3
Quantum anomalies from path integral

As is well-known, the Lagrangian approach in classical field theory is very useful for constructing conserved currents associated with symmetries of the Lagrangian. Noether's theorems* describe how to construct corresponding currents and when they are conserved.

An analogous approach in quantum field theory is based on path integrals over fields. It naturally incorporates the classical results since the weight in the path integral is given by the classical action.

However, anomalous terms (i.e. those in addition to the classical ones) in the divergences of currents can appear in the quantum case owing to a contribution from regulators which make the theory finite in the ultraviolet limit. They are called *quantum anomalies*.

In this chapter we first consider the chiral anomaly, i.e. the quantum anomaly in the divergence of the axial current, which appears in the path-integral approach as a result of the noninvariance of the regularized measure. Then we briefly repeat the analysis for the scale anomaly, i.e. the quantum anomaly in the divergence of the dilatation current.

3.1 QED via path integral

Let us restrict ourselves to the case of *quantum electrodynamics* (QED), though most of the formulas will be valid for a non-Abelian Yang–Mills theory as well.

QED is described by the following partition function:

$$Z = \int \mathcal{D}A_\mu \int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \mathrm{e}^{-S[A,\psi,\bar{\psi}]}\,, \tag{3.1}$$

where A_μ is the vector-potential of the electromagnetic field, ψ_i and $\bar{\psi}_i$ are the Grassmann variables which describe the electron–positron field with

* See, for example, §2 of the book [BS76].

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i being the spinor index. They are independent but are interchangeable under involution

$$\psi \quad \stackrel{\text{inv.}}{\longleftrightarrow} \quad \bar{\psi}\,, \tag{3.2}$$

which is defined such that*

$$\psi_1\psi_2 \quad \stackrel{\text{inv.}}{\longrightarrow} \quad \bar{\psi}_2\bar{\psi}_1\,. \tag{3.3}$$

In particular, $\bar{\psi}\psi$ is invariant under involution. Therefore, $\bar{\psi}$ is an analog of $\mathrm{i}\bar{\boldsymbol{\psi}} = \mathrm{i}\boldsymbol{\psi}^\dagger\gamma_0$ in the operator formalism, while involution is analogous to Hermitian conjugation.

The Euclidean QED action in Eq. (3.1) is given by

$$S[A,\psi,\bar{\psi}] \;=\; \int \mathrm{d}^d x \left(\bar{\psi}\gamma_\mu\nabla_\mu\psi + m\bar{\psi}\psi + \frac{1}{4}F_{\mu\nu}^2\right), \tag{3.4}$$

where $\nabla_\mu = \partial_\mu - \mathrm{i}eA_\mu(x)$ is the covariant derivative as before,

$$F_{\mu\nu} \;=\; \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.5}$$

is the field strength, and γ_μ are the Euclidean γ-matrices which are discussed in Sect. 1.2.

3.2 Chiral Ward identity

Let us perform the local *chiral transformation* (c.t.)

$$\left.\begin{aligned} \psi(x) \quad &\stackrel{\text{c.t.}}{\longrightarrow} \quad \psi'(x) = \mathrm{e}^{\mathrm{i}\alpha(x)\gamma_5}\psi(x)\,, \\ \bar{\psi}(x) \quad &\stackrel{\text{c.t.}}{\longrightarrow} \quad \bar{\psi}'(x) = \bar{\psi}(x)\,\mathrm{e}^{\mathrm{i}\alpha(x)\gamma_5}. \end{aligned}\right\} \tag{3.6}$$

Here the parameter of the transformation $\alpha(x)$ is a function of x and γ_5 is the Hermitian matrix

$$\gamma_5 \;=\; \gamma_1\gamma_2\gamma_3\gamma_4 \tag{3.7}$$

in $d = 4$ dimensions. Note that both ψ and $\bar{\psi}$ have the same transformation law since in Minkowski space

$$\bar{\psi} \;=\; \psi^\dagger\gamma_0\,, \qquad \gamma_5^\dagger \;=\; \gamma_5\,, \quad \gamma_0^\dagger \;=\; \gamma_0\,, \tag{3.8}$$

while γ_5 and γ_0 anticommute.

* See the book by Berezin [Ber86] (§3.5 of Part I). Sometimes involution is defined with an opposite sign (i.e. $\bar{\psi}$ is substituted by $\mathrm{i}\bar{\psi}$) which results in a multiplication of the fermionic part of the action (3.4) by an extra factor of i.

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The variation of the classical action (3.4) under the chiral transformation (3.6) reads as

$$\delta S \;=\; \int \mathrm{d}^d x \left[\partial_\mu \alpha(x)\, J^{\mathrm{A}}_\mu(x) + 2\mathrm{i}m\alpha(x)\bar{\psi}(x)\gamma_5\psi(x)\right], \qquad (3.9)$$

where the axial current

$$J^{\mathrm{A}}_\mu \;=\; \mathrm{i}\bar{\psi}\gamma_\mu\gamma_5\psi \qquad (3.10)$$

is the Noether current associated with the chiral transformation.

It follows from Eq. (3.9) that the divergence of the axial current (3.10) is given by

$$\partial_\mu J^{\mathrm{A}}_\mu \;=\; 2\mathrm{i}m\,\bar{\psi}\gamma_5\psi\,, \qquad (3.11)$$

so that it is conserved in the massless case ($m = 0$) at the classical level:

$$\partial_\mu J^{\mathrm{A}}_\mu \;\stackrel{m=0}{=}\; 0\,. \qquad (3.12)$$

Problem 3.1 Verify Eq. (3.11) using the classical Dirac equation

$$(\widehat{\nabla} + m)\psi(x) \;=\; 0\,, \qquad \widehat{\nabla} \;= \gamma_\mu \nabla_\mu\,. \qquad (3.13)$$

Solution Calculate the divergence of the axial current (3.10) using Eq. (3.13) and the conjugate one

$$\bar{\psi}(x)\left(\overleftarrow{\widehat{\nabla}} - m\right) \;=\; 0 \qquad (3.14)$$

with

$$\overleftarrow{\nabla}_\mu \;=\; \overleftarrow{\partial}_\mu +\, \mathrm{i}eA_\mu(x)\,. \qquad (3.15)$$

Let us now discuss how the measure in the path integral changes under the chiral transformation (3.6). The old and new measures are related by

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi \;=\; \mathcal{D}\bar{\psi}'\mathcal{D}\psi' \det\left[\mathrm{e}^{2\mathrm{i}\alpha(x)\gamma_5}\delta^{(d)}(x-y)\right], \qquad (3.16)$$

where the determinant is over the space indices x and y, as well as over the γ-matrix indices i and j. Note that the determinant, which is nothing but the Jacobian of the transformation (3.6), emerges for the Grassmann variables to the positive rather than the negative power as for commuting variables. This is a known property of the integrals (2.17) over Grassmann variables [Ber86] which look more like derivatives.

The logarithm of the Jacobian in Eq. (3.16) can be calculated as

$$\begin{aligned}
&\ln\det\left[\mathrm{e}^{2\mathrm{i}\alpha(x)\gamma_5}\delta^{(d)}(x-y)\right] \\
&\quad = \mathrm{Tr}\ \ln\left(\mathrm{e}^{2\mathrm{i}\boldsymbol{\alpha}\gamma_5}\right) \ = \ \mathrm{Tr}\,(2\mathrm{i}\,\boldsymbol{\alpha}\,\gamma_5) \\
&\quad = 2\mathrm{i}\int \mathrm{d}^d x\,\alpha(x)\ \delta^{(d)}(0)\ \mathrm{sp}\,\gamma_5 \ = \ 0\,,
\end{aligned} \tag{3.17}$$

where sp is the trace only over the γ-matrix indices i and j. The RHS vanishes naively since the trace vanishes. A subtlety with the appearance of the infinite factor of $\delta^{(d)}(0)$ will be discussed in the next section.

Note that the infinitesimal version of the transformation (3.6) is a particular case of the more general one

$$\left.\begin{aligned}
\psi(x) \quad &\longrightarrow \quad \psi'(x) = \psi(x) + \delta\psi(x)\,, \\
\bar\psi(x) \quad &\longrightarrow \quad \bar\psi'(x) = \bar\psi(x) + \delta\bar\psi(x)\,,
\end{aligned}\right\} \tag{3.18}$$

which is an analog of the transformation (2.25) and leaves the measure invariant. The calculation given in Eq. (3.17) is an explicit illustration of this fact.

The general transformation (3.18) leads, when applied to the path integral in Eq. (3.1), to the Schwinger–Dyson equations

$$\left.\begin{aligned}
\left(\widehat{\nabla} + m\right)\psi(x) \quad &\stackrel{\text{w.s.}}{=} \quad \frac{\delta}{\delta\bar\psi(x)}\,, \\
\bar\psi(x)\left(\overleftarrow{\widehat{\nabla}} - m\right) \quad &\stackrel{\text{w.s.}}{=} \quad \frac{\delta}{\delta\psi(x)}\,,
\end{aligned}\right\} \tag{3.19}$$

which hold in the weak sense, i.e. under the averaging over $\bar\psi$ and ψ.

More restrictive transformations of the same type as (3.6), which are associated with symmetries of the classical action and result in conserved currents, lead to some (less restrictive) relations between correlators which are called *Ward identities*. This terminology goes back to the 1950s when a proper relation between the two- and three-point Green functions was first derived for the gauge symmetry in QED.

The simplest Ward identity, which is associated with the chiral transformation (3.6), is given as

$$\begin{aligned}
&\langle\, \partial_\mu J^{\mathrm{A}}_\mu(0)\,\psi_i(x)\,\bar\psi_j(y)\,\rangle \\
&\qquad \stackrel{m=0}{=} \mathrm{i}\,\delta^{(d)}(x)\langle\,(\gamma_5\psi)_i(0)\,\bar\psi_j(y)\,\rangle - \mathrm{i}\,\delta^{(d)}(y)\langle\,\psi_i(x)\,\left(\bar\psi\gamma_5\right)_j(0)\,\rangle.
\end{aligned} \tag{3.20}$$

It is clear from the way in which Eq. (3.20) was derived, that it is always satisfied as a consequence of the quantum equations of motion (3.19).

Problem 3.2 Derive Eq. (3.20) in the operator formalism when the averages are substituted by the vacuum expectation values of the T-products.

Solution Equation (3.13) acquires an extra $-\mathrm{i}$ in Minkowski space, where the spatial γ-matrices are anti-Hermitian rather than Hermitian as in Euclidean space, and holds in the quantum case in the weak sense, i.e. when applied to a state. Using it and the canonical equal-time anticommutation relations for ψ and $\bar{\psi}$ with the only nonvanishing anticommutator being

$$\delta(x_0 - y_0)\left\{\bar{\boldsymbol{\psi}}_i(y), \boldsymbol{\psi}_j(x)\right\} \;=\; \delta_{ij}\delta^{(d)}(x-y)\,, \tag{3.21}$$

we reproduce Eq. (3.20) in the operator formalism.

Remark on γ_5 in d dimensions

Let us recall that

$$\gamma_5 \;=\; \gamma_1\gamma_2\cdots\gamma_d \tag{3.22}$$

only exists for even d when the size of the γ-matrices is $2^{d/2} \times 2^{d/2}$. For this reason the dimensional regularization is not applicable in calculations of the chiral anomaly.

Remark on gauge-fixing

Note that we did not add a gauge-fixing term to the action (3.4). It is harmless to do that since the gauge-fixing term does not contribute to the variation of the action under the chiral transformation. Moreover, all gauge-invariant quantities do not depend on the gauge-fixing. How one can quantize a gauge theory without adding a gauge-fixing term will be explained in Part 2.

3.3 Chiral anomaly

As has already been mentioned, Eq. (3.17) involves the uncertainty

$$\delta^{(d)}(0)\cdot \operatorname{sp}\gamma_5 \;=\; \infty\cdot 0\,. \tag{3.23}$$

To regularize $\delta^{(d)}(0)$, one needs [Ver78, Fuj79] to regularize the measure in the path integral over ψ and $\bar{\psi}$, since this term comes from the change of the measure under the chiral transformation.

Let us expand the fields ψ and $\bar{\psi}$ over some set of the orthogonal basis functions, similarly to Eq. (1.82):

$$\psi^i(x) \;=\; \sum_n c_n^i \phi_n^i(x)\,, \qquad \bar{\psi}^i(x) \;=\; \sum_n \bar{c}_n^i \phi_n^{i\,\dagger}(x)\,, \tag{3.24}$$

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where there is no summation over the spinor index i. Here c_n^i and $\bar{c}_n^i$ are Grassmann variables. The measure is then similar to that of Eq. (1.83) and reads explicitly as

$$\mathcal{D}\bar{\psi}\mathcal{D}\psi = \prod_{n=1}^{\infty}\prod_i \mathrm{d}\bar{c}_n^{\,i} \prod_{m=1}^{\infty}\prod_j \mathrm{d}c_m^{\,j}\,. \tag{3.25}$$

The idea of regularizing the measure is to restrict ourselves to a large but finite number of basis functions. This is analogous to the discretization of Sect. 1.4. We therefore define the regularized measure as

$$(\mathcal{D}\bar{\psi})_R(\mathcal{D}\psi)_R = \prod_{n=1}^{M}\prod_i \mathrm{d}\bar{c}_n^{\,i} \prod_{m=1}^{M}\prod_j \mathrm{d}c_m^{\,j}\,. \tag{3.26}$$

The change of the measure under the chiral transformation is

$$(\mathcal{D}\bar{\psi})_R(\mathcal{D}\psi)_R = (\mathcal{D}\bar{\psi}')_R(\mathcal{D}\psi')_R \det\left[\int \mathrm{d}^d x\, \phi_n^{k\,\dagger}(x)\, \mathrm{e}^{2\mathrm{i}\alpha(x)\gamma_5^{kj}}\, \phi_m^j(x)\right], \tag{3.27}$$

where the determinant is over both the n and m indices and the spinor indices k and j. This is the regularized analog of the nonregularized expression (3.16).

Using the orthogonality of the basis functions:

$$\int \mathrm{d}^d x\, \phi_n^{j\,\dagger}(x)\, \phi_m^i(x) = \delta_{nm}\delta^{ij}, \tag{3.28}$$

and Eq. (2.20), we rewrite the determinant on the RHS of Eq. (3.27) for an infinitesimal parameter α as

$$\det \int \mathrm{d}^d x\, \phi_n^{k\,\dagger}(x)\, \mathrm{e}^{2\mathrm{i}\alpha(x)\gamma_5^{kj}}\, \phi_m^j(x) = 1 + 2\mathrm{i}\sum_{n=1}^{M} \int \mathrm{d}^d x\, \phi_n^{\dagger}(x)\alpha(x)\gamma_5\phi_n(x)\,, \tag{3.29}$$

where the spinor indices are contracted in the usual way.

It is easy to see how this formula recovers Eq. (3.17) since

$$\sum_{n=1}^{\infty} \phi_n^i(x)\, \phi_n^{j\,\dagger}(y) = \delta^{(d)}(x-y)\,\delta^{ij} \tag{3.30}$$

in the nonregularized case owing to the completeness of the basis functions.

In the regularized case, the sum over n on the LHS of Eq. (3.30) is restricted by M from above so that the RHS is no longer equal to the delta-function. We substitute

$$\sum_{n=1}^{M} \phi_n^i(x)\,\phi_n^{j\,\dagger}(y) \;=\; R^{ij}(x,y)\,, \tag{3.31}$$

with the RHS being the matrix element of some regularizing operator $\boldsymbol{R}$.

It can be chosen in many ways. We shall work with several forms:

$$\boldsymbol{R} \;=\; \mathrm{e}^{a^2\hat{\nabla}^2}\,, \tag{3.32}$$

or

$$\boldsymbol{R} \;=\; \frac{1}{1-a^2\hat{\nabla}^2}\,, \tag{3.33}$$

or

$$\boldsymbol{R} \;=\; \frac{1}{1+a\hat{\nabla}}\,, \tag{3.34}$$

etc., where again $\hat{\nabla} = \gamma_\mu \nabla_\mu$. The parameter a is the ultraviolet cutoff. The cutoff disappears as $a \to 0$ when Eq. (2.57) holds.

These regularizations (3.32)–(3.34) are nonperturbative, and preserve gauge invariance since they are constructed from the covariant derivative ∇_μ. A consistent regularization occurs when $\boldsymbol{R}$ commutes with $\hat{\nabla}$, which is obviously true for the regularizations (3.32)–(3.34).*

Therefore, we find

$$\begin{aligned}\int \mathrm{d}^d x\,\alpha(x)\,\partial_\mu J_\mu^{\mathrm{A}} \;&=\; 2\mathrm{i}\,\mathrm{Tr}\,(\boldsymbol{\alpha}\gamma_5\boldsymbol{R}) \\ &=\; 2\mathrm{i}\int \mathrm{d}^d x\,\alpha(x)\,\mathrm{sp}\,[\gamma_5 R(x,x)]\,.\end{aligned} \tag{3.35}$$

It is worth noting that the extra R in Eq. (3.35) is a consequence of the more general formula

$$\mathrm{Tr}\,\boldsymbol{O} \;\longrightarrow\; \mathrm{Tr}\,\boldsymbol{O}\boldsymbol{R}\,, \tag{3.36}$$

which describes how to regularize the traces of operators.

* This can be shown by choosing the basis functions to be eigenfunctions of the Hermitian operator $\mathrm{i}\hat{\nabla}$ ($\mathrm{i}\hat{\nabla}\phi_n = E_n\phi_n$) and applying $\hat{\nabla}^{ki}(x)[\hat{\nabla}^{-1}(y)]^{jl}$ to Eq. (3.31). Then the LHS does not change (because $E_n E_n^{-1} = 1$), while $\langle x|\hat{\nabla}\boldsymbol{R}\hat{\nabla}^{-1}|y\rangle$ appears on the RHS. It coincides with the RHS of Eq. (3.31) when $\hat{\nabla}$ and $\boldsymbol{R}$ commute.

Remark on regularization of the measure

The regularization of the measure in the path integral using Eq. (3.26) is equivalent to the point-splitting procedure where the delta-function in the commutator term is smeared according to Eq. (2.55).

To show this, let us note that the variational derivative can be approximated for a finite number of basis functions by

$$\frac{\delta}{\delta \psi^j_R(y)} \;=\; \sum_{n=1}^{M} \phi^{j\,\dagger}_n(y) \sum_k \frac{\partial}{\partial c^k_n} \,. \tag{3.37}$$

This definition extends the standard mathematical one* to the case of spinor indices. The sum over k is included in order for the regularized variational derivative to reflect variations of all the spinor components of c_n when the variation is not diagonal in the spinor indices.

When applied to

$$\psi^i_R(x) \;=\; \sum_{n=1}^{M} c^i_n \phi^i_n(x) \,, \tag{3.38}$$

it yields

$$\frac{\delta \psi^i_R(x)}{\delta \psi^j_R(y)} \;=\; \sum_{n=1}^{M} \phi^i_n(x)\, \phi^{j\,\dagger}_n(y) \;=\; R^{ij}(x,y) \,, \tag{3.39}$$

or, equivalently,

$$\delta^{ij} \delta^{(d)}(x-y) \;\stackrel{\text{reg.}}{\Longrightarrow}\; R^{ij}(x,y) \,, \tag{3.40}$$

which is the fermionic analog of Eq. (2.55).

Thus, we conclude that the regularization of the measure in the path integral is equivalent to smearing the delta-function in commutator terms.

Remark on regularized Schwinger–Dyson equations

The procedure from the previous Remark results in the following regularized Schwinger–Dyson equations:

$$\left.\begin{aligned} \left(\widehat{\nabla} + m\right) \psi(x) \;&\stackrel{\text{w.s.}}{=}\; \int \mathrm{d}^d y\, R(x,y) \frac{\delta}{\delta \bar{\psi}(y)} \,, \\ \bar{\psi}(x) \left(\overleftarrow{\widehat{\nabla}} - m\right) \;&\stackrel{\text{w.s.}}{=}\; \int \mathrm{d}^d y\, R(x,y) \frac{\delta}{\delta \psi(y)} \,. \end{aligned}\right\} \tag{3.41}$$

These equations are understood again in the weak sense, i.e. under the averaging over $\bar{\psi}$ and ψ and obviously reproduce Eq. (3.19) as $a \to 0$.

* See, for example, the book by Lévy [Lev51].

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Problem 3.3 Derive Eq. (3.35) using the regularized Schwinger–Dyson equation (3.41).

Solution The calculation is similar to that of Problem 3.1 except for the additional terms arising from the RHS of Eq. (3.41). For $m = 0$ one finds

$$\begin{aligned} \partial_\mu J^{\mathrm{A}}_\mu \overset{\text{w.s.}}{=} & \; \mathrm{i} \int \mathrm{d}^d y \, \frac{\delta}{\delta \psi(y)} R(x,y) \gamma_5 \psi(x) - \mathrm{i} \bar{\psi}(x) \gamma_5 \int \mathrm{d}^d y \, R(x,y) \frac{\delta}{\delta \bar{\psi}(y)} \\ = & \; 2\mathrm{i} \, \mathrm{sp} \left[\gamma_5 R(x,x) \right], \end{aligned} \tag{3.42}$$

which is equivalent to Eq. (3.35) since there $\alpha(x)$ is an arbitrary function.

3.4 Chiral anomaly (calculation)

In order to derive an explicit expression for the chiral anomaly, we should calculate the RHS of Eq. (3.35) for some choice of the regularizing operator $\boldsymbol{R}$. Let us choose $\boldsymbol{R}$ given by Eq. (3.33). The operator $\widehat{\nabla}^2$ in the denominator can be transformed as

$$\begin{aligned} \widehat{\nabla}^2 &= \nabla^2 + \frac{1}{2} \left[\gamma_\mu, \gamma_\nu \right] \nabla_\mu \nabla_\nu \\ &= \nabla^2 - \frac{\mathrm{i}e}{4} \left[\gamma_\mu, \gamma_\nu \right] F_{\mu\nu} \\ &= \nabla^2 + \frac{e}{2} \Sigma_{\mu\nu} F_{\mu\nu} , \end{aligned} \tag{3.43}$$

where the trace of the spin matrices

$$\Sigma_{\mu\nu} = \frac{1}{2\mathrm{i}} \left[\gamma_\mu, \gamma_\nu \right] \tag{3.44}$$

is given by

$$\mathrm{sp} \left(\Sigma_{\mu\nu} \Sigma_{\lambda\rho} \gamma_5 \right) = -4 \epsilon_{\mu\nu\lambda\rho} . \tag{3.45}$$

Expanding in e,

$$\boldsymbol{R} = \boldsymbol{R}_0 + \boldsymbol{R}_0 \left(\cdots \right) \boldsymbol{R}_0 + \cdots \tag{3.46}$$

with

$$\boldsymbol{R}_0 = \frac{1}{1 - a^2 \partial^2} , \tag{3.47}$$

we find schematically

$$\begin{aligned} \mathrm{Tr} \left(\boldsymbol{\alpha} \gamma_5 \boldsymbol{R} \right) &= a^4 \, \mathrm{Tr} \left[\boldsymbol{\alpha} \gamma_5 \boldsymbol{R}_0 \left(\frac{e \Sigma F}{2} \right) \boldsymbol{R}_0 \left(\frac{e \Sigma F}{2} \right) \boldsymbol{R}_0 \right] \\ &= - \int \mathrm{d}^4 x \, \alpha(x) \, \frac{e^2}{16 \pi^2} F_{\mu\nu} \widetilde{F}_{\mu\nu} , \end{aligned} \tag{3.48}$$

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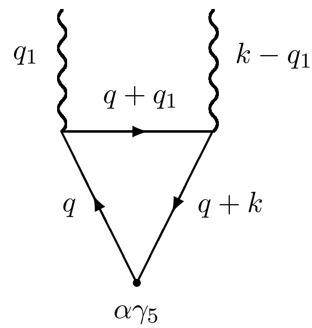

Fig. 3.1. Triangular diagram associated with chiral anomaly in $d = 4$. The solid lines correspond to R_0 given by Eq. (3.50). The wavy lines correspond to the field strength.

where

$$\widetilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda\rho}F_{\lambda\rho} \tag{3.49}$$

is the dual field strength.

The calculation described in Eq. (3.48) is most easily performed in momentum space where it is associated with one-loop diagrams. The analytic expression to be calculated can be represented in $d = 4$ graphically as the triangular diagram in Fig. 3.1. The solid lines are associated with R_0 given by Eq. (3.47), which reads in momentum space as

$$R_0(p) = \frac{1}{1 + a^2p^2}\,. \tag{3.50}$$

The wavy lines correspond to the field strength. The lower vertex is associated with $\alpha\gamma_5$.

The integral over the four-momentum q, which circulates along the triangular loop, can be easily calculated by introducing $\omega = aq$ and transforming the integral as

$$\int \mathrm{d}^4q\, f(q) \to \frac{1}{a^4}\int \mathrm{d}^4\omega\, f\left(\frac{\omega}{a}\right). \tag{3.51}$$

Note that the integral involves a^{-4} which cancels a^4 coming from the expansion in e, for which the proper term is given by the intermediate expression in Eq. (3.48). Therefore, the result is nonvanishing and a-independent as $a \to 0$. Higher terms of the expansion in e are proportional to higher powers in a and vanish as $a \to 0$.

Finally, from Eqs. (3.35) and (3.48) we obtain

$$\partial_\mu J^{\mathrm{A}}_\mu = -\frac{\mathrm{i}e^2}{8\pi^2}F_{\mu\nu}\widetilde{F}_{\mu\nu}\,. \tag{3.52}$$

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The anomaly on the RHS is known as the Adler–Bell–Jackiw anomaly. Its appearance is usually related to the fact that any regularization cannot be simultaneously gauge and chiral invariant.

Problem 3.4 Calculate the coefficient in Eq. (3.48) and show that it is regulator-independent.

Solution The contribution of the triangular diagram of Fig. 3.1, which represents the intermediate expression in Eq. (3.48), reads explicitly as

$$\begin{aligned} 2\,\mathrm{Tr}\,(\boldsymbol{\alpha}\gamma_5\boldsymbol{R}) \;&=\; -4e^2a^4\int \mathrm{d}^4x\int \mathrm{d}^4y\int \mathrm{d}^4z \\ &\quad\times\,\alpha(x)R_0(x,y)F_{\mu\nu}(y)R_0(y,z)\tilde{F}_{\mu\nu}(z)R_0(z,x)\,. \end{aligned} \tag{3.53}$$

In momentum space, it becomes

$$\begin{aligned} & -2e^2a^4\int \mathrm{d}^4x\,\alpha(x)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}kx}\int\frac{\mathrm{d}^4q_1}{(2\pi)^4}F_{\mu\nu}(q_1)\tilde{F}_{\mu\nu}(k-q_1) \\ &\qquad\times\int\frac{\mathrm{d}^4q}{(2\pi)^4}\frac{1}{(1+a^2q^2)\,(1+a^2(q+q_1)^2)\,(1+a^2(q+k)^2)} \\ &\overset{(3.51)}{=}\; -\frac{2e^2}{16\pi^2}\int \mathrm{d}^4x\,\alpha(x)\int\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{e}^{\mathrm{i}kx} \\ &\qquad\times\int\frac{\mathrm{d}^4q_1}{(2\pi)^4}F_{\mu\nu}(q_1)\tilde{F}_{\mu\nu}(k-q_1)\int_0^\infty\frac{\omega^2\mathrm{d}\omega^2}{(1+\omega^2)^3} \end{aligned} \tag{3.54}$$

which recovers the RHS of Eq. (3.48).

An analogous calculation can be repeated for other regulators (3.32) and (3.34). Let us denote

$$r(a^2p^2) \;\equiv\; R_0(p)\,. \tag{3.55}$$

Then the only difference with Eq. (3.54) is that the last integral over ω^2 is replaced by

$$\int_0^\infty \mathrm{d}\omega^2\,\omega^2 r''(\omega^2) \;=\; r(0) \;=\; 1 \tag{3.56}$$

for reasonable functions r which look like those given by Eqs. (3.32)–(3.34).

An anomaly which is analogous to the Adler–Bell–Jackiw anomaly (3.52) exists in $d=2$ where

$$\partial_\mu J_\mu^{\mathrm{A}} \;=\; -\frac{e}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}\,. \tag{3.57}$$

This anomaly is given by the diagram depicted in Fig. 3.2. It involves only two lines with the regulator $R_0(p)$ since in $d=2$

$$\int \mathrm{d}^2q\,f(q) \;\to\; \frac{1}{a^2}\int \mathrm{d}^2\omega\,f\Big(\frac{\omega}{a}\Big) \tag{3.58}$$

so that all terms with more lines vanish as $a\to 0$.

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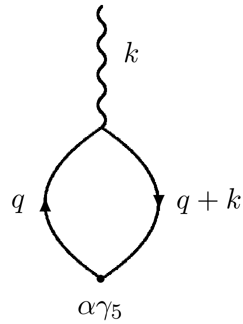

Fig. 3.2. The diagram associated with the chiral anomaly in $d = 2$. The solid lines correspond to R_0 given by Eq. (3.50). The wavy line corresponds to the field strength.

Problem 3.5 Calculate $2\,\mathrm{Tr}\,(\boldsymbol{\alpha}\gamma_5\boldsymbol{R})$ in $d = 2$.

Solution Proceeding as before, we see that only the diagram of Fig. 3.2 is essential in $d = 2$ which yields

$$
\begin{aligned}
&2\mathrm{i}ea^2 \int \mathrm{d}^2x \int \mathrm{d}^2y\, \alpha(x) R_0(x,y) F_{\mu\nu}(y)\epsilon_{\mu\nu} R_0(y,x) \\
&\quad = \; 2\mathrm{i}ea^2 \int \mathrm{d}^2x\, \alpha(x) \int \frac{\mathrm{d}^2k}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}kx} F_{\mu\nu}(k)\epsilon_{\mu\nu} \int \frac{\mathrm{d}^2q}{(2\pi)^2} \frac{1}{(1+a^2q^2)\,[1+a^2(q+k)^2]} \\
&\quad \overset{(3.58)}{=} \; 2\frac{\mathrm{i}e}{4\pi} \int \mathrm{d}^2x\, \alpha(x) \int \frac{\mathrm{d}^2k}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}kx} F_{\mu\nu}(k)\epsilon_{\mu\nu} \int_0^\infty \frac{\mathrm{d}\omega^2}{(1+\omega^2)^2} \\
&\quad = \; \int \mathrm{d}^2x\, \alpha(x) \frac{\mathrm{i}eF_{\mu\nu}(x)\epsilon_{\mu\nu}}{2\pi}\,.
\end{aligned}
\tag{3.59}
$$

The linear-in-$F_{\mu\nu}$ term is nonvanishing since

$$
\mathrm{sp}\,(\Sigma_{\mu\nu}\gamma_5) \;=\; 2\mathrm{i}\epsilon_{\mu\nu} \tag{3.60}
$$

in $d = 2$.

The result is again regulator-independent since the integral over ω is replaced for an arbitrary $R_0(p)$ by

$$
-\int_0^\infty \mathrm{d}\omega^2 r'(\omega^2) \;=\; r(0) \;=\; 1 \tag{3.61}
$$

where Eq. (3.55) has been used.

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Remark on the non-Abelian chiral anomaly

Equation (3.52) also holds in the case of a non-Abelian gauge group where $F^a_{\mu\nu}$ is the non-Abelian field strength

$$F^a_{\mu\nu}(x) \quad = \quad \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x) + g f^{abc} A^b_\mu(x) A^c_\nu(x) \,. \tag{3.62}$$

Here f^{abc} are the structure constants of the gauge group and g is the coupling constant. The non-Abelian analog of Eq. (3.52) for the axial current, which is a singlet with respect to the gauge group, is given by*

$$\partial_\mu J^{\mathrm{A}}_\mu \quad = \quad -\frac{\mathrm{i} g^2}{8\pi^2} \sum_a F^a_{\mu\nu} \widetilde{F}^a_{\mu\nu} \,. \tag{3.63}$$

The $d = 2$ anomaly (3.57) exists for the singlet axial current only in the Abelian case.

A description of the chiral anomaly in non-Abelian gauge theories is given, for example, in Chapter 22 of the book by Weinberg [Wei98].

3.5 Scale anomaly

The scale transformation is defined by

$$x_\mu \quad \longrightarrow \quad x'_\mu \;=\; \rho\, x_\mu \,, \tag{3.64}$$

$$\varphi(x) \quad \longrightarrow \quad \varphi'\big(x'\big) \;=\; \rho^{l_\varphi} \varphi\big(x'\big) \,. \tag{3.65}$$

The index l_φ is called the *scale dimension* of the field φ. The value of l_φ in a free theory is called the canonical dimension, which equals $(d-2)/2$ for bosons (scalar or vector fields) and $(d-1)/2$ for the spinor Dirac field, i.e. 1 and 3/2 in $d = 4$, respectively. Sometimes l_φ is called, for historical reasons, the anomalous dimension. More often the term "anomalous dimension" is used for the difference between l_φ and the canonical value.

The proper Noether current, which is called the *dilatation current*, is expressed via the energy–momentum tensor $\theta_{\mu\nu}$ as

$$D_\mu \quad = \quad x_\nu \theta_{\mu\nu} \tag{3.66}$$

so that its divergence equals the trace of the energy–momentum tensor over the spatial indices:

$$\partial_\mu D_\mu \quad = \quad \theta_{\mu\mu} \,, \tag{3.67}$$

* The coefficient in this formula is the same as in Eq. (3.52) and is twice as large as the conventional one. This is owing to our normalization, which is described in Sect. 5.1.

since the energy–momentum tensor is conserved. For the action (3.4) one finds

$$\theta_{\mu\mu} \quad = \quad -m\bar{\psi}\psi \tag{3.68}$$

at the classical level.

The above formulas can be obtained from the Noether theorems which state

$$\delta S \quad = \quad \int \mathrm{d}^d x\, \rho(x)\, \partial_\mu D_\mu(x) \tag{3.69}$$

or

$$\partial_\mu D_\mu(x) \quad = \quad \frac{\delta S}{\delta \rho(x)}\,. \tag{3.70}$$

In the massless case, $m = 0$, the RHS of Eq. (3.68) vanishes and the dilatation current is conserved. This is a well-known property of electrodynamics with a massless electron that is scale invariant at the classical level. A generic scale-invariant theory does not depend on parameters of the dimension of mass or length. This usual dimension is to be distinguished from the scale dimension which is defined by Eq. (3.65). The dimensional parameters do not change under the scale transformation (3.64).

In the quantum case, the scale invariance is broken by the (dimensional) cutoff a. The energy–momentum tensor is no longer traceless owing to loop effects. The relation (3.67) holds in the quantum case in the weak sense, i.e. for the averages

$$\left\langle\, \partial_\mu D_\mu F[A,\psi,\bar{\psi}]\,\right\rangle \quad = \quad \left\langle\, \theta_{\mu\mu} F[A,\psi,\bar{\psi}]\,\right\rangle, \tag{3.71}$$

where $F[A,\psi,\bar{\psi}]$ is a gauge-invariant functional of A, ψ and $\bar{\psi}$.

For a renormalizable theory such as QED, the RHS of Eq. (3.71) is proportional to the Gell-Mann–Low function $\mathcal{B}(e^2)$ which is defined by

$$-a\frac{\mathrm{d}e^2}{\mathrm{d}a} \quad = \quad \mathcal{B}\big(e^2\big)\,. \tag{3.72}$$

A nontrivial property of a renormalizable theory is that the RHS in this formula is a function solely of e^2 – the bare charge.

The meaning of the *renormalizability* is very simple: physical quantities do not depend on the cutoff a, provided the bare charge e is chosen to be cutoff-dependent according to Eq. (3.72). This dependence of e on a effectively accounts for distances smaller than a, which are excluded from the theory.

The precise relation between the trace of the energy–momentum tensor and the Gell-Mann–Low function is given by

$$\theta_{\mu\mu} \quad \stackrel{\text{w.s.}}{=} \quad \frac{\mathcal{B}\big(e^2\big)}{4e^2} F^2_{\mu\nu}\,, \tag{3.73}$$

where the equality is understood again in the weak sense. This formula was first obtained in [Cre72, CE72] to leading order in e^2 and proven in [ACD77] to all orders in e^2.

Note that this formula holds in the operator formalism only when applied to a gauge-invariant state. The reason is that otherwise a contribution from a gauge-fixing term in the action would be essential. It does not contribute, however, to gauge-invariant averages which can be formally proven using the gauge Ward identity.

Problem 3.6 Prove the relation (3.73).

Solution Let us absorb the coupling e into A_μ introducing

$$\left.\begin{array}{rcl} \mathcal{A}_\mu & = & eA_\mu\,, \\ \mathcal{F}_{\mu\nu} & = & eF_{\mu\nu}\,. \end{array}\right\} \tag{3.74}$$

The Lagrangian density of massless QED then reads as

$$\mathcal{L} \;=\; \bar{\psi}\left(\widehat{\partial} - \mathrm{i}\,\widehat{\mathcal{A}}\right)\psi + \frac{1}{4e^2}\mathcal{F}^2. \tag{3.75}$$

To prove Eq. (3.73), let us use the chain of Eqs. (3.67) and (3.70). It is crucial that in the absence of other dimensional parameters the derivative $\partial/\partial\rho$ can be replaced by $\partial/\partial a$, since all dimensionless quantities in a theory with a cutoff depend only on ratios of the type x/a.* Since the dependence on the cutoff a enters in Eq. (3.75) formally only via e^{-2} in front of $\mathcal{F}^2_{\mu\nu}$, Eq. (3.73) can be proven heuristically by first differentiating with respect to a and then expressing the result via $F_{\mu\nu}$ again. Here we have used the fact that $\mathcal{F}_{\mu\nu}$ is invariant under the renormalization-group transformation and, therefore, does not depend on a.

In the path-integral approach, a contribution to the scale anomaly comes from the regularized quantum measure. Proceeding as in Sect. 3.3, we obtain

$$\partial_\mu D_\mu(x) \;=\; -\mathrm{sp}\left[R(x,x)\right] \tag{3.76}$$

which determines the scale anomaly.

Problem 3.7 Derive Eq. (3.76) using the regularized Schwinger–Dyson equations (3.41).

Solution The energy–momentum tensor of QED is given by

$$\theta_{\mu\nu} \;=\; F_{\mu\lambda}F_{\nu\lambda} - \frac{1}{4}\delta_{\mu\nu}F^2_{\rho\lambda} + \frac{1}{4}\left(\bar{\psi}\gamma_\mu\overset{\leftrightarrow}{\nabla}_\nu\psi + \bar{\psi}\gamma_\nu\overset{\leftrightarrow}{\nabla}_\mu\psi\right). \tag{3.77}$$

Taking the trace, one obtains

$$\theta_{\mu\mu} \;=\; \frac{1}{2}\bar{\psi}\gamma_\mu\overset{\leftrightarrow}{\nabla}_\mu\psi\,. \tag{3.78}$$

* This is the reason why the Callan–Symanzik equations, which are nothing but the dilatation Ward identities, coincide with the renormalization-group equations.

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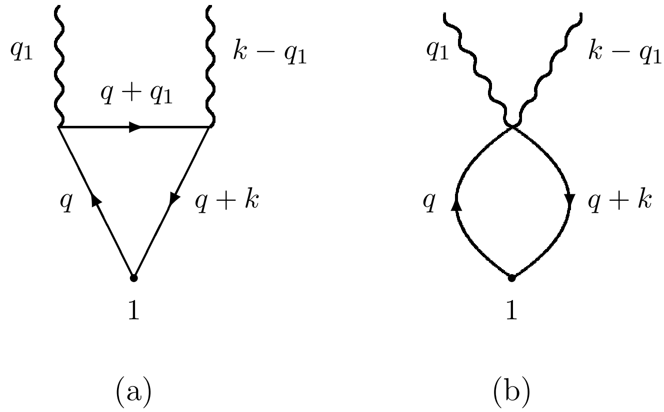

Fig. 3.3. The diagrams which contribute to the scale anomaly in $d = 4$. The wavy line corresponds to the field strength.

Using Eq. (3.41), it can be transformed as

$$\begin{aligned}\theta_{\mu\mu} &= -m\bar{\psi}\psi + \frac{1}{2}\left[\int \mathrm{d}^d y\, R(x,y)\frac{\delta}{\delta\psi(y)}\psi(x) - \bar{\psi}(x)\int \mathrm{d}^d y\, R(x,y)\frac{\delta}{\delta\bar{\psi}(y)}\right] \\ &= -m\bar{\psi}\psi - \mathrm{sp}\,[R(x,x)]\,, \qquad (3.79)\end{aligned}$$

which reproduces Eq. (3.76) as $m \to 0$.

To calculate the scale anomaly we should therefore perform a one-loop calculation of

$$\begin{aligned}\mathrm{sp}\,[R(x,x)] &= \mathrm{sp}\left\langle x\left|\frac{1}{1+a^2(\mathrm{i}\widehat{\nabla}^2)}\right|x\right\rangle \\ &= \mathrm{sp}\left\langle x\left|\frac{1}{1+a^2\left(\mathrm{i}\partial_\mu + eA_\mu\right)^2 - \frac{1}{2}a^2 e\Sigma_{\mu\nu}F_{\mu\nu}}\right|x\right\rangle \qquad (3.80)\end{aligned}$$

which is again most convenient to do in momentum space. The propagator is given by Eq. (3.47), while the vertices, which emerge in the corresponding Feynman rules for the expansion in e, come from the operators

$$-2\mathrm{i}ea^2 A_\mu\partial_\mu\,, \qquad -e^2a^2A_\mu^2\,, \qquad \tfrac{1}{2}ea^2\Sigma_{\mu\nu}F_{\mu\nu}\,.$$

The only diagrams which survive as $a \to 0$ in $d = 4$ are depicted in Fig. 3.3. The calculation of the diagram of Fig. 3.3a is the same as in Sect. 3.3 while the diagram of Fig. 3.3b gives a total derivative which does not contribute to the scale anomaly.

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The calculation of the diagram of Fig. 3.3a yields

$$\mathrm{sp}\,[R(x,x)] \;=\; -\frac{e^2 F_{\mu\nu}^2(x)}{24\pi^2}\,. \tag{3.81}$$

The one-loop Gell-Mann–Low function can now be calculated using Eqs. (3.76) and (3.73), which reproduces the known result for QED. The higher-order corrections in e do not vanish for the scale anomaly.

Remark on the non-Abelian scale anomaly

Equation (3.73) holds in the non-Abelian Yang–Mills theory as well if $F_{\mu\nu}$ is substituted by the non-Abelian field strength $F_{\mu\nu}^a$ given by Eq. (3.62). The corresponding formula, is given as

$$\theta_{\mu\mu} \;\stackrel{\text{w.s.}}{=}\; \frac{\mathcal{B}(g^2)}{4g^2}\sum_a F_{\mu\nu}^a F_{\mu\nu}^a\,. \tag{3.82}$$

A heuristic proof, presented in Problem 3.6 for the Abelian case, can be repeated. The equality is again understood in the weak sense when averaged between gauge-invariant states. The contribution of gauge-fixing and ghost terms are then canceled owing to the gauge Ward identity which is called in this case the Slavnov–Taylor identity. The proof of Eq. (3.82) was given in [CDJ77, Nie77].