to \mathfrak{M}$ of P is an embedding, then Theorem A is valid with constant $C = 1$. So it is natural to ask whether this would also be true in the case of Theorem 1.1.

To formulate our second result, let us recall the following definitions.

($\triangledown$) A complex manifold $\mathcal{M}$ is said to be *Oka* if every holomorphic map $f : K \to \mathcal{M}$ from a neighborhood of a compact convex set $K \subset \mathbb{C}^k$, $k \in \mathbb{N}$, can be approximated uniformly on K by entire maps $\mathbb{C}^k \to \mathcal{M}$.

We refer to the book [7] and the paper [13] for examples and basic results of the theory of Oka manifolds.

($\triangledown$) A path-connected topological space X is *i-simple* if for each $x \in X$ the fundamental group $\pi_1(X, x_0)$ acts trivially on the i-homotopy group $\pi_i(X, x)$ (see, e.g., [12, Chapter IV.16] for the corresponding definitions and results).

For instance, X is i-simple if the group $\pi_i(X)$ is trivial and 1-simple if and only if the group $\pi_1(X)$ is abelian. Also, every path-connected topological group is i-simple for all i. The same is true for a complex manifold biholomorphic to the quotient of a connected complex Lie group by a *connected* closed Lie subgroup (see, e.g., [11, (3.2)]).

Let $\mathcal{M}$ be a complex manifold, and let $P \subset \mathfrak{M}_a$ be a Gleason part with parameterization $L : \mathbb{D} \to P$. A continuous map $F \in C(\bar{P}, \mathcal{M})$ is said to be *holomorphic* (written, $F \in \mathcal{O}(\bar{P}, \mathcal{M})$) if $F \circ L : \mathbb{D} \to \mathcal{M}$ is a holomorphic map of complex manifolds.

Let $\mathscr{O}$ be the class of connected Oka manifolds $\mathcal{M}$ embeddable as complex submanifolds into complex Euclidean spaces and having i-simple for $i = 1, 2$ finite unbranched coverings. The analog of Theorem A for holomorphic maps with values in manifolds of class $\mathscr{O}$ is as follows:

Theorem 1.2 *Let $\mathcal{M} \subset \mathbb{C}^n$ be of class $\mathscr{O}$, and let K be a compact subset of $\mathcal{M}$. There is a constant $C = C(\mathcal{M}, K, n)$[1] such that for every map $F \in \mathcal{O}(\bar{P}, \mathcal{M})$ with image in K, there exists a map $G = (g_1, \ldots, g_n) \in (H^\infty)^n \subset C(\mathbb{D}, \mathbb{C}^n)$ with a relatively compact image in $\mathcal{M}$ such that*

$$\widehat{G}|_{\bar{P}} = F \quad \text{and} \quad \|G\|_{(H^\infty)^n} \leq C.$$

Here, $\widehat{G} := (\hat{g}_1, \ldots, \hat{g}_n)$.

Remark 1.3 (1) The class $\mathscr{O}$ was originally introduced in [3] in connection with Theorem 1.6 of that paper. The proof of this theorem invokes an extension result

[1] We write $C = C(\alpha_1, \alpha_2, \ldots)$ if the constant C depends only on $\alpha_1, \alpha_2, \ldots$.

for continuous maps defined on certain subsets of the maximal ideal space of the algebra $H^\infty(\mathbb{D}\times\mathbb{N})$. The extension result is obtained using some obstruction theory and depends on the fact that the covering dimension of the maximal ideal space $\mathfrak{M}(H^\infty(\mathbb{D}\times\mathbb{N}))$ is 2 and on the i-simplicity for $i=1,2$ of manifolds of class $\mathscr{O}$.

(2) The class $\mathscr{O}$ contains, e.g., • complements in $\mathbb{C}^k$, $k>1$, of complex algebraic subvarieties of codimension $\geq$ 2 and of compact polynomially convex sets (these manifolds are simply connected; see [6]); • connected Stein Lie groups; • quotients of connected reductive complex Lie groups by Zariski closed subgroups (these manifolds are quasi-affine algebraic (see, e.g., [1, Theorem 5.6] for the references); they have i-simple finite unbranched coverings because Zariski closed subgroups have finitely many connected components and quotients of connected complex Lie groups by connected closed Lie subgroups are i-simple for all i (see, e.g., [11, (3.2)])). Also, direct products of manifolds from class $\mathscr{O}$ belong to $\mathscr{O}$ and so forth.

2 Auxiliary results

A subset S of a metric space $(\mathcal{M},d)$ is said to be ε*-separated* if $d(x,y)\geq\varepsilon$ for all $x,y\in S$, $x\neq y$. A maximal ε-separated subset of $\mathcal{M}$ is said to be an ε*-chain*. Thus, if $S\subset\mathcal{M}$ is an ε-chain, then S is ε-separated and for every $z\in\mathcal{M}\backslash S$ there is an $x\in S$ such that $d(z,x)<\varepsilon$. The existence of ε-chains follows from the Zorn lemma.

A subset $S\subset\mathbb{D}$ is said to be *quasi-interpolating*, if an ε-chain of S, $\varepsilon\in(0,1)$, with respect to the pseudohyperbolic metric ρ (see (1.1)) is an interpolating sequence for H^∞. (In fact, in this case, every ε-chain of S, $\varepsilon\in(0,1)$, with respect to ρ is an interpolating sequence for H^∞; this easily follows from [8, Chapter X, Corollary 1.6, Chapter VII, Lemma 5.3].)

Let $K\subset\mathbb{D}$ be a Lebesgue measurable subset, and let X be a complex Banach space. Two X-valued functions on $\mathbb{D}$ are equivalent if they coincide a.e. on $\mathbb{D}$. The complex Banach space $L^\infty(K,X)$ consists of equivalence classes of Bochner measurable essentially bounded functions $f:\mathbb{D}\to X$ equal 0 a.e. on $\mathbb{D}\backslash K$ equipped with norm $\|f\|_{L^\infty(K,X)}:=\operatorname{ess\,sup}_{z\in K}\|f(z)\|_X$. Also, we denote by $C_\rho(\mathbb{D},X)$ the Banach space of bounded continuous functions $f:\mathbb{D}\to X$ uniformly continuous with respect to ρ equipped with norm $\|f\|_{C_\rho(\mathbb{D},X)}:=\sup_{z\in\mathbb{D}}\|f(z)\|_X$.

In [4], we studied the differential equation

$$\frac{\partial F}{\partial\bar z}=\frac{f(z)}{1-|z|^2},\qquad |z|<1,\quad f\in L^\infty(K,X). \tag{2.1}$$

We proved that if K is quasi-interpolating, then equation (2.1) has a weak solution $F\in C_\rho(\mathbb{D},X)$, i.e., such that for every C^∞ function s with compact support in $\mathbb{D}$,

$$\iint_{\mathbb{D}} F(z)\cdot\frac{\partial s(z)}{\partial\bar z}\,dz\wedge d\bar z=-\iint_{\mathbb{D}}\frac{f(z)}{1-|z|^2}\cdot s(z)\,dz\wedge d\bar z, \tag{2.2}$$

given by a bounded linear operator $L_K^X:L^\infty(K,X)\to C_\rho(\mathbb{D},X)$. Specifically, we obtained the following result.

Theorem 2.1 [4, Theorem 1.1] *Suppose a quasi-interpolating set $K \subset \mathbb{D}$ is Lebesgue measurable and $\zeta = \{z_j\}$ is an ε-chain of K, $\varepsilon \in (0,1)$, with respect to ρ such that*

$$\delta(\zeta) := \inf_k \prod_{j,\, j\neq k} \rho(z_j, z_k) \geq \delta > 0.$$[2]

There is a bounded linear operator $L_K^X : L^\infty(K,X) \to C_\rho(\mathbb{D},X)$ of norm

$$\|L_K^X\| \leq \frac{c\varepsilon}{1-\varepsilon} \cdot \max\left\{1, \frac{\log\frac{1}{\delta}}{(1-\varepsilon_*)^2}\right\}, \quad \varepsilon_* := \max\left\{\frac{1}{2}, \varepsilon\right\}, \tag{2.3}$$

for a numerical constant $c < 5^2 \times 10^6$ such that for every $f \in L^\infty(K,X)$ the function $L_K^X f$ is a weak solution of equation (2.1).

The operator L_K^X has the following properties:

(i) *If $T : X \to Y$ is a bounded linear operator between complex Banach spaces, then*
$$TL_K^X = L_K^Y T,$$
where $(Tf)(z) := T(f(z))$, $z \in \mathbb{D}$, $f : \mathbb{D} \to X$.

(ii) *If $f \in L^\infty(K,X)$ has a compact essential range, then the range of $L_K^X f$ is relatively compact.*

(iii) *If $f \in L^\infty(X,K)$ is continuously differentiable on an open set $U \subset \mathbb{D}$, then $L_K^X f$ is continuously differentiable on U.*

An important example of a quasi-interpolating set is a pseudohyperbolic neighborhood of a Carleson contour used in the proof of the corona theorem. In particular, using the construction from the proof of the theorem, one obtains the following (for the proof of this result, see [5, 17] and [8, Chapter VIII.5]).

Lemma 2.2 *Suppose $f \in H^\infty$ with $\|f\|_\infty \leq 1$. Given $0 < \beta < 1$, there is an $\varepsilon = \varepsilon(\beta) \in (0,\beta)$, a quasi-interpolating set $K_\beta \subset \mathbb{D}$ having a $\frac{1}{2}$-chain $\zeta \subset K_\beta$ such that $\delta(\zeta) \geq \delta = \delta(\beta) > 0$, and a function $\Phi \in C^\infty(\mathbb{D})$, $0 \leq \Phi \leq 1$, satisfying $\mathbb{D}\backslash K_\beta \subset \Phi^{-1}(\{0,1\})$ and*

(i)
$$\{z \in \mathbb{D} : |f(z)| \geq \beta\} \subset \Phi^{-1}(0) \cap (\mathbb{D}\backslash K_\beta);$$

(ii)
$$\{z \in \mathbb{D} : |f(z)| \leq \varepsilon\} \subset \Phi^{-1}(1) \cap (\mathbb{D}\backslash K_\beta);$$

(iii)
$$\frac{\partial\Phi}{\partial\bar{z}} = \frac{g(z)}{1-|z|^2}, \quad z \in \mathbb{D}, \quad \textit{where} \quad g \in L^\infty(K_\beta, \mathbb{C}), \ \|g\|_\infty \leq A = A(\beta).$$

Using the previous results, we prove the following Banach-valued version of [16, Lemma 3.9].

[2] Hence, ζ is an interpolating sequence for H^∞ by the Carleson theorem (see, e.g., [8, Chapter VII, Theorem 1.1]).

Lemma 2.3 *Let X be a complex Banach space, let u be an inner function, and let $0<\beta<1$. Put $V=\{z\in\mathbb{D} : |u(z)|<\beta\}$ and suppose that $g\in H^{\infty}_{\rm comp}(V,X)$. Then there are $\varepsilon=\varepsilon(\beta)\in(0,\beta)$, $C=C(\beta)>0$, and $G\in H^{\infty}_{\rm comp}(\mathbb{D},X)$ such that*

(i)

$$\|G\|_{H^{\infty}_{\rm comp}(\mathbb{D},X)}\le C\|g\|_{H^{\infty}_{\rm comp}(V,X)} \quad \textit{and}$$

(ii)

$$|G(z)-g(z)|\le C\|g\|_{H^{\infty}_{\rm comp}(V,X)}|u(z)| \textit{ when } |u(z)|<\varepsilon.$$

Proof We choose Φ of Lemma 2.2 for the function $f:=u$. According to Lemma 2.2(ii) and (iii) and Theorem 2.1(ii) and (iii), there is a function with relatively compact image $F\in C^{\infty}(\mathbb{D},X)$ such that for some $c=c(\beta)>0$,

$$\frac{\partial F}{\partial\bar z}=\frac{g}{u}\frac{\partial\Phi}{\partial\bar z}, \quad \text{and} \quad \|F\|_{\infty}\le\frac{cA}{\varepsilon}\|g\|_{H^{\infty}_{\rm comp}(V,X)}.$$

(Note that $\frac{g}{u}\frac{\partial\Phi}{\partial\bar z}\in L^{\infty}(K_{\beta},X)$ and has a relatively compact image.)

Consider the function $G=g\Phi-Fu$. Then the previous equation implies $\frac{\partial G}{\partial\bar z}=0$, i.e., $G\in H^{\infty}_{\rm comp}(\mathbb{D},X)$ and by Lemma 2.2(i) (since $\lim_{r\to1^-}|u(re^{i\theta})|=1$ a.e. $\theta\in[0,2\pi)$),

$$\|G\|_{H^{\infty}_{\rm comp}(\mathbb{D},X)}\le\frac{cA}{\varepsilon}\|g\|_{H^{\infty}_{\rm comp}(V,X)}.$$

On the other hand, Lemma 2.2(ii) implies that $G(z)=g(z)-F(z)u(z)$ when $|u(z)|<\varepsilon$. Thus, for such z,

$$|G(z)-g(z)|=|F(z)u(z)|\le\frac{cA}{\varepsilon}\|g\|_{H^{\infty}_{\rm comp}(V,X)}|u(z)|.$$ ■

3 Proofs

Proof of Theorem 1.1 The construction presented in [16, Section 4] is also applicable to maps $g\in\mathcal{O}(\bar P,X)$. In particular, one can define an open set $\Omega\subset\mathbb{D}$, a Blaschke product $b\in H^{\infty}$, such that $\hat b=0$ on $\bar P$ and $V=\{z\in\mathbb{D} : |b(z)|<\frac12\}\subset\Omega$, and a function $h\in H_{\rm comp}(\Omega,X)$ whose image is contained in the image of $g|_P$ such that for every point $x\in P$ and a net $\{z_\alpha\}\subset\mathbb{D}$ converging to x,

$$\lim_{\alpha} h(z_\alpha)=g(x). \tag{3.1}$$

Since V is the intersection of $\mathbb{D}$ and the open set $\widehat V:=\{x\in\mathfrak{M} : |\hat b(x)|<\frac12\}\subset\mathfrak{M}$, the function h extends to a continuous X-valued function $\hat h$ with a relatively compact image on $\widehat V$ (see [14, Theorem 3.2] and [2, Proposition 1.3]). By (3.1) and the definition of h,

$$\hat h|_{\bar P}=g, \quad \text{and} \quad \|\hat h\|_{C(\widehat V,X)}=\|g\|_{C(\bar P,X)}. \tag{3.2}$$

Further, we apply Lemma 2.3 to the inner function b and the function $h \in H^\infty_{\text{comp}}(V, X)$ with $\beta = \frac{1}{2}$. Then, due to (3.2), we get absolute constants $C > 0$ and $\varepsilon \in (0, \frac{1}{2})$, and a function $f \in H^\infty_{\text{comp}}(\mathbb{D}, X)$ such that

$$\|f\|_{H^\infty_{\text{comp}}(\mathbb{D},X)} \le C\|g\|_{H^\infty_{\text{comp}}(V,X)},$$

and

$$|f(z) - h(z)| \le C\|g\|_{H^\infty_{\text{comp}}(V,X)}|b(z)| \text{ when } |b(z)| < \varepsilon.$$

Since $\hat{b} = 0$ on $\bar{P}$, the latter inequality and (3.2) imply

$$\hat{f}|_{\bar{P}} = g,$$

as required. ∎

Proof of Theorem 1.2 Let $F \in \mathcal{O}(\bar{P}, \mathcal{M})$ have an image in K. According to Theorem 1.1, there is a map $F_1 \in (H^\infty)^n$ such that

$$\widehat{F_1}|_{\bar{P}} = F \quad \text{and} \quad \|F_1\|_{(H^\infty)^n} \le C. \tag{3.3}$$

Further, as in the previous proof, one applies a construction from [16, Section 4] to define an open set $\Omega \subset \mathbb{D}$, a Blaschke product $b \in H^\infty$, such that $\hat{b} = 0$ on $\bar{P}$ and $V = \{z \in \mathbb{D} : |b(z)| < \frac{1}{2}\} \subset \Omega$, and a holomorphic map $H \in H^\infty(\Omega, \mathcal{M})$ with image in K such that

$$\widehat{H}|_{\bar{P}} = F. \tag{3.4}$$

Since $\mathcal{M} \in \mathscr{O}$, conditions (3.3) and (3.4) allow us to apply [3, Theorem 1.11] with $c = 2$, $\delta = \frac{1}{2}$, $k = 1$, $\Pi^k_{c,\delta} = \{z \in \mathbb{D} : |2b(z)| < \delta\}$, $\Pi^k_c = \Pi^k_{c,1}$, $g = F_1$, $b = C$, and $f = H$ to get a constant $C(\mathcal{M}, K, n) := C(\mathcal{M}, K, n, b, c, k, \delta) > 1$ and a map $G \in (H^\infty)^n$ with a relatively compact image in $\mathcal{M}$ such that

$$\widehat{G}|_{\bar{P}} = F \quad \text{and} \quad \|G\|_{(H^\infty)^n} \le C(\mathcal{M}, K, n),$$

as required. ∎

4 Concluding remark

Let $\mathfrak{M}^n$ be the n-fold direct product of $\mathfrak{M}$. For Gleason parts $P_1, \dots, P_n \in \mathfrak{M}_a$ with parameterizations $L_1, \dots, L_n$, we set $P := P_1 \times \cdots \times P_n \subset \mathfrak{M}^n_a$ and $L := (L_1, \dots, L_n) := \mathbb{D}^n \to P$. As before, for a complex Banach space X, we denote by $\mathcal{O}(\bar{P}, X)$ the Banach space of continuous X-valued maps f on the closure $\bar{P} \subset \mathfrak{M}^n$ of P such that $f \circ L \in H^\infty(\mathbb{D}^n, X)$ equipped with the norm $\|f\|_{C(\bar{P},X)}$. Based on Theorem 1.1, we can prove formally a more general statement.

Theorem 4.1 *For every $f \in \mathcal{O}(\bar{P}, X)$, there is a map $F \in \mathcal{O}(\mathfrak{M}^n, X)$ such that*

$$F|_P = f \quad \textit{and} \quad \|F\|_{C(\mathfrak{M}^n,X)} \le C^n \, \|f\|_{C(\bar{P},X)},$$

where C is the constant in Theorem 1.1.

Proof The proof is by induction on n. For $n = 1$, the theorem is the content of Theorem 1.1. Assuming that Theorem 4.1 is proved for $n - 1$ with $n > 1$, let us prove it for n. To this end, we set $P' := P_1 \times \cdots \times P_{n-1}$ and $L' = (L_1, \dots, L_{n-1})$.

Lemma 4.2 *$\mathcal{O}(\bar{P}, X)$ is isometrically isomorphic to $\mathcal{O}(\bar{P}', \mathcal{O}(\bar{P}_n, X))$.*

Proof We prove that the correspondence $f \mapsto f'$, $f \in \mathcal{O}(\bar{P}, X)$, where

$$f'(x_1, \dots, x_{n-1})(x_n) := f(x_1, \dots, x_n), \quad (x_1, \dots, x_n) \in \bar{P},$$

gives the required isometry.

Indeed, it is clear that $f' \in C(\bar{P}', C(\bar{P}, X))$. Next, given $x' := (x_1, \dots, x_{n-1}) \in \bar{P}'$, let us take a net $(z'_\alpha) \subset \mathbb{D}^{n-1}$ such that the net $(L'(z'_\alpha)) \subset P'$ converges to x'. By the definition, each $(f'(L'(z'_\alpha))) \circ L_n = f(L'(z'_\alpha), L_n(\cdot))$ is an X-valued holomorphic function on $\mathbb{D}$ and

$$\begin{aligned}\lim_\alpha f'(L'(z'_\alpha))(L_n(z)) &= \lim_\alpha f(L'(z'_\alpha), L_n(z)) = f(x', L_n(z)) \\ &= f'(x')(L_n(z)), \quad z \in \mathbb{D}.\end{aligned}$$

Moreover, images of functions $f'(L'(z'_\alpha)) \circ L_n \in H^\infty(\mathbb{D}, X)$ belong to the compact set $f(\bar{P}) \subset X$. Thus, using a standard normal family argument for bounded holomorphic functions, we obtain that the net $(f'(L'(z'_\alpha)) \circ L_n)$ has a subnet converging uniformly on compact subsets of $\mathbb{D}$ to $(f'(x')) \circ L_n$. Hence, $(f'(x')) \circ L_n \in H^\infty(\mathbb{D}, X)$. This shows that $f' \in C(\bar{P}', \mathcal{O}(\bar{P}_n, X)))$.

Similarly, given $x_n \in \bar{P}_n$, one shows that $(f' \circ L')(x_n) \in H^\infty(\mathbb{D}^{n-1}, X)$ and its image belongs to the compact set $f(\bar{P})$. Using the Bochner integral, we define

$$K_r(f' \circ L')(z_1, \dots, z_{n-1}) = \frac{1}{(2\pi i)^{n-1}} \int_{\mathbb{T}_r^{n-1}} \frac{(f' \circ L')(w_1, \dots, w_{n-1})}{(w_1 - z_1) \cdots (w_{n-1} - z_{n-1})} dw_1 \cdots dw_{n-1};$$

here, $\mathbb{T}_r^{n-1}$ is the boundary torus of the open polydisk $\mathbb{D}_r^{n-1}$, where $\mathbb{D}_r := \{z \in \mathbb{C} : |z| < r\}$, $r \in (0,1)$, and $(z_1, \dots, z_{n-1}) \in \mathbb{D}_r^{n-1}$. Then $K_r(f' \circ L')$ is a holomorphic function on $\mathbb{D}_r^{n-1}$ with values in $\mathcal{O}(\bar{P}_n, X)$ such that for each $x_n \in \bar{P}_n$,

$$(K_r(f' \circ L'))(x_n) = (f' \circ L')|_{\mathbb{D}_r^{n-1}}(x_n).$$

Thus, $K_r(f' \circ L') = f' \circ L'|_{\mathbb{D}_r^{n-1}}$ for all $r \in (0,1)$. This shows that $f' \circ L' \in H^\infty(\mathbb{D}^{n-1}, X)$; hence, the correspondence $f \mapsto f'$ determines an isometrical isomorphism between $\mathcal{O}(\bar{P}, X)$ and $\mathcal{O}(\bar{P}', \mathcal{O}(\bar{P}_n, X))$, as required. ∎

Using the lemma, let us continue the proof of the theorem. To this end, let $f \in \mathcal{O}(\bar{P}, X)$ and $f' \in \mathcal{O}(\bar{P}', \mathcal{O}(\bar{P}_n, X))$ be as in Lemma 4.2. Then, by the induction hypothesis, there is an $F' \in \mathcal{O}(\mathfrak{M}^{n-1}, \mathcal{O}(\bar{P}_n, X))$ such that

$$F'|_{\bar{P}'} = f' \quad \text{and} \quad \|F'\|_{\mathcal{O}(\mathfrak{M}^{n-1}, \mathcal{O}(\bar{P}_n, X))} \le C^{n-1} \|f'\|_{\mathcal{O}(\bar{P}', \mathcal{O}(\bar{P}_n, X))} = C^{n-1} \|f\|_{\mathcal{O}(\bar{P}, X)}.$$

We define

$$g'(x_n)(x_1, \dots, x_{n-1}) := F'(x_1, \dots, x_{n-1})(x_n), \quad (x_1, \dots, x_{n-1}) \in \mathfrak{M}^{n-1}, \ x_n \in \bar{P}_n.$$

As in Lemma 4.2, one proves that $g' \in \mathcal{O}(\bar{P}_n, \mathcal{O}(\mathfrak{M}^{n-1}, X))$. Applying Theorem 1.1 to g', we construct a function $G' \in \mathcal{O}(\mathfrak{M}, \mathcal{O}(\mathfrak{M}^{n-1}, X))$ such that

$$G'|_{\bar{P}_n} = g' \quad \text{and} \quad \|G'\|_{\mathcal{O}(\mathfrak{M},\mathcal{O}(\mathfrak{M}^{n-1},X))} \le C\|G'\|_{\mathcal{O}(\bar{P}_n,\mathcal{O}(\mathfrak{M}^{n-1},X))} \le C^n\|f\|_{\mathcal{O}(\bar{P},X)}.$$

We set

$$F(x_1,\dots,x_n) := G'(x_n)(x_1,\dots,x_{n-1}), \quad (x_1,\dots,x_n) \in \mathfrak{M}^n.$$

Then F satisfies the required conditions. ∎

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