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GENERAL HEREDITY FOR RADICAL THEORY

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Let W be a class of not necessarily associative rings which is universal in the sense that it is closed under homomorphic images and is hereditary to subrings. All rings considered will be assumed to belong to W. The notation $I \triangleleft R$ will mean I is an ideal of R. A relation σ on W will be called an H-relation if σ satisfies the properties:

(1) $I\sigma R$ implies I is a subring of R.

(2) If $I\sigma R$ and $\emptyset$ is a homomorphism of R, then $I\emptyset\sigma R\emptyset$.

(3) If $I\sigma R$ and J is an ideal of R, then $I\cap J\sigma J$.

Examples of H-relations are "subring of", "left ideal of" and "ideal of". A large class of examples is provided by the following propositions, the proofs of which are elementary.

Proposition 1. *Let $M \subseteq W$ be closed under homomorphisms and hereditary to ideals. Then the following are H-relations*:

(i) $\{(I, R) | I$ *is an ideal of* R *and* $R/I \in M\}$.

(ii) $\{(I, R) | I$ *is a subring of* R *and* $I \in M\}$.

(iii) $\{(I, R) | I$ *is a subring of* R *and the ideal* $G(I, R)$ *of* R *generated by* I *is in* $M\}$.

Proposition 2. *Any union or intersection of H-relations is again an H-relation.*

Thus for example $\{(I, R) | I$ is a commutative left ideal of $R\}$ and $\{(I, R) | I$ is an ideal of finite index in $R\}$ are H-relations.

If σ and τ are H-relations, define $\sigma \circ \tau$ as follows: $I\sigma \circ \tau R$ if and only if there exists $J \subseteq R$ such that $I\sigma J$ and $J\tau R$. Also define $I\sigma \wedge \tau R$ if and only if there exists $A\sigma R$ and $B\tau R$ such that $I = A \cap B$.

Proposition 3. *If σ and τ are H-relations, then $\sigma \circ \tau$ and $\sigma \wedge \tau$ are also H-relations.*

For example, let σ = "is an accessible subring of"; then $\sigma = \cup_{n=1}^{\infty} \triangleleft^n$.

If M is any subclass of W we denote by LM the lower radical class determined in W by M. It is proved in (**3**) that if M is hereditary to ideals, so is LM. This result is reproved in (**4**), and it is observed in (**7**) that with slight modifications the latter argument shows that if M is hereditary to left ideals, right ideals or subrings, then so is LM. A good summary of all these results may be found in (5).

For our purposes here the lower radical construction of (4) is useful, and we sketch it for the reader's convenience. Let M_1 be the homomorphic closure of a given class $M \subseteq W$. Suppose β is an ordinal greater than one and that the classes M_α have been defined for all $\alpha < \beta$. If β is a limit ordinal, admit R to M_β if and only if R is the union of a chain of ideals each belonging to one of the classes M_α for $\alpha < \beta$. If $\beta - 1$ exists, set $M_\beta = \{R|R$ has an ideal $I \in M_{\beta-1}$ such that $R/I \in M_{\beta-1}\}$. Then, as is shown in (4), each class M_β is homomorphically closed and LM is the union of all of them. The argument used in (4) to show that M hereditary implies LM hereditary easily applies to prove the following more general result.

Theorem 4. *Let σ be an H-relation and let $M \subseteq W$ be a homomorphically closed class which is σ-hereditary in the sense that $R \in M$ and $I\sigma R$ imply $I \in M$. Then LM is also σ-hereditary.*

Each nonempty subclass M of W determines a class

$$UM = \{R \in W | I \neq R \Rightarrow R/I \notin M\}$$

The classes M for which UM is a hereditary radical class are characterised in (2). for each H-relation σ the σ-hereditary upper radicals are characterised by the following result. For each class M let SUM denote the class $\{R \in W | 0 \neq I \lhd R \Rightarrow I \notin UM\}$.

Theorem 5. *The class UM is a σ-hereditary radical class in W if and only if M has the properties:–*

(i) *If $0 \neq R \in M$, there exists $0 \neq R/I \in SUM$.*

(ii) *If $0 \neq I\sigma R \in W$ is such that some $I/J \neq 0$ is in M, then there exists $H \lhd R$ such that $0 \neq R/H \in SUM$.*

Proof. By (2), UM is a radical class if and only if (i). If UM is σ-hereditary, suppose $0 \neq I\sigma R$ and $I/J \neq 0$ is in M. Then $I \notin UM$ since UM is homomorphically closed, so that $R \notin UM$ since UM is σ-hereditary. But then since UM is a radical class, R must have a nonzero UM-semisimple homomorphic image. This establishes (ii).

On the other hand if M satisfies (i) and (ii), then let $0 \neq R \in UM$ and let $0 \neq I\sigma R$. If $I \notin UM$, then there exists $I/J \neq 0$ in M, so by (ii) R has a UM-semisimple image, i.e. $R \notin UM$. This is a contradiction. Therefore UM is σ-hereditary.

Another application of the notion of H-relation occurs in relation to the study of strongly hereditary radical classes, which was introduced in (6). Let M be a homomorphically closed class such that if $J \lhd \circ \lhd R$ then the ideal $G(J, R)$ of R generated by J is again in M. Then Theorem 2.4 of (9) states that LM has the same property. A one-sided version of this result with a different proof appears in (**8**). This latter proof applies almost word for word to give the following more general theorem. Most of the results of (**8**) can be extended in a similar way.

Theorem 6. *Let σ be an H-relation such that $I\sigma R$ wherever I is an ideal of R. Let $M \subseteq W$ be homomorphically closed. Suppose that if $J \triangleleft \circ \sigma R$ and $J \in M$, then $G(J, R) \in M$. Then LM satisfies the same property.*

One further way to generalise the idea of heredity is to involve a second class. Let M and N be subclasses of W and let σ be an H-relation. We say that M is *σ-transfer hereditary* to N if $R \in M$ and $I\sigma R$ imply $I \in N$.

Proposition 7. *If M_1 is σ_1-transfer hereditary to N_1 and M_2 is σ_2-transfer hereditary to N_2, then $M_1 \cap M_2$ is $\sigma_1 \cap \sigma_2$-transfer hereditary to $N_1 \cap N_2$ and $M_1 \cup M_2$ is $\sigma_1 \cup \sigma_2$-transfer hereditary to $N_1 \cup N_2$.*

Proposition 8. *If M_1 is σ_1-transfer hereditary to M_2 and M_2 is σ_2-transfer hereditary to M_3 then M_1 is $\sigma_2 \circ \sigma_1$-transfer hereditary to M_3.*

Theorem 9. *Let σ be an H-relation and M and N homomorphically closed classes such that M is σ-transfer hereditary to N. Then LM is σ-transfer hereditary to LN.*

Proof. We prove by induction that M_β is σ-transfer hereditary to N_β for each ordinal β. The theorem follows easily. The result is true for $\beta = 1$. Suppose $\beta > 1$ and that M_α is σ-transfer hereditary to N_α for each $\alpha < \beta$. Let $R \in M_\beta$ and $I\sigma R$. First let β be a limit ordinal. Then $R = \cup I_\gamma$ where $\{I_\gamma\}$ is a chain of ideals each belonging to some M_α with $\alpha < \beta$. Since $I\sigma R$, $I \cap I_\gamma \sigma I_\gamma$ for each γ. Since $I_\gamma \in M_\alpha$ for some $\alpha < \beta$, the induction hypothesis implies that $I \cap I_\gamma \in N_\alpha$. Thus I, being the union of its chain of ideals $I \cap I_\gamma$, is a member of N_β.

Now suppose $\beta - 1$ exists. Then R has an ideal J with J and $R/J \in M_{\beta-1}$. Since $I \cap J\sigma J$ we have $I \cap J \in N_{\beta-1}$. Also $(I+J)/J\sigma R/J$ so $(I+J)/J \cong I/(I \cap J) \in N_{\beta-1}$. This means $I \in N_\beta$, finishing the proof.

Of course if $M = N$ this reduces to Theorem 4. We list a few other corollaries as examples of ways in which Theorem 9 can be applied. In these results rings are restricted to be associative and M is assumed to be a homomorphically closed class.

Corollary 10. *Suppose $R \in M$ implies every countable ideal of R is nil. Then the same is true of LM.*

Proof. M is σ-transfer hereditary to the class N of nil rings, where $\sigma = \{(I, R) | I$ is a countable ideal of $R\}$. Hence LM is σ-transfer hereditary to $LN = N$.

Corollary 11. *Suppose $R \in M$ implies every nonzero countable ideal of R is non-nil. Then the same is true of LM.*

Proof. Let $\sigma = \{(I, R) | I$ is a countable nil ideal of $R\}$. Then σ is an H-relation by Propositions 1 and 2, and M is σ-transfer hereditary to the class $\{(0)\}$. Thus so is LM.

Corollary 12. *If $R \in M$ implies every nil ideal of R is locally nilpotent, then the same is true of LM.*

Proof. Local nilpotence is a radical property.

It is also of interest to consider hypotheses of the form $A \sigma R$, $R \in M \Rightarrow P(A)$, where P is a predicate. Let A be a subring of a ring R; we define $I(A, R) = \{x \in R | xA \subseteq A$ and $Ax \subseteq A\}$. If R is an associative ring, $I(A, R)$ is a subring of R called the idealizer of A. The next result, however, does not assume associativity.

Theorem 13. *Let σ be an H-relation and M a homomorphically closed class. Suppose that whenever $A \sigma R \in M$, then $I(A, R) = A$. Then the same is true of LM.*

Proof. Again we proceed by induction as in the proof of Theorem 4, showing that $I(A, R) = A$ for each $A \sigma R \in M_\beta$ for each ordinal β. The result is true for $\beta = 1$. Suppose that $\beta > 1$ and that if $\alpha < \beta$ and $A \sigma R \in M_\alpha$, then $I(A, R) = A$.

If β is a limit ordinal then $R = \cup J_\gamma$ where $\{J_\gamma\}$ is a chain of ideals contained in $\cup_{\alpha<\beta} M_\alpha$. Since $A \cap J_\gamma \sigma J_\gamma$ for each γ, $I(A \cap J_\gamma, J_\gamma) = A \cap J_\gamma$ by the inductive hypothesis. Suppose $I(A, R) \neq A$. Then there exists $x \in R \backslash A$ such that $xA \subseteq A$ and $Ax \subseteq A$. For some γ_0, $x \in J_{\gamma_0}$. Then $x(A \cap J_{\gamma_0}) \not\subseteq A \cap J_{\gamma_0}$ or $(A \cap J_{\gamma_0})x \not\subseteq A \cap J_{\gamma_0}$ since $I(A \cap J_{\gamma_0}, J_{\gamma_0}) = A \cap J_{\gamma_0}$. Hence there exists $a \in A \cap J_{\gamma_0}$ such that either $xa \notin A \cap J_{\gamma_0}$ or $ax \notin A \cap J_{\gamma_0}$. But $ax, xa \in J_{\gamma_0}$. Hence xa or $ax \notin A$. This is a contradiction.

If, on the other hand, $\beta - 1$ exists, there is $K \lhd R$ such that R/K, $K \in M_{\beta-1}$. By the inductive hypothesis, since $(A + K)/K \sigma R/K$ and $A \cap K \sigma K$, we have $I((A + K)/K, R/K) = (A + K)/K$ and $I(A \cap K, K) = A \cap K$. If $I(A, R) \neq A$ then there exists $x \in R \backslash A$ such that $xA \cup Ax \subseteq A$. Let $\bar{x}$, $\bar{A}$ denote the canonical images of x and A in R/K. Then $\bar{x}\bar{A} \cup \bar{A}\bar{x} \subseteq \bar{A}$. Since $I(\bar{A}, \bar{R}) = \bar{A}$, $\bar{x} \in \bar{A}$. Hence $x \in A + K$. But if $x \notin A$, $x = a + k$ with $k \neq 0$. Since $(a + k)A \cup A(a + k) \subseteq A$ implies $kA \cup Ak \subseteq A$, we may take $x \in K$. But then $x \notin K \cap A$ and $xA \cup Ax \subseteq A$, which contradicts $I(K \cap A, K) = K \cap A$.

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