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A CHARACTERIZATION OF VARIETIES WITH A DIFFERENCE TERM

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ABSTRACT. We characterize, by means of congruence identities, all varieties having a weak difference term, and all neutral varieties. Our characterization of varieties with a difference term is new even in the particular case of locally finite varieties.

1. **Introduction.** The study of difference terms arose in connection with commutator theory in universal algebra. The tale begins with [Sm], who defined, for congruences, a commutator operation sharing all the good properties of the commutator of normal subgroups in a group, and of the ideal product in commutative rings as well. Smith's theory worked only for permutable varieties, but was soon generalized by [HH] to modular varieties, and played an essential role in the development of a refined theory of such varieties ([FMK], [Gu]).

Outside modular varieties, the commutator is not so well behaved; anyway, it has proved useful: see for example [HMK], [Lp1]. Moreover, a large part of the theory of the modular commutator can be carried over using only the existence of a difference term ([Lp2], [Ke2]); surprisingly, many results follow already from the existence of a weak difference term!

Roughly, a *difference term* is a ternary term satisfying the Mal'cev identities for permutability, when equality is replaced on one side by congruence modulo some commutator. When we perform the replacement on both sides, we get a *weak difference term*.

In a modular variety there always exists a difference term. A locally finite variety has a weak difference term iff it omits type 1 [HMK]. It is quite trivial to show that a variety has a weak difference term iff every block of every abelian congruence is affine (that is, has a module structure) [Lp3]. In [HMK] locally finite varieties having a weak difference term are characterized also by some congruence identity: the aim of the present work is to show that, under some mild assumptions, this characterization holds for arbitrary varieties. In a similar way, we characterize neutral varieties, and varieties with a difference term: this last characterization seems to be new even in the case of locally finite varieties (Corollary 4.5).

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2. **Preliminaries.** We assume the reader is familiar with the standard notions of universal algebra (as found, *e.g.*, in [BS] or [MKNT]). Our notation is essentially the same as in [FMK].

Strictly speaking, familiarity with commutator theory (inside or outside modular varieties) is not required; nevertheless, it could be of great help (for commutator theory, see the references quoted in the introduction).

We now recall the basic definitions:

DEFINITION 2.1. Suppose that $\mathbf{A}$ is an algebra, and $\alpha, \beta, \gamma \in \operatorname{Con}\mathbf{A}$.
$\bar{a}, \bar{b}, \ldots$ denote n-tuples of elements; and $\bar{a}\alpha\bar{b}$ means $a_1\alpha b_1, a_2\alpha b_2, \ldots$.
We let $M(\alpha, \beta)$ denote the set of all matrices of the form:

$$\begin{vmatrix} t(\bar{a}, \bar{b}) & t(\bar{a}, \bar{b}') \\ t(\bar{a}', \bar{b}) & t(\bar{a}', \bar{b}') \end{vmatrix}$$

where $\bar{a}, \bar{a}' \in A^n$, $\bar{b}, \bar{b}' \in A^m$, for some $n, m \geq 0$, t is any $m+n$-ary term operation of $\mathbf{A}$, and $\bar{a}\alpha\bar{a}'$, $\bar{b}\beta\bar{b}'$.

We put $K(\alpha, \beta; \gamma) = \{(c, d) \mid \begin{vmatrix} a & b \\ c & d \end{vmatrix} \in M(\alpha, \beta)$, for some $a\gamma b \in A\}$; and we say that α *centralizes* β *modulo* γ, $C(\alpha, \beta; \gamma)$ for short, iff $K(\alpha, \beta; \gamma) \leq \gamma$. $[\alpha, \beta]$ is the least congruence γ such that $C(\alpha, \beta; \gamma)$ holds; $[\alpha, \beta]$ is called the *commutator* of α and β.

The condition $C(1, 1; 0)$ is sometimes called the *Term Condition.*

A congruence α on an algebra $\mathbf{A}$ is *abelian* iff $[\alpha, \alpha] = 0_{\mathbf{A}}$, and an algebra $\mathbf{A}$ is *abelian* if $1_{\mathbf{A}}$ is an abelian congruence. $\mathbf{A}$ is *affine* iff $\mathbf{A}$ has the same polynomial operations of a module over some ring.

An algebra $\mathbf{A}$ is *neutral* iff $\mathbf{A} \models_{\mathrm{Con}} [\alpha, \alpha] = \alpha$, or, equivalently, $\mathbf{A} \models_{\mathrm{Con}} [\alpha, \beta] = \alpha\beta$. A variety V is *neutral* iff every algebra in V is neutral.

A ternary term t is a *weak difference term* for an algebra $\mathbf{A}$ iff $a[\alpha, \alpha]t(a, b, b)$ and $t(a, a, b)[\alpha, \alpha]b$, for every $\alpha \in \operatorname{Con}\mathbf{A}$ and $a\alpha b \in A$. t is a *difference term* for $\mathbf{A}$ iff $a = t(a, b, b)$ and $t(a, a, b)[\alpha, \alpha]b$, for every $\alpha \in \operatorname{Con}\mathbf{A}$ and $a\alpha b \in A$. V has a (weak) difference term iff there is a term which is a (weak) difference term for every algebra in V.

When we speak of *the* commutator, we always refer to $[\alpha, \beta]$, the operation just introduced (the TC-commutator). It must be remarked, however, that many other operations have some right to receive the title of *commutators*: generally, they coincide with the preceding one only assuming congruence modularity. We believe that those alternative commutators will play an important role in subsequent developments of the theory. Hints can be perceived in [Qu], [Ki, § 3], [Lp4] and [Kel, § 5].

Each of those commutators has its own salient features; for example, what makes the TC-commutator useful is mainly left semidistributivity: $[\alpha, \beta] = [\gamma, \beta]$ implies $[\alpha, \beta] = [\alpha + \gamma, \beta]$. It seems that a decent commutator theory can be developed only under the assumption that some of the various commutators coincide: as we mentioned above, there is a quite good commutator theory for varieties having a weak difference term: but what this assumption really means is that, for congruences, "abelian" (in the sense of

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the TC-commutator) is the same as "affine", a structural characterization of, say, abelian groups.

Another important example can be found in [Ki]: many fundamental properties of the TC-commutator in modular varieties follow from the fact that it is identical with the 2TC-commutator (see also [DG], [KMK], [Lp4]). Indeed, in the present paper, too, we need the 2TC-commutator (it is still open whether or not its use can be avoided).

DEFINITION 2.2. Suppose that $\mathbf{A}$ is an algebra, and $\alpha, \beta, \gamma \in \operatorname{Con}\mathbf{A}$.

We put $K_{2T}(\alpha,\beta;\gamma) = \{(d,d'') \mid \begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a' & b' \\ c' & d' \end{vmatrix} \in M(\alpha,\beta)$, for some $a\gamma a'$, $b\gamma b'$ and $c\gamma c' \in A\}$; and we say that α *centralizes* β *modulo* γ *in the sense of the Two Terms Commutator*, $C_{2T}(\alpha,\beta;\gamma)$ for short, iff $K_{2T}(\alpha,\beta;\gamma) \leq \gamma$. $[\alpha,\beta]_{2T}$ is the least congruence γ such that $C_{2T}(\alpha,\beta;\gamma)$ holds; $[\alpha,\beta]_{2T}$ is called the *2TC-commutator* of α and β.

The condition $C_{2T}(1,1;0)$ is called the *Two Terms Condition* in [Ki]; but it originates from [Qu]. It is quite easy to show that $[\alpha,\beta] \leq [\alpha,\beta]_{2T}$ ([Ki]).

DEFINITION 2.3. If P is any property which can be satisfied by congruences of algebras, we write $\mathbf{A} \models_{\mathrm{Con}} P$ iff the set of all congruences of $\mathbf{A}$ satisfies P. If V is a variety, $V \models_{\mathrm{Con}} P$ means that $A \models_{\mathrm{Con}} P$ for every $\mathbf{A} \in \mathbf{V}$.

The operations and relations we shall use in building P will be the commutators, composition, denoted by $\circ$, intersection (of binary relations) denoted by juxtaposition; and inclusion, denoted by $\leq$.

Given α, β, γ congruences, define recursively:

$$\beta_0 = \gamma_0 = 0; \quad \beta_{n+1} = \beta + \alpha\gamma_n; \quad \gamma_{n+1} = \gamma + \alpha\beta_n.$$

■ denotes the end of proof.

3. A characterization of varieties with a weak difference term.

THEOREM 3.1. *A variety V has a weak difference term if and only if:*

(a) $V \models_{\mathrm{Con}} [\alpha,\alpha] = 0 \Rightarrow [\alpha,\alpha]_{2T} = 0$*; and*

(b) there is some positive integer n such that $V \models_{\mathrm{Con}} \alpha(\beta\circ\gamma) \leq \alpha\beta_n \circ \gamma \circ \beta \circ \alpha\beta_n$.

PROOF. Suppose that V has a weak difference term t. We first prove (a).

If $[\alpha,\alpha] = 0$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix}, \begin{vmatrix} a & b \\ c & d' \end{vmatrix} \in M(\alpha,\alpha)$, then $d\alpha c\alpha d'$, so that $t(d,d,d') = d'$, and

$$\begin{vmatrix} t(b,a,a) & t(b,b,b) \\ t(d,c,c) & t(d,d,d') \end{vmatrix} = \begin{vmatrix} b & b \\ d & d' \end{vmatrix} \in M(\alpha,\alpha),$$

so that $d = d'$, since $[\alpha,\alpha] = 0$. This proves $[\alpha,\alpha]_{2T} = 0$.

In order to prove (b), observe that, if $a\beta b\gamma c$ and $a\alpha c$, then

$$a[\alpha(\beta+\gamma),\alpha(\beta+\gamma)]t(a,c,c)\gamma t(a,b,c)\beta t(a,a,c)[\alpha(\beta+\gamma),\alpha(\beta+\gamma)]c,$$

so that

$$(*) \qquad \alpha(\beta\circ\gamma) \leq (\cup_{n\in\omega}\alpha\beta_n)\circ\gamma\circ\beta\circ(\cup_{n\in\omega}\alpha\beta_n)$$

follows from [Lp1, Lemma 1(ii)]. It is then a standard argument to show that, for a whole variety, (*) implies (b).

Indeed, let $F_V(3) = F_V(x,y,z)$ be the free algebra in V generated by three elements, and let $\alpha = Cg(x,z)$, $\beta = Cg(x,y)$, $\gamma = Cg(y,z)$. (*) now yields ternary terms s, t, u such that x and $s(x,y,z)$ ($u(x,y,z)$ and z, respectively) are congruent modulo $\bigcup_{n\in\omega}\alpha\beta_n$ in $F_V(3)$; and moreover $s\gamma t$ and $t\beta u$.

This implies that, for some n, x and $s(x,y,z)$ are congruent modulo $\alpha\beta_n$; and the same for $u(x,y,z)$ and z. This furnishes a finite number of terms which, in a standard way, can be used to show that (b) holds throughout V.

Suppose now that (b) holds. Let $F_V(3)$, α, β, γ and s, t, u be as above. We want to show that t is a weak difference term for V.

So suppose that $\mathbf{A} \in V$, and $\delta \in \operatorname{Con}\mathbf{A}$. It is enough to show that, if $[\delta,\delta] = 0$, and $a\delta b\delta c \in A$, then $a = s(a,b,c)$, so that $a = t(a,b,b)$, since $s(x,y,y) = t(x,y,y)$ is an identity in V. (Assuming $[\delta,\delta] = 0$ is no loss of generality, since, if $[\delta,\delta] \neq 0$, we could have worked in $\mathbf{A}/[\delta,\delta]$).

$t(a,a,b) = b$ is obtained in a symmetrical way.

CLAIM. Suppose that (a) holds, $\mathbf{A} \in V$, $\delta \in \operatorname{Con}\mathbf{A}$, $[\delta,\delta] = 0$ and that v,w are ternary terms such that $v(e,f,e) = w(e,f,e)$ and $v(e,e,f) = w(e,e,f)$, for every $e\delta f \in A$. Then $v = w$ on every δ-block, that is, $v(e,f,g) = w(e,f,g)$ for every $e\delta f\delta g$.

PROOF OF THE CLAIM. By (a), $[\delta,\delta]_{2T} = 0$, so that, for every $e\delta f\delta g \in A$:

$$\begin{vmatrix} v(f,e,f) & v(e,e,g) \\ v(f,f,f) & v(e,f,g) \end{vmatrix} \text{ and } \begin{vmatrix} w(f,e,f) & w(e,e,g) \\ w(f,f,f) & w(e,f,g) \end{vmatrix} \in M(\delta,\delta);$$

this implies $v(e,f,g) = w(e,f,g)$; that is, v and w coincide within any δ-block. ∎

END OF PROOF OF THEOREM 3.1. Now, suppose that $v\alpha(\gamma\circ\alpha\beta\circ\cdots\circ\alpha\beta\circ\gamma)w$ in $F_V(3)$. This means that $v\gamma v_1\alpha\beta w_1 \cdots v_m\alpha\beta w_m\gamma w$, for appropriate terms $v_1, w_1, \ldots, v_m, w_m$. This means that, *e.g.*, $v_1(x,y,x) = w_1(x,y,x)$, and $v_1(x,x,y) = w_1(x,x,y)$ is an identity in V. Then, by the Claim, $v_1 = w_1, \ldots, v_m = w_m$ on every δ-block, so that $v(e,f,f) = w(e,f,f)$, for every $e\delta f \in A$.

But $v\alpha w$, hence $v(x,y,x) = w(x,y,x)$ holds in V; whence, by the symmetric version of the Claim, $v = w$ on every δ-block.

Proceeding in such a way by induction (starting from the expressions inside the internal parenthesis of $\alpha\beta_n$), we get the desired conclusion that $a = s(a,b,c)$ on every δ-block. ∎

MAIN PROBLEM 3.2. We do not know whether condition (a) can be omitted in Theorem 3.1. That is, does (b) alone imply the existence of a weak difference term? (This is true in locally finite varieties by [HMK, Theorem 9.6]. It is also true if everywhere we deal with tolerances rather than congruences, except that the commutator of tolerances is required to be a congruence).

We can show that $\alpha(\beta\circ\gamma) \leq \alpha(\beta+\alpha\gamma)\circ\gamma\circ\beta\circ\alpha(\gamma+\alpha\beta)$ implies the existence of a weak difference term, but the argument does not seem to carry over for, say, $\alpha\beta_3$.

REMARKS 3.3. (a) It is not difficult to show that if V has a weak difference term, then $V \models_{\mathrm{Con}} [\alpha,\alpha] = [\alpha,\alpha]_L$, this last being the *linear* commutator ([Qu], [KQ]). Indeed, if $[\alpha,\alpha] = 0$, and $\begin{vmatrix} a & b \\ c & d \end{vmatrix} \in M(\alpha,\alpha)$, then

$$\begin{vmatrix} t(a,b,b) & t(a,a,b) \\ t(c,b,b) & t(c,a,b) \end{vmatrix} = \begin{vmatrix} a & b \\ c & t(c,a,b) \end{vmatrix} \in M(\alpha,\alpha),$$

so that $d = t(c,a,b)$ by the proof of (a) in Theorem 3.1. That $[\alpha,\alpha]_L = 0$ follows now from the fact that a group operation can be added to every α-block in such a way that $t(c,a,b) = c - a + b$: use, *e.g.*, Herrmann's Theorem [FMK, Corollary 5.9] applied to each α-block, considered as an algebra, under restriction (it belongs to a permutable variety since t is a Mal'cev term on every α-block, by above and $[\alpha,\alpha] = 0$).

On the contrary, there are examples of varieties with a weak difference term and with congruences α and β such that $[\alpha,\beta] \neq [\beta,\alpha]$ ([KP], [Lp2]). This shows that the commutator may be distinct from the linear commutator, in a variety with a weak difference term, since $[\alpha,\beta]_L = [\beta,\alpha]_L$ holds for every α and β.

(b) The existence of a weak difference term, and 3.1(b) as well, are Mal'cev conditions ([Lp1], [Lp3]); but 3.1(a) is not equivalent to a Mal'cev condition: every algebra with only unary operations satisfies 3.1(a).

(c) The proof of Theorem 3.1 shows that V has a weak difference term iff $F_V(3)$ has. Indeed, it can be shown that V has a weak difference term iff $F_V(2)$ has a weak difference term [Lp3].

(d) For every even $m \geq 2$ condition (b) in 3.1 can be replaced by

(b_m) there is some positive integer n such that:

$$\alpha\,\underbrace{(\beta \circ \gamma \circ \beta \circ \cdots)}_{m \text{ factors}} \leq \alpha\beta_n \circ \underbrace{\gamma \circ \beta \circ \cdots}_{m \text{ factors}} \circ \alpha\beta_n.$$

(thus, (b) is (b_2)).

Indeed, if V has a weak difference term, the same argument used in proving (b) proves (b_m).

On the other side, if V satisfies (b_m) then the arguments of [HM] yield ternary terms $t_1, \ldots, t_{m-1}$ such that $t_1(x,x,y) = t_2(x,y,y), \ldots, t_{m-2}(x,x,y) = t_{m-1}(x,x,y)$ hold in V, and the proof of Theorem 3.1 shows that $a[\alpha,\alpha]t_1(a,b,b)$ and $t_{m-1}(a,a,b)[\alpha,\alpha]a$ whenever $a\alpha b$.

The existence of a weak difference term follows now from the methods of [Ta] (see [Lp5] for more details).

The proof of Theorem 3.1 gives something more:

THEOREM 3.4. *Suppose that $F_1(\alpha,\beta,\gamma), \ldots, F_i(\alpha,\beta,\gamma)$ are lattice terms such that in every lattice $\alpha\beta = \alpha\gamma = \beta\gamma = 0$ implies that $F_1(\alpha,\beta,\gamma) = \cdots = F_i(\alpha,\beta,\gamma) = 0$; and let p be a term built using $\circ$, + and $\cap$. Then for every variety V (a) $\Rightarrow$ (b), (c) $\Rightarrow$ (d), and there is n such that (b) implies the particular case of (c) in which $F_1 = \cdots = F_i = \alpha\beta_n$ where:*

(a) $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq p(\alpha, \beta, \gamma, [\alpha, \alpha], \ldots, [\alpha, \alpha])$.
(b) $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq p(\alpha, \beta, \gamma, [\beta + \gamma, \alpha], \ldots, [\beta + \gamma, \alpha])$.
(c) $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq p\big(\alpha, \beta, \gamma, F_1(\alpha, \beta, \gamma), \ldots, F_i(\alpha, \beta, \gamma)\big)$.
(d) $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq p(\alpha, \beta, \gamma, [\alpha, \alpha]_{2T}, \ldots, [\alpha, \alpha]_{2T})$.

Theorem 3.4 generalizes Theorem 3.1 since it is quite easy to show that a variety V has a weak difference term if and only if $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq [\alpha, \alpha] \circ \gamma \circ \beta \circ [\alpha, \alpha]$ (and also if and only if $V \models_{\mathrm{Con}} \beta \circ \gamma \leq [\beta, \beta] \circ \gamma \circ \beta \circ [\gamma, \gamma]$. Compare [Ke2, Lemma 2.7]). Also Theorems 4.1 and 4.3 could be obtained as consequences of Theorem 3.4, in a similar fashion.

PROBLEM 3.5. Prove or disprove: if V satisfies some nontrivial congruence identity (in the language with $\cap$, $\circ$, $+$) then V has a weak difference term (true for locally finite varieties [HMK, Remark 9.7]).

4. **Some variations.**

The proof of Theorem 3.1 gives, without any essential change:

THEOREM 4.1. *A variety V has a difference term if and only if:*
(a) $V \models_{\mathrm{Con}} [\alpha, \alpha] = 0 \Rightarrow [\alpha, \alpha]_{2T} = 0$; *and*
(b) there is some positive integer n such that $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \gamma \circ \beta \circ \alpha\beta_n$. ∎

REMARK 4.2. By [Lp2, Lemma 3.1(ii), and proof of Theorem 3.2], for every $j \geq 2$ condition (b) in Theorem 4.1 can be equivalently replaced by:
(b^j) there is some positive integer n such that

$$V \models_{\mathrm{Con}} \alpha\underbrace{(\beta \circ \gamma \circ \beta \cdots)}_{j \text{ factors}} \leq \gamma \circ \beta \circ \alpha\beta_n.$$

(b) can also be replaced by
(b^∞) $\alpha(\beta + \gamma) \leq \gamma \circ \beta \circ \bigcup_{n \in \omega} \alpha\beta_n$.
On the contrary, it is not possible to replace (b) with

$$(*) \qquad \text{there exists } n \text{ such that } \alpha(\beta + \gamma) \leq \gamma \circ \beta \circ \alpha\beta_n$$

If a variety satisfies $(*)$, then it satisfies the nontrival lattice identity $\alpha(\beta + \gamma) \leq \alpha\big(\gamma(\alpha + \beta) + \beta(\alpha + \gamma)\big) + \alpha\beta_n$; but semilattices have a difference term and satisfy no nontrival lattice identity.

THEOREM 4.3. *A variety V is neutral if and only if:*
(a) $V \models_{\mathrm{Con}} [\alpha, \alpha] = 0 \Rightarrow [\alpha, \alpha]_{2T} = 0$; *and*
(b) there is some positive integer n such that $V \models_{\mathrm{Con}} \alpha(\beta \circ \gamma) \leq \alpha\beta_n$.
For every $j \geq 2$ condition (b) could be equivalently replaced by:
(b^j) there is some positive integer n such that

$$V \models_{\mathrm{Con}} \alpha\underbrace{(\beta \circ \gamma \circ \beta \cdots)}_{j \text{ factors}} \leq \alpha\beta_n.$$

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Condition (b) can also be replaced by:

(b^∞) $V \models_{\mathrm{Con}} \alpha(\beta+\gamma) \leq \bigcup_{n\in\omega}\alpha\beta_n$.

PROOF. $\alpha(\beta\circ\gamma) \leq \alpha(\beta+\gamma) = [\beta+\gamma,\alpha] \leq \bigcup_{n<\omega}\alpha\beta_n$. The rest is similar to Theorem 3.1. ∎

Compare Theorem 4.3 with [HMK, Theorem 9.10].

PROBLEM 4.4. Again, we do not know whether (a) can be omitted in Theorems 4.1 and 4.3.

Notice that an affirmative answer would imply that the existence of a difference term is "permutability composed with neutrality" in a sense very similar to Gumm's result that modularity is permutability composed with distributivity.

Using a result by Hobby and McKenzie, however, we can show that (a) can be actually omitted in the particular case of locally finite varieties.

COROLLARY 4.5. *If V is a locally finite variety, then the following are equivalent:*

(a) V has a difference term;

(b) $V \models_{\mathrm{Con}} \alpha(\beta+\gamma) \leq \gamma\circ\beta\circ\bigcup_{n\in\omega}\alpha\beta_n$.

(c) For every (equivalently, some) $j \geq 2$ there is a positive integer n such that

$$V \models_{\mathrm{Con}} \alpha\underbrace{(\gamma\circ\beta\circ\gamma\cdots)}_{j\ \text{factors}} \leq \beta\circ\gamma\circ\alpha\beta_n.$$

(d) There is some positive integer n such that $V \models_{\mathrm{Con}} \alpha(\gamma\circ\beta) \leq \beta\circ\gamma\circ\alpha\beta_n$.

PROOF. (a) $\Rightarrow$ (b) is from Remark 4.2.

(b) $\Rightarrow$ (c) is an argument in the proof of Theorem 3.1(b).

(c) $\Rightarrow$ (d) is trivial. (Anyway, (a) $\Rightarrow$ (d) is one half of Theorem 4.1).

(d) $\Rightarrow$ (a) By [HMK, Theorem 9.6] V has a weak difference term. By Theorem 3.1, $V \models_{\mathrm{Con}} [\alpha,\alpha] = 0 \Rightarrow [\alpha,\alpha]_{2T} = 0$; and by Theorem 4.1, V has a difference term. ∎

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