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CONJUGACY CLASS REPRESENTATIVES IN FISCHER'S BABY MONSTER

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Abstract

A set of conjugacy class representatives is given in this paper for the elements in Fischer's Baby Monster simple group, up to inversion.

1. *Introduction*

Fischer's Baby Monster group is the second-largest of the 26 sporadic simple groups, and has order greater than 4×10^{33}. Many of its basic properties were described by Fischer, and its character table was computed by Hunt (see [**1**]). It was first constructed by Leon and Sims [**2**], essentially as a permutation group on some 10^{10} points.

In [**9**], the author constructed the 4370-dimensional representation of the Baby Monster over GF(2). In [**5**], the 4371-dimensional representation was constructed over GF(3) and GF(5). The method could in principle be applied also to construct the representation over any field of odd characteristic.

These matrix constructions now give almost enough invariants to distinguish all conjugacy classes of elements. In this paper we use this information to produce a complete set of conjugacy class representatives for the Baby Monster.

We note first that it is sufficient to find representatives for the maximal cyclic subgroups, because conjugacy class representatives for the elements can then be found as suitable powers of the given generators of the maximal cyclic subgroups. It is easy to calculate from the class list and power maps that there are just 76 classes of maximal cyclic subgroups. The Atlas names for these are listed in Table 2. We also include in this table the main result of the paper, namely words for generators of such maximal cyclic subgroups, using the elements a to z defined in Table 1. This table also gives the orders of these elements. The derivation of the words in Table 2 is the main purpose of this paper.

2. *Distinguishing conjugacy classes*

Our main tool here is the character table (see [**1**]). All the classes of maximal cyclic subgroups of odd order are determined by the order. These are $25A$, $27A$, $31AB$, $39A$, $47AB$, and $55A$. Thus we concentrate on the classes of elements of even order from now on. We work as far as possible in the 2-modular representation, as that is much faster to work in than the others. As well as the order, we use the trace as a cheap but not very useful class invariant. We also calculate the dimension of the fixed space, which can be used later as a more discriminating invariant for even-order elements. (This is of no use for odd-order elements, as it can already be calculated from the character table.) It is possible to calculate the full Jordan block structure, but this turns out to be of little use.

In addition to the classes of odd-order elements, the types $38A$, $44A$, $46AB$, $52A$, $56AB$, $66A$, and $70A$ are determined by their orders. Thus we need only to find elements of all

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these orders in order to find representatives for these classes and all their powers. Using the trace mod 2 as well, we can characterise the classes $60C$, $40E$, and $36C$. Using the trace mod 3, we can distinguish $48A$ and $48B$, $42B$, $32\,AB$ and $32\,CD$, $28A$, and $40B$. Using both traces, we can also distinguish $40A$, $30C$ and $30\,GH$.

Since by this stage we know involutions of each class, as suitable powers of elements that have already been identified, we can calculate the codimensions of their fixed spaces modulo 2 as 1860, 2048, 2158 and 2168 for classes $2A$, $2B$, $2C$ and $2D$, respectively. This enables us to distinguish certain classes by the class of involution up to which they power. This deals with $26B$, $42C$, $34A$, $34\,BC$, $28E$, $20H$ and $18F$, and also $42A$, if we use the trace mod 3 as well.

For elements of order 60, we can distinguish between $60A$ and $60B$ by the trace mod 3 of the 5th power. For elements of order 40, we distinguish $40C$ and $40D$ from the rest by the trace mod 2 and mod 3, and then distinguish them from each other by the trace mod 3 of the 10th power. For elements of order 36, we distinguish $36A$ and $36B$ from $36C$ by the trace mod 2, and from each other by the trace mod 5 of the 9th power.

For elements of order 30, we distinguish $30D$ and $30E$ from the rest by trace mod 3, and from each other by the trace mod 3 of the fifth power. Similarly, $30A$ and $30B$ can be distinguished from the rest by the trace mod 2 and 3, and from each other by the trace mod 5. For elements of order 28, the classes $28C$, $28D$ and $28E$ are distinguished by having trace 0 mod 3, and then the traces of their 7th powers are 1, 0 and 2, respectively, mod 3.

For elements of order 24, the traces mod 2 and 5 separate them into seven sets: $24A/I$, $24B/E/J$, $24C/G$, $24D$, $24F$, $24H/K/L$ and $24M/N$. Each of these sets can then be resolved by the trace mod 3 of the square. This deals with all elements of order 24. For elements of order 20, the trace mod 2 and mod 3 distinguishes $20B$, $20C$ and $20I$, and separates the rest into three classes, namely $20A/E/H$, $20D/F$ and $20G/J$, all of which can be distinguished by the trace of the fifth power mod 3.

For elements of order 18, the traces mod 2 and 3, and the 9th power, distinguish the classes except for $18A$ and $18B$. The elements of order 16 are distinguished up to ambiguities $16A/D/F$, $16C/E$ and $16B/G/H$, by traces mod 3 alone. The traces of the squares mod 3 resolve all of these except the pair $16B/H$ (which can be resolved by the trace mod 5) and the one remaining problem, that of distinguishing $16D/F$ (see below).

For elements of order 12, the traces mod 2, 3, and 5 distinguish classes except for the following ambiguities: $12B/K/Q$, $12F/O$, $12H/P$, $12J/L$, $12M/R$ and $12N/T$. The first is not required for our purposes, while the rest can be distinguished by the class of the 6th power (determined as above by the codimension of its fixed space).

3. *Difficult cases*

While most classes are fairly easily found by the above methods, we encountered a few problem cases. These are the classes $18A$ and $18B$, which cannot be distinguished by traces and power maps alone; $16F$, which similarly cannot be distinguished from $16D$; and $12A$, which is such a small conjugacy class that finding a $12A$-element at random is impracticable.

3.1. *The classes* $18A$ *and* $18B$

We found two elements, each of which is in either $18A$ or $18B$, but one has fixed space of dimension 280 in the 4370-space over GF(2), while the other has fixed space dimension 282. Thus one of them is in $18A$ and the other is in $18B$, but without extra information

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we cannot tell which is which. For example, we might look inside the involution centralizer: each powers up to a $2A$-element, and inside the $2A$-centralizer $2^{\cdot 2}E_6(2){:}2$, one maps to a $9A$-element and the other maps to a $9B$-element. However, these elements are still not easy to distinguish in this subgroup.

An alternative approach is to look inside a subgroup Fi_{23}. We first find such a subgroup, with standard generators $(e^{20})^{c^{10}}$ and $(g^4)^{d^9}$ given in terms of the words in Table 1 in generators a and b of the Baby Monster. Words for representatives of all the conjugacy classes of elements in Fi_{23} are given in [8] (see also [11]), and we can test the elements of order 18. We found that an element of Fi_{23}-class $18A$ has fixed space dimension 282. This class corresponds to class $-9A$ in $2^{\cdot}\,\mathrm{Fi}_{22}$, which fuses to class $-9A$ in $2^{\cdot 2}E_6(2)$, and thence to $18A$ in B.

3.2. *The class* $16F$

To resolve the classes $16D$ and $16F$, we adopted a slightly different approach, which involved finding the centralizer order directly. Suppose that we have an element x in one of these two classes, and work in $C(x^8) \cong 2^{1+22\cdot}\,\mathrm{Co}_2$. Now x is conjugate to x^9, so $C(x)$ has the same order as the centralizer of its image in $C(x^8)/(x^8)$. We calculate the latter as follows.

First, we calculate the involution centralizer (in the 2-modular representation) by one of the standard methods, and chop the representation to obtain an irreducible 22-dimensional constituent. We use the latter to find standard generators for the quotient Co_2, and hence find words for a subgroup $\mathrm{U}_6(2){:}2$ thereof.

Next, we switch to the 3-modular representation, and again chop the restriction to the involution centralizer. We take the 2300-dimensional constituent, and find the invariant 1-space of the group $2^{22}.\,\mathrm{U}_6(2).2$ which acts on it. A vector in this 1-space has 4600 images under $2^{22}.\,\mathrm{Co}_2$, which acts faithfully on this orbit. Therefore we can convert to a permutation representation of $2^{22}.\,\mathrm{Co}_2$ on 4600 points, and then we use GAP [7] to find the order of the centralizer of our element quickly.

We have applied this to the element dej, and have found that its centralizer has order 1024; therefore, it is a $16F$-element.

3.3. *The class* $12A$

There is a problem with very rare classes, such as $12A$, where a purely random search would take a very long time. In this case, only about 1 in 4 million elements of the group is in the class $12A$, and it would take about three years of CPU time to make 4 million elements on a Pentium 4/1400. Indeed, even with a rapid screening procedure to eliminate elements of order not 12, we estimated a CPU-time requirement of many months, and therefore decided on a different strategy. The result is a significantly longer word, which requires 30 multiplications to make, rather than the 5 multiplications that we would expect from the random approach.

We first take an element that powers up to a $6C$-element, and find the centralizer of the involution to which it powers. For example, we can take the $48A$-element cg, and find the centralizer of $(cg)^{24}$ to be generated by cg and $x = (a(cg)^{24})^6$. We now have a group $2^{1+22}.\,\mathrm{Co}_2$ and an element $(cg)^{16}$ mapping to Co_2-class $3B$. We now search in the corresponding coset of 2^{1+22} for elements of order 12, and use the trace mod 2, 3 and 5 to test for membership in class $12A$.

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Table 1: The elements a to z

a	b	$c = ab$	$d = cb$	$e = cd$	$f = ce$	$g = fc$	$h = gd$
2	3	55	55	40	20	12	18
$i = ch$	$j = id$	$k = jd$	$l = ck$	$m = lc$	$n = md$	$o = nd$	$p = eo$
31	23	17	46	16	46	24	34
$q = pe$	$r = qc$	$s = dr$	$t = sc$	$u = no$	$v = il$	$w = gh$	
36	60	26	47	47	34	47	

$x = (a(cg)^{24})^6$	$y = ((cg)^6 x)^{12}$	$z = ((cg)^4 xcgx)^{15}$
2	2	2

Table 2: Maximal cyclic subgroups

$12A$	$12H$	$12I$	$12L$	$12P$	$12S$	$12T$	$16E$
$yz(cg)^{16}$	i^3vj	h^2n	$ehvk$	n^2snv	efj	cwh	api
$16F$	$16G$	$16H$	$18A$	$18B$	$18D$	$18F$	$20B$
dej	m	guk	mo	dtv	aoj	h	$clsh$
$20C$	$20H$	$20I$	$20J$	$24A$	$24B$	$24C$	$24D$
$cfvo$	fh^2	ij^2	fg^2	fmw	kuq	$cgon$	fu
$24E$	$24F$	$24H$	$24I$	$24J$	$24K$	$24L$	$24M$
efh	e^2fh	ps	fhj	fhg	cig	wgl	wgk
$24N$	$25A$	$26B$	$27A$	$28A$	$28C$	$28D$	$28E$
bv	cfg	cfh	hj	afr	h^2i	uw	gj
$30A$	$30B$	$30C$	$30E$	$30\,GH$	$31\,AB$	$32\,AB$	$32\,CD$
lum	agq	guq	hg^2	cjg	i	ci	cn
$34A$	$34\,BC$	$36A$	$36B$	$36C$	$38A$	$39A$	$40A$
v	fg	tw	hl	ehg	ij	cfj	nv^2
$40B$	$40C$	$40D$	$40E$	$42A$	$42B$	$42C$	$44A$
gj^2	op	e	cgh	amu	amo	gn	gi
$46\,AB$	$47\,AB$	$48A$	$48B$	$52A$	$55A$	$56\,AB$	$60A$
l	fi	cg	kn	e^2i	c	eih	djf
$60B$	$60C$	$66A$	$70A$				
nv	cg^2	gji	cih				

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Table 3: Codimensions of fixed spaces

$2A$	1860	$8G$	3810	$12Q$	4002	$20H$	4144	$30E$	4214
$2B$	2048	$8H$	3780	$12R$	4002	$20I$	4138	$30F$	4224
$2C$	2158	$8I$	3786	$12S$	4004	$20J$	4150	$30\,GH$	4216
$2D$	2168	$8J$	3812	$12T$	4002	$22A$	4140	$32\,AB$	4222
$4A$	3114	$8K$	3818	$14A$	3996	$22B$	4158	$32\,CD$	4222
$4B$	3114	$8L$	3786	$14B$	4008	$24A$	4152	$34A$	4238
$4C$	3192	$8M$	3818	$14C$	4048	$24B$	4152	$34\,BC$	4220
$4D$	3192	$8N$	3818	$14D$	4034	$24C$	4152	$36A$	4226
$4E$	3256	$10A$	3860	$14E$	4052	$24D$	4152	$36B$	4238
$4F$	3202	$10B$	3896	$16A$	4072	$24E$	4164	$36C$	4248
$4G$	3204	$10C$	3918	$16B$	4072	$24F$	4170	$38A$	4236
$4H$	3266	$10D$	3908	$16C$	4074	$24G$	4164	$40A$	4242
$4I$	3264	$10E$	3920	$16D$	4074	$24H$	4182	$40B$	4242
$4J$	3266	$10F$	3932	$16E$	4074	$24I$	4176	$40C$	4242
$6A$	3486	$12A$	3936	$16F$	4074	$24J$	4178	$40D$	4250
$6B$	3510	$12B$	3942	$16G$	4094	$24K$	4174	$40E$	4258
$6C$	3566	$12C$	3936	$16H$	4094	$24L$	4186	$42A$	4242
$6D$	3534	$12D$	3936	$18A$	4088	$24M$	4176	$42B$	4242
$6E$	3606	$12E$	3958	$18B$	4090	$24N$	4186	$42C$	4258
$6F$	3604	$12F$	3996	$18C$	4110	$26A$	4196	$44A$	4254
$6G$	3596	$12G$	3962	$18D$	4124	$26B$	4176	$46\,AB$	4270
$6H$	3610	$12H$	3962	$18E$	4128	$28A$	4188	$48A$	4266
$6I$	3636	$12I$	3964	$18F$	4122	$28B$	4200	$48B$	4266
$6J$	3638	$12J$	3986	$20A$	4114	$28C$	4188	$52A$	4284
$6K$	3634	$12K$	3978	$20B$	4114	$28D$	4200	$56\,AB$	4284
$8A$	3774	$12L$	3966	$20C$	4114	$28E$	4210	$60A$	4280
$8B$	3738	$12M$	3978	$20D$	4114	$30A$	4190	$60B$	4286
$8C$	3738	$12N$	4000	$20E$	4128	$30B$	4190	$60C$	4296
$8D$	3778	$12O$	3982	$20F$	4132	$30C$	4212	$66A$	4286
$8E$	3738	$12P$	3988	$20G$	4148	$30D$	4206	$70A$	4292
$8F$	3780								

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4. *Further remarks*

In fact, we found our elements in various classes by employing methods other than those described above. The point is that if we can guess the class correctly, then it is easy to prove that our guess is correct by using the criteria in the previous sections. We actually used various tables of partial information collected over the years (with a few mistakes in) to help us find elements in various classes, and then proved them as above.

Finally, having produced elements in each of the conjugacy classes, we could tabulate information that may not be easily obtainable by other means. In particular, we have calculated the dimensions of the fixed spaces of all the elements of even order on the 4370-dimensional module over GF(2). This is a useful conjugacy-class invariant, which can often be used to identify the conjugacy class of a given element more quickly than the methods described above. We tabulate this information in Table 3.

References

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker and R. A. Wilson, *An ATLAS of Finite Groups* (Oxford University Press, 1985). 175, 175

2. J. S. Leon and C. C. Sims, 'The existence and uniqueness of a simple group generated by {3, 4}-transpositions', *Bull. Amer. Math. Soc.* 83 (1977) 1039–1040. 175

3. R. A. Parker, 'The computer calculation of modular characters (The "Meat-axe")', *Computational group theory* (ed. M. D. Atkinson, Academic Press, 1984) 267–274.

4. R. A. Parker and R. A. Wilson, 'Computer construction of matrix representations of finite groups over finite fields', *J. Symbolic Comput.* 9 (1990) 583–590.

5. R. A. Parker and R. A. Wilson, 'Constructions of Fischer's Baby Monster over fields of characteristic not 2', *J. Algebra* 229 (2000) 109–117. 175

6. M. Ringe, *The C Meat-axe* 2.3 *documentation* (Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1995).

7. M. Schönert *et al.*, GAP – *groups, algorithms, and programming*, 5th edn (Lehrstuhl D für Mathematik, Rheinisch Westfälische Technische Hochschule, Aachen, Germany, 1995). 177

8. I. A. I. Suleiman, P. G. Walsh and R. A. Wilson, 'Conjugacy classes in sporadic simple groups', *Comm. Algebra* 28 (2000) 3209–3222. 177

9. R. A. Wilson, 'A new construction of the Baby Monster, and its applications', *Bull. London Math. Soc.* 25 (1993) 431–437. 175

10. R. A. Wilson, 'Standard generators for sporadic simple groups', *J. Algebra* 184 (1996) 505–515.

11. R. A. Wilson *et al.*, 'A world-wide-web atlas of group representations', 1995–2002, http://www.mat.bham.ac.uk/atlas/. 177

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