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A FAMILY OF PLANE CURVES WITH MODULI $3g - 4$

ABEL CASTORENA
Instituto de Matemáticas, Unidad Morelia, Universidad Nacional Autónoma de México Apdo, Postal 61-3(Xangari), C.P. 58089, Morelia, Michoacán, MEXICO
e-mail: abel@matmor.unam.mx

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Abstract. In the moduli space $\mathcal{M}_g$ of smooth and complex irreducible projective curves of genus g, let $\mathcal{GP}_g$ be the locus of curves that do not satisfy the Gieseker-Petri theorem. Let $\mathcal{GP}^1_{g,d}$ be the subvariety of $\mathcal{GP}_g$ formed by curves C of genus g with a pencil $g^1_d = (V, L) \in G^1_d(C)$ free of base points for which the Petri map $\mu_V : V \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)$ is not injective. For $g \geq 8$, we construct in this work a family of irreducible plane curves of genus g with moduli $3g - 4$ in $\mathcal{GP}^1_{g,g-2}$.

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1. Statement of results. Let $\mathcal{M}_g$ be the moduli space of smooth and complex irreducible projective curves of genus g. Let $C \in \mathcal{M}_g$ and let K_C be the canonical bundle of C. The Gieseker-Petri theorem (cf. [**9**, p. 285]) says that for every line bundle L on a general curve $C \in \mathcal{M}_g$, the Petri map $\mu_L : H^0(C, L) \otimes H^0(K_C \otimes L^{-1}) \to H^0(C, K_C)$ is injective. This implies that the Gieseker-Petri locus defined as

$$\mathcal{GP}_g := \{C \in \mathcal{M}_g | C \text{ does not satisfy the Gieseker-Petri theorem.}\}$$

is a proper closed Zariski subset in $\mathcal{M}_g$. It is an old and open problem to show that $\mathcal{GP}_g$ is a divisor. For $g = 7$, $\mathcal{GP}_7$ is a divisor (cf. [**4**]). Other results related with some components of $\mathcal{GP}_g$ are given in ([**1**], [**6**], [**7**], [**10**], [**11**]).

Let $C \in \mathcal{M}_g$ be and $L \to C$ a line bundle of degree d with $r + 1 = h^0(C, L)$. The Brill-Noether number is defined as $\rho(g, d, r) := h^0(C, K_C) - h^0(C, L)h^0(C, K_C \otimes L^{-1}) = g - (r + 1)(g - d + r)$. Consider the varieties $W^r_d := \{L \in \mathrm{Pic}^d(C) : h^0(C, L) \geq r + 1\}$, and $G^r_d(C) := \{(V, L) : V \subseteq H^0(C, L), \dim V = r + 1\}$. Denote by $\mu_V : V \otimes H^0(C, K \otimes L^{-1}) \to H^0(C, K)$ the Petri map.

Given g, d, r, consider the variety $\mathcal{C}^r_d$ which parametrizes couples (C, g^r_d), with C a smooth curve of genus g, and $g^r_d \in G^r_d(C)$. The dimension of any component of $\mathcal{C}^r_d$ is at least $3g - 3 + \rho(g, d, r)$. (cf. [**2**]).

Let $\widetilde{\mathcal{GP}^r_{g,d}} := \{(C, (V, L)) \in \mathcal{C}^r_d : (V, L)$ is free of base points with rank $(\mu_V : V \otimes H^0(C, K_C \otimes L^{-1}) \to H^0(C, K_C)) \leq g - (\rho + 1)\}$. Let $\pi : \mathcal{C}^r_d \to \mathcal{M}_g$ be the projection. Consider the image $\pi(\widetilde{\mathcal{GP}^r_{g,d}}) := \mathcal{GP}^r_{g,d} = \{C \in \mathcal{M}_g :$ there exists a base point

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free $(V, L) \in G^r_d(C)$ with μ_V not injective.}. We have a commutative diagram:

$$\begin{array}{ccc} \widetilde{\mathcal{GP}^r_{g,d}} & \xrightarrow{\text{inclusion}} & \mathscr{C}^r_d \\ \pi\big\downarrow & & \pi\big\downarrow \\ \mathcal{GP}^r_{g,d} & \xrightarrow{\text{inclusion}} & \mathcal{M}_g \end{array}$$

The codimension of $\widetilde{\mathcal{GP}^r_{g,d}}$ is $\leq \rho + 1$.

Suppose that $g, r, d \geq 1$ such that $\rho \geq 0$. For integers $g \geq 4$ and $\frac{g+2}{2} \leq k \leq g-1$, G. Farkas showed (cf. [**8**]) that $\mathcal{GP}^1_{g,k}$ has a divisorial component Z. In such component the author describes the elements in $\overline{\mathcal{GP}^1_{g,k}} \cap \Delta_1$, where Δ_1 is the divisor in $\overline{\mathcal{M}_g}$, where a general point of Δ_1 consists of a smooth curve of genus $g-1$ joined at one point to a smooth curve of genus one.

For $g \geq 8$, we construct explicity a component of $\mathcal{GP}^1_{g-2}$ of pure codimension one in $\mathcal{M}_g$ as follow.

Let C be a smooth curve of genus $g \geq 8$ with a pencil $g^1_{g-2} = (V, L)$ free of base points on C such that the residual g^2_g of the g^1_{g-2} determines a birational map onto a plane curve Γ of degree g and geometric genus g with $\delta = \frac{(g-1)(g-2)}{2} - g$ nodes as singularities. In Lemma 2.2 we show that μ_V is not injective if and only if there exists a curve G of degree $g-5$ containing $\delta - 1$ nodes of Γ. Consider the Severi variety $\mathcal{V}^{g,g}$ of plane curves of degree g and geometric genus g having only nodes as singularities (cf. [**9**, p. 30]). We consider the subvariety $\mathcal{V}^{g,g}_\delta \subset \mathcal{V}^{g,g}/PGL(3,\mathbb{C})$ formed by plane curves with exactly $\delta = \frac{(g-1)(g-2)}{2} - g = \frac{g(g-5)}{2} + 1$ nodes. Let $\mathcal{V}_g := \{\Gamma \in \mathcal{V}^{g,g}_\delta : \delta - 1 = \frac{g(g-5)}{2}$ nodes lie on a curve of degree $g-5\}$.

Consider a curve C of genus $g \geq 8$, neither trigonal nor bi-elliptic such that C has a plane projective model as in Lemma 2.2. In Lemma 2.5 we show that there exist at most finitely many pencils $(V, L) \in G^1_{g-2}(C)$ free of base points, for which the Petri map μ_V is not injective. Let $\psi : \mathcal{V}_g \to \mathcal{M}_g$ be the natural morphism and denote $\mathcal{D}_g := \psi(\mathcal{V}_g) \subset \mathcal{GP}^1_{g,g-2}$. In this paper we prove the following theorem.

THEOREM. *$\mathcal{D}_g$ has pure codimension one in $\mathcal{M}_g$.*

2. Two basic lemmas.

2.1. Let C be a smooth curve of genus g with a pencil $g^1_{g-2} = (V, L) \in G^1_{g-2}(C)$ free of base points for which the Petri map μ_V is not injective. Assume that the residual g^2_g of the pencil g^1_{g-2} induces a birational map onto a plane curve in $\mathbb{P}^2$. Let Γ be such a curve and $f : C \to \Gamma$ the normalization of Γ. We denote by $\triangle_\Gamma$ the scheme of singular points of Γ and $\triangle : = f^*(\triangle_\Gamma)$; note that $\triangle$ is a divisor of degree 2δ. By the genus formula the length of $(\triangle_\Gamma) = \delta$, i.e. $\triangle_\Gamma$ is a curvilinear scheme consisting of δ double points which can be infinitely near. We only consider the case where all $\delta = \frac{(g-1)(g-2)}{2} - g$ singularities of Γ are distinct. The following lemma is a generalization of [**4**, Proposition 2.8].

LEMMA 2.2. *Let Γ be a plane curve of degree g and geometric genus g such that Γ has only δ double points as singularities. Let $f : C \to \Gamma$ be the normalization of Γ. Then there is a curve G of degree $g-5$ such that the scheme theoretic intersection of G with*

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$\triangle_\Gamma$ has length equal to $\delta - 1$, i.e. $f^*(G)$ contains a divisor of degree $2\delta - 2$ contained in $\triangle$ if and only if C has a pencil $g^1_{g-2} = (V, L) \in G^1_{g-2}(C)$ free of base points with μ_V not injective.

Proof. First we show the part "if". I will consider the case in which the support of $\triangle_\Gamma = \{x\}$.

If the support of $\triangle_\Gamma = \{x\}$, then Γ has δ infinitely near double points. Let $\eta :=f^*(x)$. η is a divisor of degree two and $\triangle = \delta\eta$. Our hypothesis means that the pullback f^*G on C contains $(\delta - 1)\eta$. Consider the g^1_{g-2} cut out on C by the lines through x. Let ℓ_1, ℓ_2 be general such lines, cutting out on C two effective divisors $D_1, D_2 \in g^1_{g-2}$. The pullback of $G + \ell_1 + \ell_2$ contains $(\delta + 1)\eta + D_1 + D_2 \sim (\delta + 1)\eta + 2D$. By adjunction formula (cf. [**3**, p. 53]), $K_C \sim \mathcal{O}_C(g - 3)(-\triangle)$, we have that $K_C(-2D)$ is effective where $|D| = g^1_{g-2}$. Since kernel $\mu_D \simeq H^0(C, K_C(-2D))$, (cf. [**3**, p. 126]), we have the assertion.

Other extra cases can occur. These cases depend on δ and can be proved in a similar way. For example consider the case when the support of $\triangle_\Gamma$ consists of $\delta - 2$ infinitely near singular double points and one tacnode p. By hypothesis, $f^*(G)$ contains a divisor B of degree $2\delta - 2$ contained in the divisor $\triangle$ which is of degree 2δ. Consider the g^1_{g-2} cut out on C by the lines through the tacnode p of Γ. Let ℓ_1, ℓ_2 be general such lines, cutting out on C two effective divisors $D_1, D_2 \in g^1_{g-2} = |D|$. Since ℓ_1, ℓ_2 are lines through p, note that the pullback of $F := G + \ell_1 + \ell_2$ contains $B + (f^*(p) + D_1) + (f^*(p) + D_2) \sim (B + f^*(p)) + 2D + f^*(p) \sim \triangle + 2D + f^*(p)$. Since $K_C \sim \mathcal{O}_C(g - 3)(-\triangle)$, then $K_C(-2D)$ is effective, so we have a non-zero section of $H^0(C, K_C(-2D)) \simeq \ker \mu_D$.

The same argument works when the support of $\triangle_\Gamma$ consists of $\delta - 3$ infinitely near singular double points with an ordinary singular double point and one tacnode. Another case for which the proof is valid is when Γ has $\delta - 4$ singular double points and two tacnodes. Suppose now that the support of $\triangle_\Gamma = \{x_1, \ldots, x_{\delta-k}, x\}$, where $x_j = 1, \ldots, \delta - k$ are distinct singular double points; then Γ has k infinitely near singular double points. For $k = 1$ we have δ distinct ordinary singular double points which is the case we are interested in. For $k = 2$ is when Γ has one tacnode and $\delta - 2$ infinitely near singular double points. With this notation take in general any $k \leq \delta - 1$ and consider $\eta :=f^*(x)$. So the lines through x cut out on C a $|D| = g^1_{g-2}$. Consider ℓ_1, ℓ_2 two general such lines. The pullback of $G + \ell_1 + \ell_2$ contains $\triangle + 2D + \eta$, this implies that $\ker \mu_D \simeq H^0(C, K_C(-2D)) \neq 0$. Similarly other cases can be proved in this way.

Now suppose that $\ker \mu_V \neq 0$ and consider the residual $g^2_g = |K_C \otimes L^{-1}|$, where $g^1_{g-2} = (V, L)$. This g^2_g determines a birational morphism $C \to \Gamma \subset \mathbb{P}^2$. By assumption Γ has only double points as singularities. Since C fails the Gieseker-Petri theorem for the g^1_{g-2}, we have that kernel $\mu_V \simeq H^0(C, K_C \otimes L^{-2})$, (cf. [**3**, p. 126]), but $|K_C \otimes L^{-2}| \sim g^2_g - g^1_{g-2}$ is effective, so necessarily the g^1_{g-2} is cut out by a pencil of lines through a singular double point p of Γ. By adjunction formula there is a curve G of degree $g - 5$ such that G contains $\triangle_\Gamma - \{p\}$. $\square$

2.3. In general it is complicated to construct an irreducible and reduced plane curve of degree g and geometric genus g with projective model as in Lemma 2.2. However at least for $6 \leq g \leq 10$ such kind of curves exist. In ([**5**, p. 148–156]), the author show the existence of canonical surfaces in $\mathbb{P}^3$ with $p_g = 4$, degree $d = 6, 7, 8, 9, 10$ and sectional genus $g = 7, 8, 9, 10, 11$ with ordinary singularities. The general plane section is semicanonical with number of nodes $\delta = 3, 7, 12, 18, 25$

lying respectively on a curve of degree $1, 2, 3, 4, 5$. A tangent general section has degree $d = 6, 7, 8, 9, 10$ and the corresponding genus is $g = 6, \dots, 10$. Such curves have respectively nodes $\delta = 4, 8, 13, 19, 26$ with $3, 7, 12, 18, 25$ lying respectively on a curve of degree $1, 2, 3, 4, 5$.

2.4. Let C be a smooth curve of genus g. Consider the morphism $\mathrm{Pic}^d(C) \to \mathrm{Pic}^{2d}(C)$ given by $L \to L^2$ inside the Jacobian of C, $J(C)$. Note that this morphism has finite kernel.

Suppose that C is a smooth curve of genus $g \geq 8$, neither trigonal nor bi-elliptic. By Mumford theorem (cf. [**3**, p. 193]), the dimension of $W^1_{g-2}(C)$ is exactly the Brill-Noether number $\rho(g, g-2, 1) = g - 6$. Then we have that the subvariety $X_1 := \{L^2 : L \in W^1_{g-2}(C)\}$ has dimension $\rho = g - 6$ and $T_L(W^1_{g-2}(C)) \simeq T_{L^2}X_1$ inside $H^1(C, \mathcal{O}_C)$.

Let $g^1_{g-2} = (V, L) \in G^1_{g-2}(C)$ be free of base points such that μ_V is not injective. Since C is in particular non-hyperelliptic we have that dimension of kernel $\mu_V = h^0(C, K_C \otimes L^{-2}) = 1$, so there exist points $p, q \in C$ such that $K_C \otimes L^{-2} = \mathcal{O}_C(p+q)$ with $h^0(C, \mathcal{O}_C(p+q)) = 1$, then $h^0(C, K_C(-p-q)) = g-2$. If $p = q$, then $L + p$ is a theta characteristic. So we only consider the case $p \neq q$. Define inside the Jacobian of C, $J(C)$, the subvariety $X_2 := K_C - W_2(C) = \{K_C - (p+q) : p + q \in W_2(C)\} \subset W^r_{2g-4}(C)$ for $r = h^0(C, K_C(-p-q)) - 1 = g - 3$. We have that the dimension of $X_2 = 2$. Let $\mathcal{L} = K_C - (p+q) \in X_2$ be any point, then $K_C - \mathcal{L} = p + q$. The image of $\mu_{\mathcal{L}} : H^0(C, \mathcal{L}) \otimes H^0(C, K_C \otimes \mathcal{L}^{-1}) \to H^0(C, K_C)$ is equal to $H^0(C, K_C(-p-q))$, since $h^0(C, \mathcal{O}_C(p+q)) = 1$, $\mu_{\mathcal{L}}$ is injective and $T_{\mathcal{L}}X_2$ is a two dimensional subspace of $H^1(C, \mathcal{O}_C) \simeq T_0(J(C))$.

LEMMA 2.5. *Let C be a smooth curve of genus $g \geq 8$ neither trigonal nor bi-elliptic. Suppose that there exists a pencil $g^1_{g-2} = (V, L) \in G^1_{g-2}(C)$ free of base points such that the residual $g^2_g = |K_C \otimes L^{-1}|$ induces a birational morphism from C to a plane curve Γ of degree g in $\mathbb{P}^2$ with $x_1, \dots, x_\delta$ nodes all distinct and $x_1, \dots, x_{\delta-1}$ lying on curve of degree $g-5$. Then there are at most finitely many pencils $g^1_{g-2} = (V, L)$, $L \in W^1_{g-2}(C)$, free of base points with μ_V not injective.*

Proof. We are going to show that $X_1 \cap X_2$ is a finite set, where X_1 and X_2 are the subvarieties of $J(C)$ defined in 2.4. Without loss of generality we can assume that g^1_{g-2} is complete, that is, $|L| = g^1_{g-2}$. Let L be as in the hypothesis with kernel $\mu_L \neq 0$. By Lemma 2.2 we can assume that $|L|$ is cut out by lines through the node x_δ of Γ. We have that $L^2 \in X_1 \cap X_2$. If we show that $T_{L^2}X_1 \cap T_{L^2}X_2 = \{0\}$ inside $H^1(C, \mathcal{O}_C)$ we obtain that $L \in W^1_{g-2}(C)$ is an isolated point and this implies that $X_1 \cap X_2$ is a finite set.

Consider the normalization map $f : C \to \Gamma$. Let $f^*(x_\delta) = \{p, q\}$ be for some points $p, q \in C$, where $p \neq q$ because x_δ is a node. Since $L^2 \in X_1 \cap X_2$, then $K_C \otimes L^{-2} \simeq \mathcal{O}_C(p+q)$ and $h^0(C, K_C \otimes L^{-2}) =$ dimension of kernel $\mu_L = 1$, since C is in particular non-hyperelliptic. We have that dimension of $T_{L^2}X_1 =$ dimension of $T_L(W^1_{g-2}(C)) = \rho +$ dimension of kernel $\mu_L = g - 5$, and the dimension of $T_{L^2}X_2 = 2$, so $T_{L^2}X_1 \cap T_{L^2}X_2 = \{0\}$ if and only if $(T_{L^2}X_1)^\perp + (T_{L^2}X_2)^\perp$ generates all of $H^0(C, K_C)$, where $\perp$ means orthogonal complement with respect to Serre duality pairing $<,>$ (cf. [**3**, p. 7]). The dimension of $(T_{L^2}X_1)^\perp +$ dimension of $(T_{L^2}X_2)^\perp = g + 3 = h^0(C, K_C) + 3$, that is, dimension of $(T_{L^2}X_1)^\perp +$ dimension of $(T_{L^2}X_2)^\perp - h^0(C, K_C) = 3$. So $(T_{L^2}X_1)^\perp + (T_{L^2}X_2)^\perp$ generates all of $H^0(C, K_C)$ if p and q impose independent conditions to image $\mu_L \subset H^0(C, K_C)$, that is, p and q

impose independent conditions to image μ_L, if the dimension of $\mathscr{L}(-p-q)=3$, where $\mathscr{L}(-p-q):=\text{image } \mu_L \cap H^0(C, K_C(-p-q))$. We denote by $|\text{image } \mu_L|$ the linear system determined by the subvector space (image μ_L) $\subset H^0(C, K_C)$

Claim. The dimension of $\mathscr{L}(-p-q)=3$.

Proof of the claim. Let $D \in |K_C - L|$, D not containing $p+q$, and consider $D+|L| := \{D+E : E \in |L|\} \subseteq |\text{image } \mu_L|$; then if $p+q$ imposes independent conditions to $D+|L|$, then $p+q$ imposes independent conditions to the linear system $|\text{image } \mu_L|$, and in this case the dimension of $\mathscr{L}(-p-q)=3$. Let ℓ be a line through the node x_δ determined by p and q. Then the intersection $\ell \cdot \Gamma$, of ℓ with Γ, is $\ell \cdot \Gamma = p+q+E_\ell$, where $E_\ell \in |L|$. Since x_δ is a node there exist two different lines ℓ_p, ℓ_q through the node such that $\ell_p \cdot \Gamma = 2p+q+E_p$, where q is not in the support of the divisor E_p, and $\ell_q \cdot \Gamma = 2q+p+E_q$, with p not in the support of E_q. Then $E_p + p \in |L|$ is the unique divisor given by the tangent line ℓ_p to the branch through p not containing q. Similarly $q+E_q$ is the unique divisor given by the tangent line ℓ_q through the branch q not containing the point p. So we have that $p+q$ imposes independent conditions to $D+|L|$. $\square$

3. Proof of the Theorem. Now we are going to show that $\psi(\mathcal{V}_g)=\mathcal{D}_g \subset \mathcal{GP}^1_{g,g-2}$ has pure codimension one in $\mathcal{M}_g$.

Is well known that $\mathscr{C}^1_d$ is a smooth and irreducible variety of dimension $\dim \mathcal{M}_g + \rho(g,d,1) = 2g+2d-5$, (cf. **[2]**). Let C be a smooth curve of genus $g \geq 8$ neither trigonal nor bi-elliptic and suppose that there exists a pencil $g^1_{g-2} = (V, L) \in G^1_{g-2}(C)$ free of base points such that the residual $g^2_g = |K_C \otimes L^{-1}|$ induces a projective model as in Lemma 2.5. We have that $(C,(V,L)) \in \widetilde{\mathcal{GP}}^1_{g,g-2} \subset \mathscr{C}^1_{g-2}$. Consider the subvariety of $\mathscr{C}^1_{g-2}$ defined as

$$\widetilde{\mathcal{D}} := \{(C,(V,L)) \in \widetilde{\mathcal{GP}^1_{g,g-2}} : (C,(V,L)) \text{ as in Lemma 2.5.}\}.$$

Let $(C,(V,L)) \in \widetilde{\mathcal{D}}$. We have that kernel μ_V is one dimensional, so μ_V has rank five. We can assume that C is outside a locus $\mathcal{B}$ in $\mathcal{M}_g$ of codimension ≥ 2. In a small open neighborhood $U_C \subset \mathcal{M}_g - \mathcal{B}$ containing C, we have a finite cover $\widetilde{U}$ and pairs $(\tilde{C},(\tilde{V},\tilde{L})) \in \mathscr{C}^1_{g-2}$, $\tilde{C} \in \widetilde{U}$, $\tilde{L} \in \mathrm{Pic}^{g-2}(\tilde{C})$ such that locally the Petri map is a homomorphism of vector bundles

$$\mu|_{(\tilde{C},(\tilde{V},\tilde{L}))} : \tilde{V} \otimes H^0(\tilde{C}, K_{\tilde{C}} \otimes (\tilde{L})^{-1}) \to H^0(\tilde{C}, K_{\tilde{C}}).$$

For each $(\tilde{C},(\tilde{V},\tilde{L})) \in \widetilde{\mathcal{D}}$, the homomorphism $\mu|_{(\tilde{C},(\tilde{V},\tilde{L}))}$ has rank ≤ 5, so the subvariety $\widetilde{\mathcal{D}}$ has codimension $\leq g-5=\rho+1$, that is, $\dim \widetilde{\mathcal{D}} \geq \dim \mathscr{C}^1_{g-2} - (\rho+1) = 3g-3+\rho-(\rho+1) = 3g-4$. The Lemma 2.5 implies that the projection $\pi|_{\widetilde{\mathcal{D}}} : \widetilde{\mathcal{D}} \to \mathcal{GP}^1_{g,g-2}$ is generically finite. This show that $\pi|_{\widetilde{\mathcal{D}}}(\widetilde{\mathcal{D}}) = \mathcal{D}_g \subset \mathcal{GP}^1_{g,g-2}$ has dimension $3g-4$. Let $Y \subset \mathcal{D}_g$ be an irreducible component and $C \in Y$. By Lemma 2.5 we have that $(\pi|_{\widetilde{\mathcal{D}}})^{-1}(C)$ is zero-dimensional; this implies that Y is of codimension one, so each irreducible component of $\mathcal{D}_g$ has dimension $3g-4$, that is, $\mathcal{D}_g$ has pure codimension one in $\mathcal{M}_g$. $\square$.

Remark. From 2.3 we see that the variety $\mathcal{V}_g \neq \emptyset$ for $7 \leq g \leq 10$. Suppose that for $g \geq 11$, $\mathcal{V}_g \neq \emptyset$. We are going to show that for $g' = g+1$, $\mathcal{V}_{g+1} \neq \emptyset$.

Consider $\Delta_1 \subset \overline{\mathcal{M}_{g'}}$, where a generic point of Δ_1 is obtained by identifying a point in a smooth curve of genus g with a point in a smooth curve of genus one. Take a general curve $C \in \mathcal{D}_g = \psi(\mathcal{V}_g) \subset \mathcal{GP}^1_{g,g-2}$ with birational projective model $\Gamma \in \mathcal{V}_g$. Choose a general point $p \in C$ and set $X_0 := C \cup_p E$, where E is a elliptic curve. We have that $X_0 \in \overline{\mathcal{GP}^1_{g',g'-2}} \cap \Delta_1$, and there is a smooth curve C' near X_0 such that $C' \in \mathcal{GP}^1_{g',g'-2}$, (cf. [**8**]). Consider the rational map $\psi : \overline{\mathcal{V}^{g',g'}_{\delta'}} \to \overline{\mathcal{M}_{g'}}$, and let $\mathcal{F}$ be the graph of ψ. Consider the projections π_1, π_2 from $\mathcal{F}$ to $\mathcal{V}^{g',g'}_{\delta'}$ and $\overline{\mathcal{M}_{g'}}$ respectively. Denote $\psi^{-1}(X_0) := \pi_1(\pi_2^{-1}(X_0))$, with $X_0 \in \overline{\mathcal{GP}^1_{g',g'-2}} \cap \Delta_1$ as above. We have that there is an arc $W = \{\Gamma_t\}$ in $\overline{\mathcal{V}^{g',g'}_{\delta'}}$ with parameter t such that for $t \neq 0$, $W - \{\Gamma_0\} \subset \mathcal{V}^{g',g'}_{\delta'}$, $\{\Gamma_0\} \notin \mathcal{V}^{g',g'}_{\delta'}$, and such that the stable limit of the normalization of the curves Γ_t is a curve which is stably equivalent to X_0. We have that for some $t_0 \neq 0$, there is a curve $\Gamma_{t_0} \subset W$ such that C_{t_0}, the normalization of Γ_{t_0}, is contained in $\mathcal{GP}^1_{g',g'-2}$ with C_{t_0} near X_0. Since $C_{t_0} \in \mathcal{GP}^1_{g',g'-2}$ and $\Gamma_{t_0} \in \mathcal{V}^{g',g'}_{\delta'}$, the Lemma 2.2 implies that $\delta' - 1 = \frac{g'(g'-5)}{2}$ nodes of Γ_{t_0} lie on a curve of degree $g' - 5$. This show that $\mathcal{V}_{g+1} \neq \emptyset$.

REFERENCES

1. E. Arbarello and M. Cornalba, Su una congettura di Petri, *Comment. Math Helvetici.* **56** (1981), 1–38.

2. E. Arbarello and M. Cornalba, A few remarks about the variety of irreducible plane curves of given degree and genus, *Ann. Scient. Éc. Norm. Sup.* 4 serie, **16** (1983), 467–488.

3. E. Arbarello, M. Cornalba, P. Griffiths and J. Harris, *Geometry of algebraic curves Volume I*, Grundlehrem der Mathematischen Wissenschaften No. 267 (Springer-Verlag 1984).

4. A. Castorena, Curves of genus seven that do not satisfy the Gieseker-Petri theorem, *Bolletino U.M.I.* **8** (2005), 697–706.

5. C. Ciliberto, Canonical surfaces with $p_g = p_a = 4$ and $K^2 = 5, \ldots, 10$, *Duke Math. J.* **48** (1981), 121–157.

6. D. Eisenbud and J. Harris, The Kodaira dimension of the moduli space of curves of genus $g \geq 23$, *Invent. Math.* **90** (1987), 359–387.

7. D. Eisenbud and J. Harris, Irreducibility of some families of linear series with Brill-Noether Number -1, *Ann. Scient. Éc. Norm. Sup.* 4 série t. **22** (1989) 33–53.

8. Gavril Farkas, Gaussian maps, Gieseker-Petri loci and large theta-characteristics, *J. Reine Angew. Math.* **581** (2005), 151–173.

9. J. Harris and I. Morrison, *Moduli of curves*. Graduate Texts in Mathematics No. 187 (Springer-Verlag 1998).

10. M. Teixidor i Bigas, Half-canonical series on algebraic curves, *Trans. Amer. Math. Soc.* **302** (1987) 99–115.

11. M. Teixidor i Bigas, The divisor of curves with a vanishing theta-null, *Compositio Math.* **66** (1988) 15–22.