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Abstract

We classify del Pezzo surfaces with quotient singularities and Picard rank one which admit a $\mathbb{Q}$-Gorenstein smoothing. These surfaces arise as singular fibres of del Pezzo fibrations in the 3-fold minimal model program and also in moduli problems.

1. Introduction

We give a complete classification of del Pezzo surfaces with quotient singularities and Picard rank one which admit a $\mathbb{Q}$-Gorenstein smoothing. This solves a problem posed by Kollár, cf. [Kol08, § 4].

One of the possible end products of the 3-fold minimal model program is a del Pezzo fibration. A *del Pezzo fibration* is a morphism $f: Y \to S$ with connected fibres such that Y is a 3-fold with terminal singularities, S is a smooth curve, $-K_Y$ is relatively ample, and the relative Picard number $\rho(Y/S)$ equals 1. In particular, a general fibre of f is a smooth del Pezzo surface. Typically, a singular fibre $X = Y_s$ of a del Pezzo fibration Y/S is a normal del Pezzo surface with quotient singularities. Moreover, if we work locally analytically at $s \in S$, we can run a relative minimal model program over S to reduce to the case $\rho(X) = 1$. This is a key motivation for our work.

Let X be a normal surface with quotient singularities. We say that X admits a $\mathbb{Q}$-*Gorenstein smoothing* if there exists a deformation $\mathcal{X}/(0 \in T)$ of X over a smooth curve germ such that the general fibre is smooth and $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier. (The requirement that $K_{\mathcal{X}}$ be $\mathbb{Q}$-Cartier is natural from the point of view of the minimal model program and is important in moduli problems, cf. [KS88, § 5.4]. It is automatically satisfied if X is Gorenstein.)

Theorem 1.1. *Let X be a projective surface with quotient singularities such that $-K_X$ is ample, $\rho(X) = 1$, and X admits a $\mathbb{Q}$-Gorenstein smoothing. Then X is one of the following:*

(1) *a toric surface as in Theorem 4.1;*

(2) *a deformation of a toric surface from (1), determined by specifying the subset of singularities to be partially smoothed as in Corollary 2.7;*

(3) *a sporadic surface as in Example 8.3.*

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There are 14 infinite families of toric examples; see Theorem 4.1. The surfaces in each family correspond to solutions of a Markov-type equation. The solutions of the (original) Markov equation

$$a^2 + b^2 + c^2 = 3abc$$

correspond to the vertices of an infinite tree such that each vertex has degree three. Here two vertices are joined by an edge if they are related by a so-called mutation of the form

$$(a, b, c) \mapsto (a, b, 3ab - c).$$

The solutions of the other equations are described similarly.

Given one of the toric surfaces Y, the $\mathbb{Q}$-Gorenstein deformations of Y which preserve the Picard number are as follows. First, there are no locally trivial deformations and no local-to-global obstructions to deformations. Second, for each singularity $Q \in Y$, the deformation is either locally trivial or a deformation of a singularity of index >1 to a Du Val singularity of type A; see Corollary 2.7. Moreover, in the second case, the deformation is essentially unique (it is obtained from a fixed one-parameter deformation by base change).

There are 20 isolated sporadic surfaces and one family of sporadic surfaces parametrised by $\mathbb{A}^1$; see Example 8.3. Every sporadic surface has index $\leqslant 2$. In particular, they occur in the list of Alexeev and Nikulin [AN06].

Our methods produce many examples of smoothable del Pezzo surfaces X with quotient singularities of Picard rank $\rho(X) > 1$. Indeed, let Z be one of the toric surfaces enumerated in Theorem 4.1 and let X be any partial $\mathbb{Q}$-Gorenstein smoothing of Z. Then the Picard number $\rho(X)$ can be computed by the formula in Proposition 2.6.

In the case $K_X^2 = 9$ we obtain the following stronger result. This completely solves the problem studied by Manetti in [Man91].

Corollary 1.2. *Let X be a projective surface with quotient singularities which admits a smoothing to the plane. Then X is a $\mathbb{Q}$-Gorenstein deformation of a weighted projective plane $\mathbb{P}(a^2, b^2, c^2)$, where (a, b, c) is a solution of the Markov equation.*

Proof. If X is a surface with quotient singularities which admits a smoothing to the plane, then $\rho(X) = 1$, $-K_X$ is ample, and the smoothing is $\mathbb{Q}$-Gorenstein by [Man91, § 1]. □

We note that a partial classification of the surfaces with $K_X^2 \geqslant 5$ was obtained by Manetti [Man91, Man93].

As a consequence of our techniques, we verify a particular case of Reid's general elephant conjecture (see, e.g., [Ale94]).

Theorem 1.3. *Let $f\colon V \to (0 \in T)$ be a del Pezzo fibration over the germ of a smooth curve. That is, V is a 3-fold with terminal singularities, f has connected fibres, $-K_V$ is ample over T, and $\rho(V/T) = 1$. Assume in addition that the special fibre is reduced and normal, and has only quotient singularities. Then a general member $S \in |-K_V|$ is a normal surface with Du Val singularities.*

Notation. Throughout this paper, we work over the field $k = \mathbb{C}$ of complex numbers. The symbol $\boldsymbol{\mu}_n$ denotes the group of nth roots of unity.

2. T-singularities

By definition, T-singularities are the quotient singularities of dimension two which admit a $\mathbb{Q}$-Gorenstein smoothing. We recall the classification of T-singularities from [KS88, § 3] and establish some basic results.

2.1 $\mathbb{Q}$-Gorenstein deformations

Let X be a normal surface such that K_X is $\mathbb{Q}$-Cartier. A deformation $\mathcal{X}/(0 \in S)$ of X over a germ $(0 \in S)$ is $\mathbb{Q}$-*Gorenstein* if locally analytically at each singular point $P \in X$ it is induced by an equivariant deformation of the canonical covering of $P \in X$. This definition was originally proposed by Kollár [Kol91] and the general theory was worked out in [Hac04, § 3]. We only use the explicit version in § 2.2. We note that, if X has quotient singularities and S is a smooth curve, then a deformation $\mathcal{X}/(0 \in S)$ is $\mathbb{Q}$-Gorenstein if and only if $K_{\mathcal{X}}$ is $\mathbb{Q}$-Cartier.

2.2 Definition and classification of T-singularities

Definition 2.1 [KS88, Definition 3.7]. Let $P \in X$ be a quotient singularity of dimension two. We say that $P \in X$ is a T *singularity* if it admits a $\mathbb{Q}$-Gorenstein smoothing. That is, there exists a $\mathbb{Q}$-Gorenstein deformation of $P \in X$ over a smooth curve germ such that the general fibre is smooth.

For $n, a \in \mathbb{N}$ with $(a, n) = 1$, let $(1/n)(1, a)$ denote the cyclic quotient singularity $(0 \in \mathbb{A}^2_{u,v}/\boldsymbol{\mu}_n)$ given by

$$\boldsymbol{\mu}_n \ni \zeta \colon (u, v) \mapsto (\zeta u, \zeta^a v).$$

The following result is due to Wahl [Wah81, Example 5.9.1], [LW86, Propositions 5.7 and 5.9]. It was proved by a different method in [KS88, Proposition 3.10].

Proposition 2.2. *A T singularity is either a Du Val singularity or a cyclic quotient singularity of the form $(1/dn^2)(1, dna - 1)$ for some $d, n, a \in \mathbb{N}$ with $(a, n) = 1$.*

The singularity $(1/dn^2)(1, dna - 1)$ has index n and canonical covering $(1/dn)(1, -1)$, the Du Val singularity of type A_{dn-1}. We have an identification

$$\frac{1}{dn}(1, -1) = (xy = z^{dn}) \subset \mathbb{A}^3_{x,y,z},$$

where $x = u^{dn}$, $y = v^{dn}$, and $z = uv$. Taking the quotient by $\boldsymbol{\mu}_n$, we obtain

$$\frac{1}{dn^2}(1, dna - 1) = (xy = z^{dn}) \subset \frac{1}{n}(1, -1, a).$$

Hence, a $\mathbb{Q}$-Gorenstein smoothing is given by

$$(xy = z^{dn} + t) \subset \frac{1}{n}(1, -1, a) \times \mathbb{A}^1_t.$$

More generally, a versal $\mathbb{Q}$-Gorenstein deformation of $(1/dn^2)(1, dna - 1)$ is given by

$$(xy = z^{dn} + t_{d-1}z^{(d-1)n} + \cdots + t_0) \subset \frac{1}{n}(1, -1, a) \times \mathbb{A}^1_{t_0,\ldots,t_{d-1}}. \tag{1}$$

We call a T singularity of the form $(1/dn^2)(1, dna - 1)$ a T_d-*singularity.*

Proposition 2.3. *Let $(P \in \mathcal{X})/(0 \in S)$ be a $\mathbb{Q}$-Gorenstein deformation of $(1/dn^2)(1, dna - 1)$. Then the possible singularities of a fibre of $\mathcal{X}/S$ are as follows: either $A_{e_1-1}, \ldots, A_{e_s-1}$ or $(1/e_1n^2)(1, e_1na - 1)$, $A_{e_2-1}, \ldots, A_{e_s-1}$, where $e_1, \ldots, e_s$ is a partition of d.*

Proof. The family $\mathcal{X}/S$ is pulled back from the versal $\mathbb{Q}$-Gorenstein deformation (1). Hence, each fibre of $\mathcal{X}/S$ has the form

$$(xy = z^{dn} + a_{d-1}z^{(d-1)n} + \cdots + a_0) \subset \frac{1}{n}(1, -1, a)$$

for some $a_0, \ldots, a_{d-1} \in k$. Write

$$z^{dn} + a_{d-1}z^{(d-1)n} + \cdots + a_0 = \prod (z^n - \gamma_i)^{e_i},$$

where the γ_i are distinct. Then the fibre has singularities as described in the statement (the second case occurs if $\gamma_i = 0$ for some i). □

2.3 Noether's formula

For $P \in X$ a T singularity, let M be the Milnor fibre of a $\mathbb{Q}$-Gorenstein smoothing. Thus, $(M, \partial M)$ is a smooth 4-manifold with boundary, and is uniquely determined by $P \in X$ since the $\mathbb{Q}$-Gorenstein deformation space of $P \in X$ is smooth. Let $\mu_P = b_2(M)$, the *Milnor number*.

Lemma 2.4 [Man91, § 3]. *If $P \in X$ is a Du Val singularity of type A_r, D_r, or E_r, then $\mu_P = r$. If $P \in X$ is of type $(1/dn^2)(1, dna - 1)$, then $\mu_P = d - 1$.*

Remark 2.5. If M is the Milnor fibre of a smoothing of a normal surface singularity $P \in X$, then M has the homotopy type of a CW complex of real dimension two by Morse theory and $b_1(M) = 0$ [GS83]. In particular, the Euler number $e(M) = 1 + \mu_P$.

Proposition 2.6. *Let X be a projective surface with T-singularities. Then*

$$K_X^2 + e(X) + \sum_{P \in \operatorname{Sing} X} \mu_P = 12\chi(\mathcal{O}_X)$$

and

$$\chi(\mathcal{O}_X(mK_X)) = \chi(\mathcal{O}_X) + \tfrac{1}{2}m(m-1)K_X^2$$

for $m \in \mathbb{Z}$.

In particular, if X is rational, then

$$K_X^2 + \rho(X) + \sum_{P \in \operatorname{Sing} X} \mu_P = 10$$

and if $-K_X$ is big and nef, then

$$h^0(\mathcal{O}_X(-nK_X)) = 1 + \tfrac{1}{2}n(n+1)K_X^2$$

for $n \in \mathbb{Z}_{\geqslant 0}$.

Proof. For X a projective normal surface with quotient singularities, there is a singular Noether formula

$$K_X^2 + e(X) + \sum_P c_P = 12\chi(\mathcal{O}_X),$$

where the sum is over the singular points $P \in X$, and the correction term c_P depends only on the local analytic isomorphism type of the singularity $P \in X$. (Indeed, let $\pi \colon \tilde{X} \to X$ be the minimal resolution of X and $E_1, \ldots, E_n$ the exceptional curves. Noether's formula on $\tilde{X}$ gives $K_{\tilde{X}}^2 + e(\tilde{X}) = 12\chi(\mathcal{O}_{\tilde{X}})$. Write $K_{\tilde{X}} = \pi^*K_X + \sum a_i E_i = \pi^*K_X + A$. Then $K_{\tilde{X}}^2 = K_X^2 + A^2$, $e(\tilde{X}) = e(X) + n$ (by the Mayer–Vietoris sequence), and $\chi(\mathcal{O}_{\tilde{X}}) = \chi(\mathcal{O}_X)$ (because X has rational singularities). Hence, $K_X^2 + e(X) + (A^2 + n) = 12\chi(\mathcal{O}_X)$.) Similarly, if D is a Weil divisor on X we have a singular Riemann–Roch formula

$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \tfrac{1}{2}D(D - K_X) + \sum c_P(D),$$

where the sum is over points $P \in X$ where the divisor D is not Cartier and the correction term $c_P(D)$ depends only on the local analytic isomorphism type of the singularity $P \in X$ and the local analytic divisor class of D at $P \in X$ [Bla95, § 1.2].

For each T singularity $P \in X$, there exist a projective surface Y with a unique singularity isomorphic to $P \in X$ and a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{Y}/(0 \in T)$ by Looijenga's globalisation theorem [Loo85, Appendix]. We use $\mathcal{Y}/T$ to compute the correction terms c_P and $c_P(mK_X)$. Let Y' denote the general fibre. We have $K_{Y'}^2 = K_Y^2$, $\chi(\mathcal{O}_{Y'}) = \chi(\mathcal{O}_Y)$, and $e(Y') = e(Y) + \mu_P$ (because the Milnor fibre of the smoothing has Euler number $1 + \mu_P$). Hence, $K_Y^2 + e(Y) + \mu_P = 12\chi(\mathcal{O}_Y)$, so $c_P = \mu_P$. The Riemann–Roch formula for the line bundle $\mathcal{O}_{Y'}(mK_{Y'})$ on Y' gives $\chi(\mathcal{O}_{Y'}(mK_{Y'})) = \chi(\mathcal{O}_{Y'}) + (1/2)m(m-1)K_{Y'}^2$. We have $\chi(\mathcal{O}_{Y'}(mK_{Y'})) = \chi(\mathcal{O}_Y(mK_Y))$ (because $\omega_{\mathcal{Y}/T}^{[m]}$ is flat over T and commutes with base change since $\mathcal{Y}/T$ is $\mathbb{Q}$-Gorenstein). So, $\chi(\mathcal{O}_Y(mK_Y)) = \chi(\mathcal{O}_Y) + \frac{1}{2}m(m-1)K_Y^2$ and $c_P(mK_X) = 0$.

Finally, if $-K_X$ is nef and big, then $H^i(\mathcal{O}_X(-nK_X)) = 0$ for $i > 0$ and $n \geqslant 0$ by Kawamata–Viehweg vanishing, so $h^0(\mathcal{O}_X(-nK_X)) = 1 + \frac{1}{2}n(n+1)K_X^2$, as required. □

COROLLARY 2.7. *Let X be a projective surface with T-singularities and X' a fibre of a $\mathbb{Q}$-Gorenstein deformation $\mathcal{X}/(0 \in T)$ of X over a smooth curve germ. Then $e(X) = e(X')$ if and only if, at each singular point $P \in X$, the deformation is either locally trivial or a deformation of a T_d singularity to an A_{d-1} singularity.*

Proof. This follows immediately from Propositions 2.3 and 2.6. □

2.4 Minimal resolutions of T-singularities

Given a cyclic quotient singularity $(1/n)(1, a)$, let $[b_1, \ldots, b_r]$ be the expansion of n/a as a Hirzebruch–Jung continued fraction [Ful93, p. 46]. Then the exceptional locus of the minimal resolution of $(1/n)(1, a)$ is a chain of smooth rational curves with self-intersection numbers $-b_1, \ldots, -b_r$. The strict transforms of the coordinate lines $(u = 0)$ and $(v = 0)$ intersect the right and left end components of the chain, respectively.

Remark 2.8. Note that $[b_r, \ldots, b_1]$ corresponds to the same singularity as $[b_1, \ldots, b_r]$ with the roles of the coordinates u and v interchanged. Thus, if $[b_1, \ldots, b_r] = n/a$, then $[b_r, \ldots, b_1] = n/a'$, where a' is the inverse of a modulo n.

We recall the description of the minimal resolution of the cyclic quotient singularities of class T due to Wahl. Let a T_d *string* be a string $[b_1, \ldots, b_r]$ which corresponds to a T_d singularity.

PROPOSITION 2.9 ([KS88, Proposition 3.11], [Man91, Theorem 17]).

(1) $[4]$ *is a T_1-string and, for $d \geqslant 2$, $[3, 2, \ldots, 2, 3]$ (where there are $(d-2)$ 2's) is a T_d-string.*

(2) *If $[b_1, \ldots, b_r]$ is a T_d-string, then so are $[b_1 + 1, b_2, \ldots, b_r, 2]$ and $[2, b_1, \ldots, b_r + 1]$.*

(3) *For each d, all T_d-strings are obtained from the example in (1) by iterating the steps in (2).*

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3. Unobstructedness of deformations

Proposition 3.1. *Let X be a projective surface with log canonical singularities such that $-K_X$ is big. Then there are no local-to-global obstructions to deformations of X. In particular, if X has T-singularities, then X admits a $\mathbb{Q}$-Gorenstein smoothing.*

Proof. The local-to-global obstructions to deformations of X lie in $H^2(T_X)$, where $T_X = \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ is the tangent sheaf of X. This follows from either a direct cocycle computation (cf. [Wah81, Proposition 6.4]) or the theory of the cotangent complex [Ill71, ch. III, Proposition 2.1.2.3]. Since $H^2(T_X) = \mathrm{Hom}(T_X, \mathcal{O}_X(K_X))^*$ by Serre duality, it suffices to show that $\mathrm{Hom}(T_X, \mathcal{O}_X(K_X)) = 0$ or, equivalently, $\mathrm{Hom}(\mathcal{O}_X(-K_X), \Omega_X^{\vee\vee}) = 0$. If $L \subset \Omega_X^{\vee\vee}$ is a rank-one reflexive subsheaf, then the Kodaira–Iitaka dimension $\kappa(X, L) \leqslant 1$ by Bogomolov–Sommese vanishing for log canonical varieties [GKK08, Theorem 1.4]. So, $-K_X$ big implies that $\mathrm{Hom}(\mathcal{O}_X(-K_X), \Omega_X^{\vee\vee}) = 0$, as required. □

4. Toric surfaces

Theorem 4.1. *The projective toric surfaces with T-singularities and Picard rank one are as follows. There are 14 infinite families $(1), \dots, (8.4)$ which we list in Tables 1 and 2. In cases $(1), \dots, (4)$, the surface X is a weighted projective plane $\mathbb{P}(w_0, w_1, w_2)$, and the weights w_0, w_1, w_2 are determined by a solution (a, b, c) of a Markov-type equation. In the remaining cases, the surface X is a quotient of one of the above weighted projective planes Y by $\boldsymbol{\mu}_e$ acting freely in codimension one. The action is diagonal with weights (m_0, m_1, m_2), i.e.,*

$$\boldsymbol{\mu}_e \ni \zeta \colon (X_0, X_1, X_2) \mapsto (\zeta^{m_0} X_0, \zeta^{m_1} X_1, \zeta^{m_2} X_2),$$

where X_0, X_1, X_2 are homogeneous coordinates on Y. We also record K_X^2 and the values of $d = \mu + 1$ for the singularities of X.

Table 1. Toric surfaces with simply connected smooth locus.

X	w_0, w_1, w_2	Markov-type equation	K_X^2	d
(1)	a^2, b^2, c^2	$a^2 + b^2 + c^2 = 3abc$	9	1, 1, 1
(2)	$a^2, b^2, 2c^2$	$a^2 + b^2 + 2c^2 = 4abc$	8	1, 1, 2
(3)	$a^2, 2b^2, 3c^2$	$a^2 + 2b^2 + 3c^2 = 6abc$	6	1, 2, 3
(4)	$a^2, b^2, 5c^2$	$a^2 + b^2 + 5c^2 = 5abc$	5	1, 1, 5

Remark 4.2. With notation as above, let $X^0 \subset X$ be the smooth locus and $p^0 \colon Y^0 \to X^0$ the restriction of the cover $Y \to X$. Then p^0 is the universal cover of X^0. In particular, $\pi_1(X^0)$ is cyclic of order e.

The solutions of the Markov-type equations in Theorem 4.1 may be described as follows [KN98, Proposition 3.7]. We say that a solution (a, b, c) is *minimal* if $a + b + c$ is minimal. The equations (1), (2), and (3) have a unique minimal solution $(1, 1, 1)$, and (4) has minimal solutions $(1, 2, 1)$ and $(2, 1, 1)$. Given one solution, we obtain another by regarding the equation

TABLE 2. Toric surfaces with non simply connected smooth locus.

X	Y	e	m_0, m_1, m_2	K_X^2	d
(5)	(2)	2	$0, 1, -1$	4	2, 2, 4
(6.1)	(1)	3	$0, 1, -1$	3	3, 3, 3
(6.2)	(3)	2	$0, 1, -1$	3	1, 2, 6
(7.1)	(2)	4	$0, 1, 1$	2	1, 1, 8
(7.2)	(2)	4	$0, 1, -1$	2	2, 4, 4
(7.3)	(3)	3	$0, 1, -1$	2	1, 3, 6
(8.1)	(1)	9	$0, 1, -1$	1	1, 1, 9
(8.2)	(2)	8	$0, 1, -1$	1	1, 2, 8
(8.3)	(3)	6	$0, 1, -1$	1	2, 3, 6
(8.4)	(4)	5	$0, 1, -1$	1	1, 5, 5

as quadratic in one of the variables, c (say), and replacing c by the other root. Explicitly, if the equation is $\alpha a^2 + \beta b^2 + \gamma c^2 = \lambda abc$, then

$$(a, b, c) \mapsto \left(a, b, \frac{\lambda}{\gamma} ab - c\right). \tag{2}$$

This process is called a *mutation*. Every solution is obtained from a minimal solution by a sequence of mutations.

For each equation, we define an infinite graph Γ such that the vertices are labelled by the solutions and two vertices are joined by an edge if they are related by a mutation. For (1), Γ is an infinite tree such that each vertex has degree three, and there is an action of S_3 on Γ given by permuting the variables a, b, c. The other cases are similar; see [KN98, § 3.8] for details.

Proof of Theorem 4.1. Let X be a projective toric surface such that X has only T-singularities and $\rho(X) = 1$. The surface X is given by a complete fan Σ in $N_{\mathbb{R}} \simeq \mathbb{R}^2$, where $N \simeq \mathbb{Z}^2$ is the group of one-parameter subgroups of the torus. The fan Σ has three rays because $\rho(X) = 1$. Let $v_0, v_1, v_2 \in N$ be the minimal generators of the rays. There is a unique relation

$$w_0 v_0 + w_1 v_1 + w_2 v_2 = 0,$$

where $w_0, w_1, w_2 \in \mathbb{N}$ are pairwise coprime. Let $N_Y \subseteq N$ denote the subgroup generated by v_0, v_1, v_2. Let $p\colon Y \to X$ be the finite toric morphism corresponding to the inclusion $N_Y \subseteq N$. Then Y is isomorphic to the weighted projective plane $\mathbb{P}(w_0, w_1, w_2)$ and p is a cyclic cover of degree $e = |N/N_Y|$, which is étale over the smooth locus $X^0 \subset X$. The surface Y has only T-singularities because a cover of a T singularity which is étale in codimension one is again a T singularity (this follows easily from the classification of T singularities).

The surface X has three cyclic quotient singularities of class T. Let the singularities of X be $(1/d_i n_i^2)(1, d_i n_i a_i - 1)$ for $i = 0, 1, 2$. Then

$$d_0 + d_1 + d_2 + K_X^2 = 12 \tag{3}$$

by Proposition 2.6. The singularities of X are quotients of the singularities $(1/w_0)(w_1, w_2)$, $(1/w_1)(w_0, w_2)$, and $(1/w_2)(w_0, w_1)$ of Y by $\boldsymbol{\mu}_e$. Hence, $d_i n_i^2 = e w_i$. Also, $K_Y^2 = e K_X^2$ because $p\colon Y \to X$ has degree e and is étale in codimension one. Let H be the ample generator of the class group of Y. Then $K_Y \sim -(w_0 + w_1 + w_2)H$ and $H^2 = 1/w_0w_1w_2$. We deduce that

$$d_0n_0^2 + d_1n_1^2 + d_2n_2^2 = \sqrt{K_X^2 d_0d_1d_2} \cdot n_0n_1n_2. \tag{4}$$

In particular,

$$\sqrt{K_X^2 d_0d_1d_2} = \sqrt{\left(12 - \sum d_i\right) d_0d_1d_2} \in \mathbb{Z}.$$

We compute all triples $d = (d_0, d_1, d_2)$ satisfying this condition. They are as listed in the last column of the tables above.

We first treat the cases $d = (1, 1, 1)$, $(1, 1, 2)$, $(1, 2, 3)$, and $(1, 1, 5)$. These are the cases for which $K_X^2 \geqslant 5$. Since $K_Y^2 = eK_X^2 \leqslant 9$ by Proposition 2.6, we deduce that $e = 1$. Thus, X is isomorphic to a weighted projective plane. The weights $d_in_i^2$ are determined by the solution (n_0, n_1, n_2) of (4), which is the Markov-type equation given in the statement. Conversely, we check that for any solution of (4) the weighted projective plane $X = \mathbb{P}(d_0n_0^2, d_1n_1^2, d_2n_2^2)$ has T-singularities and the expected value of d. We use the description of the solutions of (4) given above. We write $\lambda = \sqrt{K_X^2 d_0d_1d_2}$, and note that $d_0d_1d_2$ divides λ in each case. By induction using (2) we find that n_0, n_1, n_2 are pairwise coprime and $\gcd(n_i, \lambda/d_i) = 1$ for each i. In particular, the $d_in_i^2$ are pairwise coprime. Now consider the singularity $(1/d_0n_0^2)(d_1n_1^2, d_2n_2^2)$. We have

$$d_1n_1^2 + d_2n_2^2 = \lambda n_0n_1n_2 \mod d_0n_0^2$$

by (4), and so $\gcd(d_1n_1^2 + d_2n_2^2, d_0n_0^2) = d_0n_0$ because $\gcd((\lambda/d_0)n_1n_2, n_0) = 1$. Thus, this singularity is of type T_{d_0}.

For the remaining values of d, we determine the degree e of the cover $p\colon Y \to X$ as follows. We have $e = \gcd(d_0n_0^2, d_1n_1^2, d_2n_2^2)$. By inspecting (4) we find a factor of e and, together with the inequality $eK_X^2 = K_Y^2 \leqslant 9$, this is sufficient to determine e in each case. For example, let $d = (1, 2, 8)$. Then we find that n_0 is divisible by four and n_1 is even, so e is divisible by eight, and hence equal to eight. In each case we have $K_Y^2 \geqslant 5$, so Y is one of the surfaces classified above.

We now classify the possible actions of $\boldsymbol{\mu}_e$ on the covering surface Y. We have $Y = \mathbb{P}(d_0n_0^2, d_1n_1^2, d_2n_2^2)$, where $d = d_Y = (1, 1, 1), (1, 1, 2), (1, 2, 3)$, or $(1, 1, 5)$, and (n_0, n_1, n_2) is a solution of (4). The action is given by

$$\boldsymbol{\mu}_e \ni \zeta\colon (X_0, X_1, X_2) \mapsto (\zeta^{m_0}X_0, \zeta^{m_1}X_1, \zeta^{m_2}X_2),$$

where X_0, X_1, X_2 are the homogeneous coordinates on the weighted projective plane Y. In each case $d_0n_0^2 = n_0^2$ is coprime to e. So, we may assume that $m_0 = 0$. We may also assume that $m_1 = 1$ (because the action is free in codimension one). Consider the singularity $P_0 \in X$ below $(1:0:0) \in Y$. This singularity admits a covering by $(1/e)(1, m_2)$ (which is étale in codimension one). Hence, $(1/e)(1, m_2)$ is a T singularity. If e is square free, it follows that $m_2 = -1$. If $e = 4$, then $m_2 = \pm 1$. If $e = 8$, then $d_Y = (1, 1, 2)$ and $d_X = (1, 2, 8)$, so we may assume that $P_0 \in X$ is a T_8 singularity (note that a $\boldsymbol{\mu}_8$-quotient of a T_2 singularity cannot be a T_8 singularity). Thus, $P_0 \in X$ is covered by $(1/8)(1, -1)$ and so $m_2 = -1$. Similarly, if $e = 9$, then $d_Y = (1, 1, 1)$ and $d_X = (1, 1, 9)$, so we may assume that $P_0 \in X$ is a T_9 singularity and $m_2 = -1$. This gives the list of group actions above. Finally, it remains to check that for each such quotient $X = Y/\boldsymbol{\mu}_e$, the surface X has T-singularities with the expected values of d. This is a straightforward toric calculation, so we omit it. □

5. Surfaces with a D or E singularity

A *log del Pezzo surface* is a normal projective surface X such that X has only quotient singularities and $-K_X$ is ample.

Theorem 5.1. *Let X be a log del Pezzo surface such that $\rho(X)=1$, and assume that* $\dim|-K_X|\geqslant 1$.

(1) *If X has a Du Val singularity of type E, then K_X is Cartier.*
(2) *If X has a Du Val singularity of type D, then either K_X is Cartier or there is a unique non Du Val singularity of type $(1/m)(1,1)$ for some $m\geqslant 3$.*

Moreover, in both cases, a general member of $|-K_X|$ is irreducible and does not pass through the Du Val singularities.

Proof. Assume that X has a D or E singularity $P\in X$ and K_X is *not* Cartier. Let $\nu\colon \hat{X}\to X$ be the minimal resolution of the non Du Val singularities of X and write $\hat{P}=\nu^{-1}(P)$. So, $\hat{X}$ has only Du Val singularities and $\hat{P}\in\hat{X}$ is a D or E singularity. Let $\{E_i\}$ be the exceptional curves of ν and write $E=\sum E_i$.

Write $|-K_{\hat{X}}|=|M|+F$, where F is the fixed part and M is general in $|M|$. We have an equality

$$K_{\hat{X}}=\nu^*K_X+\sum a_iE_i,$$

where $a_i<0$ for all i because ν is minimal and we only resolve the non Du Val singularities [KM98, Lemma 3.41]. Hence, $\dim|-K_{\hat{X}}|=\dim|-K_X|$ and $F\geqslant E$.

We run the minimal model program on $\hat{X}$. We obtain a birational morphism $\phi\colon\hat{X}\to\overline{X}$ such that $\overline{X}$ has Du Val singularities and exactly one of the following holds.

(1) $K_{\overline{X}}$ is nef.
(2) $\rho(\overline{X})=2$ and there is a fibration $\psi\colon\overline{X}\to\mathbb{P}^1$ with $K_{\overline{X}}\cdot f<0$ for f a fibre.
(3) $\rho(\overline{X})=1$ and $-K_{\overline{X}}$ is ample.

Clearly, $K_{\overline{X}}$ is not nef because $\dim|-K_{\overline{X}}|\geqslant\dim|-K_{\hat{X}}|\geqslant 1$.

In the minimal model program for surfaces with Du Val singularities, the birational extremal contractions are weighted blowups $f\colon X\to Y$ of a smooth point $P\in Y$ with weights $(1,n)$ for some $n\in\mathbb{N}$. In particular, the exceptional divisor $E\subset X$ is a smooth rational curve and passes through a unique singularity of X which is of type $(1/n)(1,-1)=A_{n-1}$. See [KM99, Lemma 3.3].

Therefore, the birational morphism ϕ is an isomorphism near the D or E singularity $\hat{P}\in\hat{X}$ and $\overline{E}:=\phi_*E$ is contained in the smooth locus of $\overline{X}$. Note also that $\overline{E}\neq 0$ because $\rho(X)=1$ and X has a non Du Val singularity.

Suppose first that we are in case (3). We have $-K_{\overline{X}}\sim\overline{M}+\overline{F}$, where $\overline{M}:=\phi_*M$ is mobile and $\overline{F}:=\phi_*F\geqslant\overline{E}$. In particular, $\operatorname{Pic}(\overline{X})$ is not generated by $-K_{\overline{X}}$ because $\overline{M}+\overline{F}>\overline{E}$ and $\overline{E}$ is Cartier. Hence, $\overline{X}$ is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}(1,1,2)$ by the classification of Gorenstein log del Pezzo surfaces [Dem80]. (Indeed, if Y is a Gorenstein del Pezzo surface, let $f\colon\tilde{Y}\to Y$ be the minimal resolution. Then either Y is isomorphic to $\mathbb{P}^2$ or $\mathbb{P}(1,1,2)$, or $\tilde{Y}$ is obtained from $\mathbb{P}^2$ by a sequence of blowups. In the last case, let $C\subset\tilde{Y}$ be a (-1)-curve. Then

$$K_Y\cdot f_*C=f^*K_Y\cdot C=K_{\tilde{Y}}\cdot C=-1.$$

It follows that $-K_Y$ is a generator of $\operatorname{Pic}Y$ if $\rho(Y)=1$.) So, $\overline{X}$ does not have a D or E singularity, a contradiction.

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So, we are in case (2). Write $p=\psi\circ\phi\colon \hat{X}\to\mathbb{P}^1$. The divisor E has a p-horizontal component, say E_1 (because $\rho(X)=1$ and so there does not exist a morphism $X\to\mathbb{P}^1$). If f is a general fibre of p, then

$$2=-K_{\hat{X}}\cdot f\geqslant E_1\cdot f\geqslant 1.$$

If $E_1\cdot f=1$, then all fibres of ψ are reduced (because $\overline{E}_1$ is contained in the smooth locus of $\overline{X}$), so $\overline{X}$ is smooth [KM99, Lemma 11.5.2], a contradiction. So, $E_1\cdot f=2$. Then $(M+(F-E_1))\cdot f=0$, so M and $F-E_1$ are p-vertical. In particular, M is base-point free and E_1 has coefficient 1 in F. Since

$$2\geqslant 2-2p_a(E_1)=-(K_{\hat{X}}+E_1)\cdot E_1=(M+(F-E_1))\cdot E_1\geqslant M\cdot E_1\geqslant 2,$$

we find that $M\cdot E_1=2$ and $(F-E_1)\cdot E_1=0$. Thus, M is a fibre of ψ and the divisors $M+E_1$ and $F-E_1$ have disjoint support. But, $M+F\sim -K_{\hat{X}}$ is connected because

$$H^1(\mathcal{O}_{\hat{X}}(-M-F))=H^1(K_{\hat{X}})=H^1(\mathcal{O}_{\hat{X}})^*=0.$$

Hence, $F=E=E_1$. In particular, X has a unique non Du Val singularity of type $(1/m)(1,1)$ (where $E_1^2=-m$). Also, a general member of $|-K_X|$ is irreducible and does not pass through any Du Val singularities. Finally, $\overline{X}$ does not have a singularity of type E by the classification of fibres of $\mathbb{P}^1$ fibrations with Du Val singularities [KM99, Lemma 11.5.12]. So, X does not have an E singularity.

If K_X is Cartier, then a general member of $|-K_X|$ is smooth and misses the singular points by [Dem80]. □

6. Surfaces of index ⩽2

Alexeev and Nikulin classified log del Pezzo surfaces X of index ⩽2 [AN06]. They proved that X is a $\mathbb{Z}/2\mathbb{Z}$ quotient of a K3 surface and used the Torelli theorem for K3 surfaces to obtain the classification. In this section, we deduce the index ⩽2 case of our main theorem from their result.

We note that the quotient singularities of index ⩽2 are the Du Val singularities and the cyclic quotient singularities of type $(1/4d)(1,2d-1)$; see [AN06]. In particular, they are T-singularities.

Proposition 6.1. *Let X be a log del Pezzo surface of index ⩽2 such that $\rho(X)=1$. Then exactly one of the following holds.*

(1) *X is a $\mathbb{Q}$-Gorenstein deformation of a toric surface.*

(2) *X has a D singularity, an E singularity, or ⩾4 Du Val singularities.*

Proof. We first observe that the two conditions cannot both hold. If X is a $\mathbb{Q}$-Gorenstein deformation of a toric surface Y, then necessarily $\rho(Y)=1$ and Y has only T-singularities. In particular, Y has at most three singularities. Moreover, since the deformation preserves the Picard number, the only possible non-trivial deformation of a singularity of Y is a deformation of a T_d singularity to a A_{d-1} singularity by Corollary 2.7. Finally, note that Y does not have a D or E singularity because Y is toric. Hence, X has at most three singularities and does not have a D or E singularity.

We now use the classification of log del Pezzo surfaces of index ⩽2 and Picard rank one [AN06, Theorems 4.2 and 4.3]. We check that each such surface X which does not satisfy condition (2) is a deformation of a toric surface Y. By [AN06], X is determined up to isomorphism by its

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singularities. So, it suffices to exhibit a toric surface Y such that $\rho(Y)=1$ and the singularities of X are obtained from the singularities of Y by a $\mathbb{Q}$-Gorenstein deformation which preserves the Picard number. We list the surfaces Y in the tables below. □

In Table 3, for each log del Pezzo surface X of Picard rank one and index $\leqslant 2$ such that X does not satisfy condition (2) of Proposition 6.1, we exhibit a toric surface Y such that X is a $\mathbb{Q}$-Gorenstein deformation of Y. We give the number of the surface X in the list of Alexeev and Nikulin [AN06, pp. 93–100]. We use the description of the toric surfaces Y given in Theorem 4.1. We give the number of the infinite family to which Y belongs and the solution (a, b, c) of the Markov-type equation corresponding to Y. We record the value of $d=\mu+1$ for each singularity in the last column of the table.

Table 3. Toric degenerations of surfaces of index $\leqslant 2$.

X	Sing X	Y	Sing Y	d
1		$(1),\ (1,1,1)$		1, 1, 1
2	A_1	$(2),\ (1,1,1)$	A_1	1, 1, 2
5	A_1, A_2	$(3),\ (1,1,1)$	A_1, A_2	1, 2, 3
6	A_4	$(4),\ (1,2,1)$	$\frac{1}{4}(1,1),\ A_4$	1, 1, 5
7b	$2A_1,\ A_3$	$(5),\ (1,1,1)$	$2A_1,\ A_3$	2, 2, 4
8b	$A_1,\ A_5$	$(6.2),\ (1,1,1)$	$\frac{1}{4}(1,1),\ A_1,\ A_5$	1, 2, 6
8c	$3A_2$	$(6.1),\ (1,1,1)$	$3A_2$	3, 3, 3
9b	A_7	$(7.1),\ (1,1,1)$	$2\frac{1}{4}(1,1),\ A_7$	1, 1, 8
9c	$A_2,\ A_5$	$(7.3),\ (1,1,1)$	$\frac{1}{9}(1,2),\ A_2,\ A_5$	1, 3, 6
9d	$A_1,\ 2A_3$	$(7.2),\ (1,1,1)$	$\frac{1}{8}(1,3),\ 2A_3$	2, 4, 4
10b	A_8	$(8.1),\ (1,1,1)$	$2\frac{1}{9}(1,2),\ A_8$	1, 1, 9
10c	$A_1,\ A_7$	$(8.2),\ (1,1,1)$	$\frac{1}{16}(1,3),\ \frac{1}{8}(1,3),\ A_7$	1, 2, 8
10d	$A_1,\ A_2,\ A_5$	$(8.3),\ (1,1,1)$	$\frac{1}{18}(1,5),\ \frac{1}{12}(1,5),\ A_5$	2, 3, 6
10e	$A_4,\ A_4$	$(8.4),\ (1,2,1)$	$\frac{1}{25}(1,9),\ \frac{1}{20}(1,9),\ A_4$	1, 5, 5
11	$\frac{1}{4}(1,1)$	$1,\ (1,1,2)$	$\frac{1}{4}(1,1)$	1, 1, 1
15	$\frac{1}{4}(1,1),\ A_4$	$4,\ (1,2,1)$	$\frac{1}{4}(1,1),\ A_4$	1, 1, 5
18	$\frac{1}{4}(1,1),\ A_1,\ A_5$	$6.2,\ (1,1,1)$	$\frac{1}{4}(1,1),\ A_1,\ A_5$	1, 2, 6
19	$\frac{1}{4}(1,1),\ A_7$	$7.1,\ (1,1,1)$	$2\frac{1}{4}(1,1),\ A_7$	1, 1, 8
21	$\frac{1}{8}(1,3),\ A_2$	$3,\ (1,2,1)$	$\frac{1}{8}(1,3),\ A_2$	1, 2, 3
25	$2\frac{1}{4}(1,1),\ A_7$	$7.1,\ (1,1,1)$	$2\frac{1}{4}(1,1),\ A_7$	1, 1, 8
26	$\frac{1}{8}(1,3),\ 2A_3,$	$7.2,\ (1,1,1)$	$\frac{1}{8}(1,3),\ 2A_3$	2, 4, 4
27	$\frac{1}{8}(1,3),\ A_7$	$8.2,\ (1,1,1)$	$\frac{1}{16}(1,3),\ \frac{1}{8}(1,3),\ A_7$	1, 2, 8
30	$\frac{1}{12}(1,5),\ 2A_2$	$6.1,\ (1,1,2)$	$\frac{1}{12}(1,5),\ 2A_2$	3, 3, 3
33	$A_1,\ \frac{1}{12}(1,5),\ A_5$	$8.3,\ (1,1,1)$	$\frac{1}{18}(1,5),\ \frac{1}{12}(1,5),\ A_5$	2, 3, 6
40	$\frac{1}{20}(1,9)$	$4, (1,3,2)$	$\frac{1}{9}(1,2),\ \frac{1}{20}(1,9)$	1, 1, 5
44	$\frac{1}{20}(1,9),\ A_4$	$8.4,\ (1,2,1)$	$\frac{1}{25}(1,9),\ \frac{1}{20}(1,9),\ A_4$	1, 5, 5
46	$A_2,\ \frac{1}{24}(1,11)$	$7.3,\ (1,2,1)$	$\frac{1}{9}(1,2),\ A_2,\ \frac{1}{24}(1,11),$	1, 3, 6
50	$\frac{1}{36}(1,17)$	$8.1,\ (2,1,1)$	$2\frac{1}{9}(1,2),\ \frac{1}{36}(1,17)$	1, 1, 9

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7. Existence of special fibrations

Let X be a log del Pezzo surface such that $\rho(X)=1$ and let $\pi\colon \tilde{X}\to X$ be its minimal resolution. We show that, under certain hypotheses, $\tilde{X}$ admits a morphism $p\colon \tilde{X}\to\mathbb{P}^1$ with general fibre a smooth rational curve such that the exceptional locus of π has a particularly simple form with respect to the ruling p. When X has only T-singularities (and satisfies the hypotheses), we use this structure to construct a toric surface Y such that X is a $\mathbb{Q}$-Gorenstein deformation of Y; see § 8.

We first establish the existence of a so-called 1-complement of K_X. We recall the definition and basic properties. For more details and motivation, see [Kol92, § 19], [Pro01]. Let X be a projective surface with quotient singularities. A *1-complement of* K_X is a divisor $D\in|-K_X|$ such that the pair (X,D) is log canonical. In particular, by the classification of log canonical singularities of pairs [KM98, Theorem 4.15], D is a nodal curve and, at each singularity $P\in X$, either $D=0$ and $P\in X$ is a Du Val singularity, or the pair $(P\in X, D)$ is locally analytically isomorphic to the pair $((1/n)(1,a),(uv=0))$ for some n and a. Moreover, D has arithmetic genus 1 because $2p_a(D)-2=(K_X+D)\cdot D=0$ (note that the adjunction formula holds because K_X+D is Cartier [Kol92, Proposition 16.4.3]). Thus, D is either a smooth elliptic curve or a cycle of smooth rational curves.

Theorem 7.1. *Let X be a log del Pezzo surface such that $\rho(X)=1$. Assume that* $\dim|-K_X|\geqslant 1$ *and every singularity of X is either a cyclic quotient singularity or a Du Val singularity. Then there exists a 1-complement of K_X, i.e., a divisor $D\in|-K_X|$ such that the pair (X,D) is log canonical.*

Proof. Write $-K_X\sim M+F$, where M is an irreducible divisor such that $\dim|M|>0$ and F is effective (we do not assume that F is the fixed part of $|-K_X|$). Let M be general in $|M|$.

Suppose first that (X,M) is purely log terminal (plt). Then M is a smooth curve. We may assume that $F\neq 0$ (otherwise M is a 1-complement). Then $-(K_X+M)\sim F$ is ample (because $\rho(X)=1$). Recall that for X a normal variety and $S\subset X$ an irreducible divisor the *different* $\operatorname{Diff}_S(0)$ is the effective $\mathbb{Q}$-divisor on S defined by the equation

$$(K_X+S)|_S=K_S+\operatorname{Diff}_S(0).$$

That is, $\operatorname{Diff}_S(0)$ is the correction to the adjunction formula for $S\subset X$ due to the singularities of X at S. See [Kol92, § 16]. If S is a normal variety and B is an effective $\mathbb{Q}$-divisor on S with coefficients less than one, a *1-complement* of K_S+B is a divisor $D\in|-K_S|$ such that (S,D) is log canonical and $D\geqslant\lfloor 2B\rfloor$. By [Pro01, Proposition 4.4.1], it is enough to show that $K_M+\operatorname{Diff}_M(0)$ has a 1-complement.

The curve M is smooth and rational and $\deg(K_M+\operatorname{Diff}_M(0))<0$ because

$$2p_a(M)-2\leqslant\deg(K_M+\operatorname{Diff}_M(0))=(K_X+M)\cdot M=-F\cdot M<0.$$

Moreover, at each singular point P_i of X on M, the pair (X,M) is of the form $((1/m_i)(1,a_i),$ $(x=0))$ and

$$\operatorname{Diff}_M(0)=\sum_i\left(1-\frac{1}{m_i}\right)P_i$$

by [Kol92, Proposition 16.6.3]. So, if $K_M+\operatorname{Diff}_M(0)$ does not have a 1-complement, then, by [Kol92, Corollary 19.5] or direct calculation, there are exactly three singular points of X on M, and (m_1, m_2, m_3) is a Platonic triple $(2, 2, m)$ (for some $m \geqslant 2$), $(2, 3, 3)$, $(2, 3, 4)$, or $(2, 3, 5)$. The divisor F passes through each singular point P_i because $F \sim -(K_X + M)$ is not Cartier there. So, $F \cdot M \geqslant \sum 1/m_i$, and

$$0 = (K_X + M + F) \cdot M = \deg(K_M + \mathrm{Diff}_M(0)) + F \cdot M \geqslant 1,$$

a contradiction.

Now suppose that the pair (X, M) is not plt, and let c be its log canonical threshold, i.e.,

$$c = \sup\{t \in \mathbb{Q}_{\geqslant 0} \mid (X, tM) \text{ is log canonical}\}.$$

Then there exists a projective birational morphism $f \colon Y \to X$ with exceptional locus an irreducible divisor E such that the discrepancy $a(E, X, cM) = -1$ and (Y, E) is plt. See [Pro01, Proposition 3.1.4]. So,

$$K_Y + cM' + E = f^*(K_X + cM),$$

where M' is the strict transform of M. Now

$$-(K_Y + E) = cM' - f^*(K_X + cM)$$

is nef (note that M' is nef because it moves). Moreover, $-(K_Y + E)$ is big unless $M'^2 = 0$ and $K_X + cM \sim_{\mathbb{Q}} 0$, in which case $c = 1$, $F = 0$, and M is a 1-complement. So, we may assume that $-(K_Y + E)$ is nef and big. Thus, by [Pro01, Proposition 4.4.1] again, it is enough to show that $K_E + \mathrm{Diff}_E(0)$ has a 1-complement. Suppose not. Then E passes through three cyclic quotient singularities on Y, as above. Let $\tilde{Y} \to Y$ be the minimal resolution of Y, E' the strict transform of E, and consider the composition $g \colon \tilde{Y} \to X$. Let $P \in X$ be the point $f(E)$. Then $g^{-1}(P)$ is the union of E' and three chains of smooth rational curves (the exceptional loci of the minimal resolutions of the cyclic quotient singularities), and E' meets each chain in one of the end components. Let $-b_i$ be the self-intersection number of the end component F_i of the ith chain that meets E'. Then $b_i \leqslant m_i$, where m_i is the order of the cyclic group for the ith quotient singularity. If we contract the F_i and let $\overline{E}'$ denote the image of E', then

$$0 > \overline{E}'^2 = E'^2 + \sum \frac{1}{b_i} \geqslant E'^2 + \sum \frac{1}{m_i} > E'^2 + 1.$$

Hence, $E'^2 \leqslant -2$ and g is the minimal resolution of X. So, $P \in X$ is a D or E singularity by our assumption. But, $P \in X$ is a base point of $|-K_X|$, so this contradicts Theorem 5.1. □

We describe the types of degenerate fibres which occur in the ruling we construct. We first introduce some notation.

Definition 7.2. Let $a, n \in \mathbb{N}$ with $a < n$ and $(a, n) = 1$. We say that the fractions n/a and $n/(n-a)$ are *conjugate*.

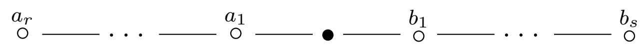
Lemma 7.3. *If $[b_1, \ldots, b_r]$ and $[c_1, \ldots, c_s]$ are conjugate, then so are $[b_1 + 1, b_2, \ldots, b_r]$ and $[2, c_1, \ldots, c_s]$. Conversely, every conjugate pair can be constructed from $[2], [2]$ by a sequence of such steps. Also, if $[b_1, \ldots, b_r]$ and $[c_1, \ldots, c_s]$ are conjugate, then so are $[b_r, \ldots, b_1]$ and $[c_s, \ldots, c_1]$.*

Proof. If $[b_1, \ldots, b_r] = n/a$ and $[c_1, \ldots, c_s] = n/(n-a)$, then $[b_1 + 1, b_2, \ldots, b_r] = (n+a)/a$ and $[2, c_1, \ldots, c_s] = (n+a)/n$. The last statement follows immediately from Remark 2.8. □

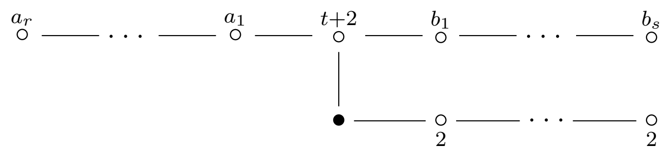
Proposition 7.4. *Let S be a smooth surface, T a smooth curve, and $p \colon S \to T$ a morphism with general fibre a smooth rational curve. Let f be a degenerate fibre of p. Suppose that f*

contains a unique (-1)-curve and the union of the remaining irreducible components of f is a disjoint union of chains of smooth rational curves. Then the dual graph of f has one of the following forms.

(I)



(II)

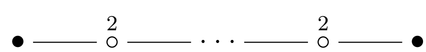


Here the black vertex denotes the (-1)-curve and a white vertex with label $a \geqslant 2$ denotes a smooth rational curve with self-intersection number $-a$. In both types the strings $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ are conjugate. In type (II) there are t (-2)-curves in the branch containing the (-1)-curve.

Conversely, any configuration of curves of this form is a degenerate fibre of a fibration $p\colon S \to T$, as above.

Proof. The morphism $p\colon S \to T$ is obtained from a $\mathbb{P}^1$-bundle $F \to T$ by a sequence of blowups. The statements follow by induction on the number of blowups. □

We refer to the fibres above as fibres of types (I) and (II). We also call a fibre of the form



a fibre of type (O).

Remark 7.5. The curves of multiplicity one in the fibre are the ends of the chain in types (O) and (I) and the ends of the branches not containing the (-1)-curve in type (II). In particular, a section of the fibration meets the fibre in one of these curves.

Theorem 7.6. *Let X be a log del Pezzo surface such that $\rho(X) = 1$. Assume that* $\dim |-K_X| \geqslant 1$ *and every singularity of X is either a cyclic quotient singularity or a Du Val singularity. Let $\pi\colon \tilde{X} \to X$ be the minimal resolution of X. Then one of the following holds.*

(1) *There exists a morphism $p\colon \tilde{X} \to \mathbb{P}^1$ with general fibre a smooth rational curve satisfying one of the following.*
   (a) *Exactly one component $\tilde{E}_1$ of the exceptional locus of π is p-horizontal. The curve $\tilde{E}_1$ is a section of p. The fibration p has at most two degenerate fibres and each is of type* (I) *or* (II).
   (b) *Exactly two components $\tilde{E}_1, \tilde{E}_2$ of the exceptional locus of π are p-horizontal. The curves $\tilde{E}_1, \tilde{E}_2$ are sections of p. Either $\tilde{E}_1$ and $\tilde{E}_2$ are disjoint and p has two degenerate fibres of types* (O) *and either* (I) *or* (II), *or $\tilde{E}_1 \cdot \tilde{E}_2 = 1$ and p has a single degenerate fibre of type* (O). *The sections $\tilde{E}_1$ and $\tilde{E}_2$ meet distinct components of the degenerate fibres.*

(2) *The surface X has at most two non Du Val singularities and each is of the form $(1/m)(1,1)$ for some $m \geqslant 3$.*

Proof. Assume that K_X is not Cartier. As in the proof of Theorem 5.1, let $\nu\colon \hat{X} \to X$ be the minimal resolution of the non Du Val singularities, $\{E_i\}$ the exceptional divisors, and $E = \sum E_i$.

Write $|-K_{\hat{X}}| = |M| + F$, where F is the fixed part and $M \in |M|$ is general. Then $F \geqslant E$ and $\dim |M| = \dim |-K_X| \geqslant 1$.

We run the minimal model program (MMP) on $\hat{X}$. We obtain a birational morphism $\phi\colon \hat{X} \to \overline{X}$ such that $\overline{X}$ has Du Val singularities and either $\rho(\overline{X}) = 2$ and there is a fibration $\psi\colon \overline{X} \to \mathbb{P}^1$ such that $-K_{\overline{X}}$ is ψ-ample or $\rho(\overline{X}) = 1$ and $-K_{\overline{X}}$ is ample. Moreover, ϕ is a composition

$$\hat{X} = X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} \cdots \xrightarrow{\phi_n} X_{n+1} = \overline{X},$$

where ϕ_i is a weighted blowup of a smooth point of X_{i+1} with weights $(1, n_i)$ (by the classification of birational extremal contractions in the MMP for surfaces with Du Val singularities).

CLAIM 7.7. Given $\phi\colon \hat{X} \to \overline{X}$, we can direct the MMP so that the components of E contracted by ϕ are contracted last. That is, for some $1 \leqslant m \leqslant n$, the exceptional divisor of ϕ_i is (the image of) a component of E if and only if $i > m$.

Proof. We have $K_{\hat{X}} = \nu^* K_X + \sum a_i E_i$, where $-1 < a_i < 0$ for each i. Write $\Delta = \sum (-a_i) E_i$. So, $\nu^* K_X = K_{\hat{X}} + \Delta$ and Δ is an effective divisor such that $\lfloor \Delta \rfloor = 0$ and $\operatorname{Supp} \Delta = E$. Hence, $-(K_{\hat{X}} + \Delta)$ is nef and big and $(\hat{X}, \Delta)$ is Kawamata log terminal (klt). These properties are preserved under the $K_{\hat{X}}$-MMP.

Let $R = \sum R_i$ be the sum of the ϕ-exceptional curves that are not contained in E and $R' \subset R$ a connected component. Then $R' \cdot E > 0$ (otherwise ν is an isomorphism near R' which contradicts $\rho(X) = 1$). Let R_i be a component of R' such that $R_i \cdot E > 0$. Then $(K_{\hat{X}} + \Delta) \cdot R_i \leqslant 0$ and $R_i \cdot \Delta > 0$. So, $K_{\hat{X}} \cdot R_i < 0$, and we can contract R_i first in the $K_{\hat{X}}$-MMP. Repeating this procedure, we contract all of R, obtaining a birational morphism $\hat{X} \to \hat{X}'$. Finally, we run the MMP on $\hat{X}'$ over $\overline{X}$ to contract the remaining curves. □

CLAIM 7.8. We may assume that $\rho(\overline{X}) = 2$.

Proof. Suppose that $\rho(\overline{X}) = 1$. Write $\overline{M} = \phi_* M$, etc. Then $-K_{\overline{X}} \sim \overline{M} + \overline{F}$, $\overline{F} \geqslant \overline{E} > 0$, and $\overline{E}$ is contained in the smooth locus of $\overline{X}$. Thus, as in the proof of Theorem 5.1, $-K_{\overline{X}}$ is not a generator of $\operatorname{Pic} \overline{X}$, so $\overline{X} \simeq \mathbb{P}^2$ or $\overline{X} \simeq \mathbb{P}(1,1,2)$ by the classification of log del Pezzo surfaces with Du Val singularities. In particular, it follows that $\overline{E}$ has at most two components.

Suppose first that ϕ does not contract any component of E. Then E has at most two components. So, either we are in case (2) or $E = E_1 + E_2$, $E_1 \cap E_2 \neq \emptyset$, $\overline{X} \simeq \mathbb{P}^2$, and $\overline{M}, \overline{E}_1, \overline{E}_2 \sim l$, where l is the class of a line. In this case $\rho(\hat{X}) = \rho(X) + 2 = 3$, so $\phi\colon \hat{X} \to \overline{X}$ is a composition of two weighted blowups of weights $(1, n_1)$ and $(1, n_2)$. These must have as centres two distinct points $P_1 \in \overline{E}_1$ and $P_2 \in \overline{E}_2$, and in each case the local equation of $\overline{E}_i$ is a coordinate with weight n_i (because E_i is contained in the smooth locus of $\hat{X}$). Let l_{12} be the line through P_1 and P_2. Then these blowups are toric with respect to the torus $\overline{X} \setminus l_{12} + \overline{E}_1 + \overline{E}_2$. We find that the minimal resolution $\tilde{X}$ is a toric surface with boundary divisor a cycle of smooth rational curves with self-intersection numbers

$$-2, \ldots, -2, -1, -(n_1 - 1), -(n_2 - 1), -1, -2, \ldots, -2, -1,$$

where $\tilde{E}_1$ and $\tilde{E}_2$ are the curves with self-intersection numbers $-(n_1 - 1)$, $-(n_2 - 1)$, the first two (-1)-curves are the strict transforms of the exceptional curves of the blowups of P_1 and P_2, the last (-1)-curve is the strict transform of l_{12}, and the chains of (-2)-curves are the exceptional loci of the resolutions of the singularities of $\hat{X}$ and have lengths $(n_1 - 1)$ and $(n_2 - 1)$. In particular, there is a fibration $p\colon \tilde{X} \to \mathbb{P}^1$ with two degenerate fibres of types $-1, -2, \ldots, -2, -1$

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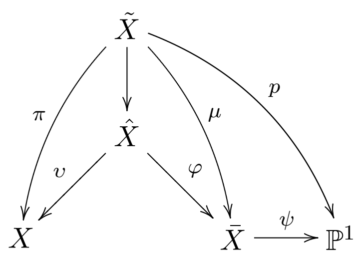
(where there are (n_2-1) (-2)-curves) and $-2, \ldots, -2, -1, -(n_1-1)$ (where there are (n_1-2) (-2)-curves), and two π-exceptional sections with self-intersection numbers $-(n_2-1)$ and -2. So, we are in case (1b).

Now suppose that ϕ contracts some component of E. Then $\phi_n \colon X_n \to X_{n+1} = \overline{X}$ is an (ordinary) blowup of a smooth point $Q \in \overline{X}$. If $\overline{X} \simeq \mathbb{P}^2$, then $X_n \simeq \mathbb{F}_1$ and there is a fibration $\psi \colon X_n \to \mathbb{P}^1$. So, we may assume that $\rho(\overline{X}) = 2$. If $\overline{X} \simeq \mathbb{P}(1,1,2)$, the quadric cone, let L be the ruling of the cone through Q. Then the strict transform L' of L on X_n satisfies $K_{X_n} \cdot L' < 0$ and $L'^2 < 0$. Contracting L', we obtain a morphism $\phi'_n \colon X_n \to \overline{X}' \simeq \mathbb{P}^2$. So, replacing ϕ_n by ϕ'_n, we may assume that $\overline{X} \simeq \mathbb{P}^2$. □

We now assume that $\rho(\overline{X}) = 2$. We have a diagram



where $\pi \colon \tilde{X} \to X$ is the minimal resolution. Let l be a general fibre of p and $\tilde{E}$ the strict transform of E on $\tilde{X}$. Note that, by construction, the components of the exceptional locus of π over Du Val singularities are contained in fibres of p. Write $|-K_{\tilde{X}}| = |\tilde{M}| + \tilde{F}$, where $\tilde{F}$ is the fixed part and $\tilde{M} \in |\tilde{M}|$ is general. Then $\tilde{F} \geqslant \tilde{E}$.

There is a 1-complement of K_X by Theorem 7.1. This can be lifted to $\tilde{X}$. (Indeed, if D is a 1-complement of K_X, define $\tilde{D}$ by $K_{\tilde{X}} + \tilde{D} = \pi^*(K_X + D)$ and $\pi_* \tilde{D} = D$. Note that $\tilde{D}$ is an effective $\mathbb{Z}$-divisor because $K_{\tilde{X}}$ is π-nef and $K_X + D$ is Cartier. Then $\tilde{D}$ is a 1-complement of $K_{\tilde{X}}$.) Hence, $(\tilde{X}, \tilde{M} + \tilde{F})$ is log canonical. In particular, $\tilde{F}$ is reduced and $\tilde{M} + \tilde{F}$ is a cycle of smooth rational curves.

There exists a p-horizontal component $\tilde{E}_1$ of $\tilde{E}$ (because $\rho(X) = 1$). Then

$$1 \leqslant \tilde{E}_1 \cdot l \leqslant (\tilde{F} + \tilde{M}) \cdot l = -K_{\tilde{X}} \cdot l = 2.$$

Suppose first that $\tilde{E}_1 \cdot l = 2$. Then $\tilde{M}$ and $\tilde{F} - \tilde{E}_1$ are p-vertical. Hence, $\tilde{M} \sim l$ and $\tilde{F} = \tilde{E}_1$, so $\tilde{E} = \tilde{E}_1$ and we are in case (2).

Suppose now that $\tilde{E}_1 \cdot l = 1$. Since $\mu(\tilde{E}_1)$ is contained in the smooth locus of $\overline{X}$, the fibres of ψ have multiplicity one, so ψ is smooth by [KM99, Lemma 11.5.2]. Thus, $\overline{X} \simeq \mathbb{F}_n$ for some $n \geqslant 0$.

If $\tilde{E}_1$ is the only p-horizontal component of $\tilde{E}$, we are in case (1a). Suppose that there is another p-horizontal component $\tilde{E}_2$. Then, since $-K_{\tilde{X}} \cdot l = 2$, we have $\tilde{E}_2 \cdot l = 1$ and $\tilde{M}$ and $\tilde{F} - \tilde{E}_1 - \tilde{E}_2$ are contained in fibres of p. If $M \sim 2l$, then $\tilde{F} = \tilde{E} = \tilde{E}_1 + \tilde{E}_2$ and $\tilde{E}_1 \cap \tilde{E}_2 = \emptyset$, so we are in case (1b). So, we may assume that $\tilde{M} \sim l$. Then the components of $\tilde{F}$ form a chain, with ends $\tilde{E}_1$ and $\tilde{E}_2$.

We note that a component Γ of a degenerate fibre of p that is not contracted by π is necessarily a (-1)-curve, because $K_{\tilde{X}} = \pi^* K_X - \tilde{\Delta}$, where $\tilde{\Delta}$ is effective and π-exceptional, so

$$K_{\tilde{X}} \cdot \Gamma \leqslant \pi^* K_X \cdot \Gamma = K_X \cdot \pi_* \Gamma < 0.$$

Hence, since $\rho(X)=1$, there exists a unique degenerate fibre of p containing exactly two (-1)-curves, and any other degenerate fibres contain exactly one (-1)-curve. Let $\tilde{G}$ denote the reduction of the fibre containing two (-1)-curves.

If $\tilde{F}=\tilde{E}_1+\tilde{E}_2$, then $\tilde{E}_1\cdot\tilde{E}_2=1$ and any degenerate fibre of p consists of (-1)-curves and (-2)-curves. It follows that $\tilde{G}$ is of type (O) and there are no other degenerate fibres, so we are in case (1b). So, assume that $\tilde{F}>\tilde{E}_1+\tilde{E}_2$. Then $\tilde{E}_1\cap\tilde{E}_2=\emptyset$.

Suppose first that $\tilde{G}$ is the only degenerate fibre. Then $\tilde{F}\leqslant\tilde{G}+\tilde{E}_1+\tilde{E}_2$. Write $\tilde{G}=\tilde{G}'+\tilde{G}''$, where $\tilde{G}'=\tilde{F}-\tilde{E}_1-\tilde{E}_2$. So, $\tilde{G}'$ is a chain of smooth rational curves. It follows that each connected component of $\tilde{G}''$ is a chain of smooth rational curves such that one end component is a (-1)-curve adjacent to $\tilde{G}'$ and the remaining curves are (-2)-curves. We construct an alternative ruling $p'\colon\tilde{X}\to\mathbb{P}^1$ with only one horizontal π-exceptional curve by inductively contracting (-1)-curves as follows. First, contract the components of $\tilde{G}''$. Second, contract (-1)-curves in $\tilde{G}'$ until the image of $\tilde{E}_1$ or $\tilde{E}_2$ is a (-1)-curve. Now contract this curve, and continue contracting (-1)-curves until we obtain a ruled surface $\overline{X}'\simeq\mathbb{F}_m$. Then $\tilde{M}\sim l$ is horizontal for the induced ruling p'. Moreover, if C is a p'-horizontal π-exceptional curve, then $C\not\subset\tilde{G}''$ by construction. Hence, $C\subset\tilde{F}$. Thus, there exists a unique such C, and C is a section of p'. So, we are in case (1a).

Finally, suppose that there is another degenerate fibre of p, and let $\tilde{V}$ denote its reduction. Then $\tilde{V}$ contains a unique (-1)-curve C. The surface X has only cyclic quotient singularities by assumption. Therefore, $\tilde{V}-C$ is a union of chains of smooth rational curves. It follows that $\tilde{V}$ is a fibre of type (I) or (II). Now $\tilde{E}_1\cdot C=\tilde{E}_2\cdot C=0$ because C has multiplicity greater than one in the fibre. So, $\tilde{V}$ contains a component of $\tilde{F}$ (because $1=-K_{\tilde{X}}\cdot C=(\tilde{M}+\tilde{F})\cdot C$). Hence, $\tilde{F}-\tilde{E}_1-\tilde{E}_2\leqslant\tilde{V}$ (because $\tilde{M}+\tilde{F}$ is a cycle of rational curves and $\tilde{M}\sim l$). In particular, $\tilde{G}$ consists of two (-1)-curves and some (-2)-curves. Hence, $\tilde{G}$ is of type (O) and we are in case (1b). This completes the proof. □

8. Proof of main theorem

Theorem 8.1. *Let X be a log del Pezzo surface such that $\rho(X)=1$ and X has only T-singularities. Then exactly one of the following holds:*

(1) *X is a $\mathbb{Q}$-Gorenstein deformation of a toric surface Y; or*

(2) *X is one of the sporadic surfaces listed in Example 8.3.*

Remark 8.2. Note that the surface Y in Theorem 8.1(1) necessarily has only T-singularities and $\rho(Y)=1$. Thus, Y is one of the surfaces listed in Theorem 4.1.

Example 8.3. We list the log del Pezzo surfaces X such that X has only T-singularities and $\rho(X)=1$, but X is not a $\mathbb{Q}$-Gorenstein deformation of a toric surface. In each case X has index $\leqslant 2$. If X is Gorenstein, the possible configurations of singularities are

$$D_5,\ E_6,\ E_7,\ A_1D_6,\ 3A_1D_4,\ E_8,\ D_8,\ A_1E_7,$$
$$A_2E_6,\ 2A_1D_6,\ A_3D_5,\ 2D_4,\ 2A_12A_3,\ 4A_2.$$

The configuration determines the surface uniquely with the following exceptions: there are two surfaces for E_8, A_1E_7, and A_2E_6, and a $\mathbb{A}^1$ of surfaces for $2D_4$. See [AN06, Theorem 4.3]. If X has index two, the possible configurations of singularities are

$$\tfrac{1}{4}(1,1)D_8,\ \tfrac{1}{4}(1,1)2A_1D_6,\ \tfrac{1}{4}(1,1)A_3D_5,\ \tfrac{1}{4}(1,1)2D_4,$$

and the configuration determines the surface uniquely. See [AN06, Theorem 4.2].

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Remark 8.4. Note that the case $K_X^2 = 7$ does not occur. This may be explained as follows. If X is a del Pezzo surface with T-singularities such that $\rho(X) = 1$, then there exists a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X}/T$ of X over $T := \operatorname{Spec} k[[t]]$ such that the generic fibre $\mathcal{X}_K$ is a smooth del Pezzo surface over $K = k((t))$ with $\rho(\mathcal{X}_K) = 1$. (Indeed, if $\mathcal{X}/T$ is a smoothing of X over T, the restriction map $\mathrm{Cl}(\mathcal{X}) \to \mathrm{Cl}(\mathcal{X}_K) = \mathrm{Pic}(\mathcal{X}_K)$ is an isomorphism because the closed fibre X is irreducible and the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(X)$ is an isomorphism because $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$. Thus, $\rho(\mathcal{X}_K) \geqslant \rho(X) = 1$ with equality if and only if the total space $\mathcal{X}$ of the deformation is $\mathbb{Q}$-factorial. Since there are no local-to-global obstructions for deformations of X, there exists a $\mathbb{Q}$-Gorenstein smoothing $\mathcal{X}/T$ such that $P \in \mathcal{X}$ is smooth for $P \in X$ a Du Val singularity and $P \in \mathcal{X}$ is of type $(1/n)(1, -1, a)$ for $P \in X$ a singularity of type $(1/dn^2)(1, dna - 1)$ (see § 2.2). In particular, $\mathcal{X}$ is $\mathbb{Q}$-factorial.) Note that $K_{\mathcal{X}_K}^2 = K_X^2$. If Y is a smooth del Pezzo surface with $K_Y^2 = 7$ over a field (not necessarily algebraically closed), then $\rho(Y) > 1$; see, e.g., [Man86]. Hence, there is no X with $K_X^2 = 7$.

Proof of Theorem 8.1. First assume that X does not have a D or E singularity. Note that $\dim |-K_X| = K_X^2 \geqslant 1$ by Proposition 2.6, so we may apply Theorem 7.6. We use the notation of that theorem.

Suppose first that we are in case (1a). We construct a toric surface Y and prove that X is a $\mathbb{Q}$-Gorenstein deformation of Y. We first describe the surface Y. Let $\tilde{E}_1^2 = -d$. There is a uniquely determined toric blowup $\mu_Y : \tilde{Y} \to \mathbb{F}_d$ such that μ_Y is an isomorphism over the negative section $B \subset \mathbb{F}_d$, and the degenerate fibres of the ruling $p_Y : \tilde{Y} \to \mathbb{P}^1$ are fibres of type (I) associated with the degenerate fibres of $p : \tilde{X} \to \mathbb{P}^1$ as follows. Let f be a degenerate fibre of p of type (I) or (II) as in Proposition 7.4, and assume that $\tilde{E}_1$ intersects the left end component. If f is of type (I), then the associated fibre f_Y of p_Y has the same form. If f is of type (II), then f_Y is a fibre of type (I) with self-intersection numbers

$$-a_r, \dots, -a_1, -t-2, -b_1, \dots, -b_s, -1, -d_1, \dots, -d_u.$$

Note that the sequence $d_1, \dots, d_u$ is uniquely determined (see Proposition 7.4). In each case the strict transform B' of B again intersects the left end component of f_Y.

Let Y be the toric surface obtained from $\tilde{Y}$ by contracting the strict transform of the negative section of $\mathbb{F}_d$ and the components of the degenerate fibres of the ruling with self-intersection number at most -2. For each fibre f of p of type (II) as above, the chain of rational curves with self intersections $-d_1, \dots, -d_u$ in the associated fibre f_Y of p_Y contracts to a T_{t+1} singularity by Lemma 8.5(1). This singularity replaces the A_t singularity on X obtained by contracting the chain of t (-2)-curves in f. In particular, the surface Y has T-singularities. Moreover, $\rho(Y) = 1$, and $K_Y^2 = K_X^2$ by Proposition 2.6. A T_d singularity admits a $\mathbb{Q}$-Gorenstein deformation to an A_{d-1} singularity (see Proposition 2.3). Hence, the singularities of X are a $\mathbb{Q}$-Gorenstein deformation of the singularities of Y. There are no local-to-global obstructions for deformations of Y by Proposition 3.1. Hence, there is a $\mathbb{Q}$-Gorenstein deformation X' of Y with the same singularities as X. We prove below that $X \simeq X'$.

Let f be a degenerate fibre of p of type (II) as above and f_Y the associated fibre of p_Y. Let $P \in Y$ be the T singularity obtained by contracting the chain of rational curves in f_Y with self intersections $-d_1, \dots, -d_u$. Let X' be the general fibre of a $\mathbb{Q}$-Gorenstein deformation of Y over the germ of a curve which deforms $P \in Y$ to an A_t singularity and is locally trivial elsewhere. Let $\hat{Y} \to Y$ and $\hat{X}' \to X'$ be the minimal resolutions of the remaining singularities (where the deformation is locally trivial). Thus, $\hat{Y}$ has a single T singularity and $\hat{X}'$ a single A_t singularity. The ruling $p_Y : \tilde{Y} \to \mathbb{P}^1$ descends to a ruling $\hat{Y} \to \mathbb{P}^1$; let A be a general fibre of

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this ruling. Then A deforms to a 0-curve A' in $\hat{X}'$ (because $H^1(\mathcal{N}_{A/\hat{Y}}) = H^1(\mathcal{O}_A) = 0$), which defines a ruling $\hat{X}' \to \mathbb{P}^1$. Let $\tilde{X}' \to \hat{X}'$ be the minimal resolution of $\hat{X}'$ and consider the induced ruling $p_{X'}\colon \tilde{X}' \to \mathbb{P}^1$. Note that the exceptional locus of $\hat{Y} \to Y$ deforms without change by construction. Moreover, the (-1)-curve in the remaining degenerate fibre (if any) of p_Y also deforms. There is a unique horizontal curve in the exceptional locus of $\pi_{X'} : \tilde{X}' \to X'$, and $\rho(X') = 1$ by Proposition 2.6. Hence, each degenerate fibre of $p_{X'}$ contains a unique (-1)-curve, and the remaining components of the fibre are in the exceptional locus of $\pi_{X'}$. We can now describe the degenerate fibres of $p_{X'}$. If p_Y has a degenerate fibre besides f_Y, then $p_{X'}$ has a degenerate fibre of the same form. We claim that there is exactly one additional degenerate fibre of $p_{X'}$, which is of type (II) and has the same form as the fibre f of p. Indeed, the union of the remaining degenerate fibres consists of the chain of rational curves with self intersections $-a_r, \ldots, -a_1, -t-2, b_1, \ldots, b_s$ (the deformation of the chain of the same form in f_Y), the chain of (-2)-curves which contracts to the A_t singularity, and some (-1)-curves. The claim follows by the description of degenerate fibres in Proposition 7.4. If there is a second degenerate fibre of p of type (II) we repeat this process. We obtain a $\mathbb{Q}$-Gorenstein deformation X' of Y with minimal resolution $\pi_{X'}\colon \tilde{X}' \to X'$, and a ruling $p_{X'}\colon \tilde{X}' \to \mathbb{P}^1$ such that the exceptional locus of $\pi_{X'}$ has the same form with respect to the ruling $p_{X'}$ as that of π with respect to p.

We claim that $X \simeq X'$. Indeed, there is a smooth toric surface Z and, for each fibre f_i of p of type (II), an irreducible toric boundary divisor $\Delta_i \subset Z$ and points P_i, P_i' in the torus orbit $O_i \subset \Delta_i$ such that $\tilde{X}$ (respectively $\tilde{X}'$) is obtained from Z by successively blowing up the points P_i (respectively P_i') $t_i + 1$ times, where t_i is the length of the chain of (-2)-curves in f_i. It remains to prove that we may assume that $P_i = P_i'$ for each i. Let T be the torus acting on Z and N its lattice of one-parameter subgroups. Let $\Sigma \subset N_{\mathbb{R}}$ be the fan corresponding to X and $v_i \in N$ the minimal generator of the ray in Σ corresponding to Δ_i. Then $T_i = (N/\langle v_i\rangle) \otimes \mathbb{G}_m$ is the quotient torus of T which acts faithfully on Δ_i. Thus, there is an element $t \in T$ taking P_i to P_i' for each i except in the following case: there are two fibres of p of type (II), and $v_1 + v_2 = 0$. In this case, there is a toric ruling $q\colon Z \to \mathbb{P}^1$ given by the projection $N \to N/\langle v_1\rangle$. The toric boundary of Z decomposes into two sections (given by Δ_1, Δ_2) and two fibres of q. But, one of these fibres (the one containing the image of $\tilde{E}_1 \subset \tilde{X}$) is a chain of rational curves of self intersections at most -2, a contradiction.

Next assume that we are in case $(1b)$. There is a ruling $p\colon \tilde{X} \to \mathbb{P}^1$ with two π-exceptional sections $\tilde{E}_1$ and $\tilde{E}_2$. Suppose first that $\tilde{E}_1 \cap \tilde{E}_2 = \emptyset$. Then there are two degenerate fibres of types (O) and either (I) or (II). We use the notation of Proposition 7.4. The exceptional locus of π consists of the components of the degenerate fibres of self intersection $\leqslant -2$ and the two disjoint sections $\tilde{E}_1$ and $\tilde{E}_2$ of p which meet the first fibre in the two (-1)-curves and the second fibre in the components labelled $-a_r$ and $-b_s$, respectively. If the degenerate fibres are of types (O) and (I), then X is toric. So, we may assume that the degenerate fibres are of types (O) and (II). Set $\tilde{E}_1^2 = -a_{r+1}$ and $\tilde{E}_2^2 = -b_{s+1}$. Let m be the number of (-2)-curves in the fibre of type (O). Then X has singularities A_m, A_t, and the cyclic quotient singularity whose minimal resolution has exceptional locus the chain of rational curves with self intersections $-a_{r+1}, \ldots, -a_1, -(t+2), -b_1, \ldots, -b_{s+1}$.

The ruling $p\colon \tilde{X} \to \mathbb{P}^1$ is obtained from a $\mathbb{P}^1$-bundle by a sequence of blowups. It follows that $m = a_{r+1} + b_{s+1} - 2$.

We construct a toric surface Y and prove that X is a $\mathbb{Q}$-Gorenstein deformation of Y. The minimal resolution of $\tilde{Y}$ is the toric surface that fibres over $\mathbb{P}^1$ with two degenerate fibres, one

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of type (O) (where there are m (-2)-curves as above) and one of type (I) with self-intersection numbers

$$-a_r, \dots, -a_1, -(t+2), -b_1, \dots, -b_{s+1}, -1, -d_1, \dots, -d_u,$$

and two disjoint torus-invariant sections with self-intersection numbers $-a_{r+1}$ and $-b_{s+1}$ which intersect the first fibre in the two (-1)-curves and the second fibre in the end components labelled $-a_r$ and $-d_u$, respectively. Note that the sequence $d_1, \dots, d_u$ is uniquely determined. Note also that, as above, the equality $m = a_{r+1} + b_{s+1} - 2$ ensures that this does define a toric surface (it is obtained as a toric blowup of a $\mathbb{P}^1$-bundle). The surface Y has as singularities an A_m singularity and the cyclic quotient singularities obtained by contracting the chains of smooth rational curves with self-intersection numbers $-a_{r+1}, \dots, -a_1, -(t+2), -b_1, \dots, -b_{s+1}$ and $-d_1, \dots, -d_u, -b_{s+1}$. This last singularity is of type T_{t+1} by Lemma 8.5(2). Hence, the singularities of X are $\mathbb{Q}$-Gorenstein deformations of the singularities of Y: the first two singularities are not deformed, and the T_{t+1} singularity is deformed to an A_t singularity. Moreover, this deformation does not change the Picard number. Let X' be the general fibre of a one-parameter deformation of X inducing this deformation of the singularities. We show that $X' \simeq X$.

Let $\hat{Y} \to Y$ and $\hat{X}' \to X'$ be the minimal resolutions of the singularities we do not deform. Thus, $\hat{Y}$ has a single T_{t+1} singularity given by contracting the chain of smooth rational curves with self-intersection numbers $-d_1, \dots, -d_u, -b_{s+1}$ on $\tilde{Y}$. Let C_1 and C_2 be the images of the (-1)-curves on $\tilde{Y}$ incident to the ends of this chain. Then C_1 and C_2 are smooth rational curves meeting in a node at the singular point. We claim that $C = C_1 + C_2$ deforms to a smooth (-1)-curve on $\hat{X}'$ (not passing through the singular point). First, by Lemma 8.6 we have $C^2 = -1$. Second, we prove that C deforms. We work on the canonical covering stack $q\colon \hat{\mathcal{Y}} \to \hat{Y}$ of $\hat{Y}$. (Here, for a normal $\mathbb{Q}$-Gorenstein surface Z, the *canonical covering stack* is the Deligne–Mumford stack $\mathcal{Z} \to Z$ with coarse moduli space Z defined by the local canonical coverings of Z. That is, if $P \in Z$ is a point of index n, and $V \to U$ is a canonical covering of a neighbourhood U of P with group $G \simeq \mathbb{Z}/n\mathbb{Z}$, then $\mathcal{Z}|_U$ is isomorphic to $[V/G]$ over U.) Note that the deformation of $\hat{Y}$ lifts to a deformation of $\hat{\mathcal{Y}}$ (because it is a $\mathbb{Q}$-Gorenstein deformation). Let $\mathcal{C} \to C$ be the restriction of the covering $\hat{\mathcal{Y}} \to \hat{Y}$. The closed substack $\mathcal{C} \subset \hat{\mathcal{Y}}$ is a Cartier divisor. Hence, the obstruction to deforming $\mathcal{C} \subset \hat{\mathcal{Y}}$ lies in $H^1(\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}})$, where $\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}$ is the normal bundle $\mathcal{O}_{\hat{\mathcal{Y}}}(\mathcal{C})|_{\mathcal{C}}$. We compute that this obstruction group is zero. Consider the exact sequence

$$0 \to \mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}} \to \oplus \mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}|_{\mathcal{C}_i} \to \mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}} \otimes k(Q) \to 0,$$

where $\mathcal{C}_i \to C_i$ are the restrictions of q and $Q \in \hat{\mathcal{Y}}$ is the point over the singular point $P \in \hat{Y}$. Now push forward to the coarse moduli space $\hat{Y}$. (Recall that if $\mathcal{X}$ is a Deligne–Mumford stack and $q\colon \mathcal{X} \to X$ is the map to its coarse moduli space, then locally over X the map q is of the form $[U/G] \to U/G$, where U is a scheme and G is a finite group acting on U. A sheaf $\mathcal{F}$ over $[U/G]$ corresponds to a G-equivariant sheaf $\mathcal{F}_U$ over U, and $q_*\mathcal{F} = (\pi_*\mathcal{F}_U)^G$, where $\pi\colon U \to U/G$ is the quotient map.) Let n be the index of the singularity $P \in Y$. Then $n > 1$ and the $\boldsymbol{\mu}_n$ action on $\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}} \otimes k(Q)$ is non-trivial. So, $q_*(\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}} \otimes k(Q)) = 0$ and $q_*\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}} = \oplus\, q_*\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}|_{\mathcal{C}_i}$ by the exact sequence above. The sheaf $q_*\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}|_{\mathcal{C}_i}$ is a line bundle on $C_i \simeq \mathbb{P}^1$ of degree $\lfloor C \cdot C_i \rfloor$. Let $\alpha\colon \tilde{Y} \to \hat{Y}$ denote the minimal resolution of $\hat{Y}$ and C_i' the strict transform of C_i for each i. Then

$$C \cdot C_i = \alpha^* C \cdot C_i' > C_i'^2 = -1.$$

Hence, $H^1(q_*\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}|_{\mathcal{C}_i}) = 0$. We deduce that $H^1(\mathcal{N}_{\mathcal{C}/\hat{\mathcal{Y}}}) = 0$, as required.

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We now compute locally that C deforms to a smooth curve that does not pass through the singular point of $\hat{X}'$. Locally at the singular point of $\hat{Y}$, the deformation of $\hat{Y}$ is of the form

$$(xy = (z^n - w)^d) \subset \frac{1}{n}(1, -1, a) \times \mathbb{C}^1_w,$$

where $d = t + 1$. The deformation of C is given by an equation $(z + w \cdot h = 0)$, where $h \in k[[x, y, w]]$ has $\boldsymbol{\mu}_n$-weight a. So, eliminating z, the abstract deformation of C is given by $(xy = u \cdot w^d) \subset (1/n)(1, -1) \times \mathbb{C}^1_w$, where u is a unit. In particular, the general fibre is smooth and misses the singular point of the ambient surface $\hat{X}'$.

We deduce that, on $\hat{X}'$, we have a cycle of smooth rational curves of self-intersection numbers

$$-a_{r+1}, \ldots, -a_1, -(t+2), -b_1, \ldots, -b_{s+1}, -1, -2, \ldots, -2, -1$$

(where the chain of (-2)-curves has length m). Indeed, the chains $-a_{r+1}, \ldots, -b_{s+1}$ and $-2, \ldots, -2$ are the exceptional loci of the minimal resolutions of two of the singular points of X', the first (-1)-curve is the deformation of C described above, and the last (-1)-curve is the deformation of the (-1)-curve on $\hat{Y}$. Moreover, $\hat{X}'$ has a unique singular point of type A_t which does not lie on this cycle. Let $\tilde{X}' \to \hat{X}'$ be the minimal resolution. Observe that the chain $-1, -2, \ldots, -2, -1$ defines a ruling of $\tilde{X}'$. If f is another degenerate fibre, then f contains a unique (-1)-curve and its remaining components are exceptional over X' (because $\rho(X') = 1$). We deduce that there is exactly one additional degenerate fibre, which is the union of the chain $-a_r, \ldots, -b_s$, the chain $-2, \ldots, -2$ of length t (the exceptional locus of the minimal resolution of the A_t singularity), and a (-1)-curve. This determines the fibre uniquely. We conclude that $X' \simeq X$.

A similar argument works when $\tilde{E}_1 \cdot \tilde{E}_2 = 1$. In this case the ruling $p\colon \tilde{X} \to \mathbb{P}^1$ has a unique degenerate fibre of type (O) and the two sections $\tilde{E}_1$ and $\tilde{E}_2$ meet this fibre in the two (-1)-curves. Set $\tilde{E}_1^2 = -a$ and $\tilde{E}_2^2 = -b$ and let m be the number of (-2)-curves in the degenerate fibre. Then X has singularities A_m and the cyclic quotient singularity whose minimal resolution has exceptional locus $\tilde{E}_1 + \tilde{E}_2$. (In particular, $(a, b) = (2, 2)$, $(3, 3)$, or $(2, 5)$ because X has T-singularities, but we give a uniform treatment of these cases.) We compute that $m = a + b + 1$ by expressing p as a blowup of a $\mathbb{P}^1$-bundle.

We construct a toric surface Y and prove that X is a $\mathbb{Q}$-Gorenstein deformation of Y. The minimal resolution of $\tilde{Y}$ is the toric surface that fibres over $\mathbb{P}^1$ with two degenerate fibres, one of type (O) (where there are m (-2)-curves as above) and one of type (I) with self-intersection numbers

$$-b, -1, -2, \ldots, -2$$

(where the chain of (-2)-curves has length $(b-1)$) and two disjoint torus-invariant sections with self-intersection numbers $-a$ and $-(b+3)$ which intersect the first fibre in the two (-1)-curves and the second fibre in the end components with self-intersection numbers $-b$ and -2, respectively. Note that the equality $m = a + (b+3) - 2$ ensures that this does define a toric surface. The surface Y has as singularities an A_m singularity and the cyclic quotient singularities obtained by contracting the chains of smooth rational curves with self-intersection numbers $-a$, $-b$ and $-2, \ldots, -2, -(b+3)$. This last singularity is of type T_1 by Proposition 2.9. Hence, the singularities of X are deformations of the singularities of Y: the first two singularities are not deformed, and the T_1 singularity is smoothed. Moreover, this deformation does not change the Picard number. Let X' be the general fibre of a one-parameter deformation of X inducing this deformation of the singularities. Let $\hat{Y} \to Y$ and $\hat{X}' \to X'$ be the minimal resolutions of the singularities we do not deform. Thus, $\hat{Y}$ has a single T_1 singularity given by contracting the chain of smooth rational curves with self-intersection numbers $-2, \ldots, -2, -(b+3)$ on $\tilde{Y}$. Let C_1 and C_2 be the images of the (-1)-curves on $\tilde{Y}$ incident to the ends of this chain, so C_1 and C_2 are smooth rational curves meeting in a node at the singular point. Then, as above, $C = C_1 + C_2$ deforms to a smooth (-1)-curve on $\hat{X}'$. We deduce that, on $\hat{X}'$, we have a cycle of smooth rational curves of self-intersection numbers

$$-a, -b, -1, -2, \ldots, -2, -1$$

(where the chain of (-2)-curves has length m). Indeed, the chains $-a, -b$ and $-2, \ldots, -2$ are the exceptional loci of the minimal resolutions of the two singular points of X', the first (-1)-curve is the deformation of C, and the last (-1)-curve is the deformation of the (-1)-curve on $\hat{Y}$. Let $\tilde{X}' \to \hat{X}'$ be the minimal resolution. Observe that the chain $-1, -2, \ldots, -2, -1$ defines a ruling of $\tilde{X}'$. There are no other degenerate fibres of this ruling because $\rho(X') = 1$. We deduce that $X' \simeq X$.

If we are in case (2) of Theorem 7.6, then the non Du Val singularities of X are of type $\frac{1}{4}(1, 1)$. In particular, $2K_X$ is Cartier. Similarly, if X has a D or E singularity, then $2K_X$ is Cartier by Theorem 5.1. So, in these cases we can refer to the classification of log del Pezzo surfaces of Picard rank one and index $\leqslant 2$ given by Alexeev and Nikulin [AN06, Theorems 4.2 and 4.3]. By Proposition 6.1 the only such surfaces which are not $\mathbb{Q}$-Gorenstein deformations of toric surfaces are those which have a D singularity, an E singularity, or at least four Du Val singularities. These are the sporadic surfaces listed in Example 8.3. This completes the proof. □

Lemma 8.5. *Let $[a_1, \ldots, a_r]$ and $[b_1, \ldots, b_s]$ be conjugate strings.*

(1) *The conjugate of $[a_r, \ldots, a_1, t+2, b_1, \ldots, b_s]$ is a T_{t+1}-string.*
(2) *Given $b_{s+1} \geqslant 2$, let $[d_1, \ldots, d_u]$ be the conjugate of $[a_r, \ldots, a_1, t+2, b_1, \ldots, b_s, b_{s+1}]$. Then $[d_1, \ldots, d_u, b_{s+1}]$ is a T_{t+1}-string.*

Proof. Let an *S_t-string* be a string $[a_r, \ldots, a_1, t+2, b_1, \ldots, b_s]$ as above. Then, by Lemma 7.3, we have:

(a) $[2, t+2, 2]$ is an S_t-string;
(b) if $[e_1, \ldots, e_v]$ is an S_t-string, then so are $[e_1+1, \ldots, e_v, 2]$ and $[2, e_1, \ldots, e_v+1]$;
(c) every S_t-string is obtained from the example in (a) by iterating the steps in (b).

Now (1) follows from Proposition 2.9 and Lemma 7.3. To deduce (2), let $[e_1, \ldots, e_v]$ be the conjugate of $[a_r, \ldots, a_1, t+2, b_1, \ldots, b_s]$. Then

$$[d_1, \ldots, d_u, b_{s+1}] = [2, \ldots, 2, e_1+1, e_2, \ldots, e_v, b_{s+1}]$$

(where there are $(b_{s+1}-2)$ 2's) by Lemma 7.3. This string is of type T_{t+1} by (1) and Proposition 2.9. □

Lemma 8.6. *Let $(P \in S, D)$ denote the local pair $((1/dn^2)(1, dna-1), (uv=0))$. Let $\pi\colon \tilde{S} \to S$ be the minimal resolution of S and D' the strict transform of D. Write $\pi^* D = D' + F$, where F is π-exceptional. Then $F^2 = -1$.*

Proof. We may assume that S is a projective toric surface, $P \in S$ is the unique singular point, and D is the toric boundary. Then $\tilde{S}$ is toric with boundary $\tilde{D} := D' + \sum E_i$, where $E_1, \ldots, E_r$

are the exceptional divisors of π. In particular, $D \in |-K_S|$ and $\tilde{D} \in |-K_{\tilde{S}}|$. Since $P \in S$ is a T_d singularity, by Proposition 2.6 we have

$$K_{\tilde{S}}^2 + \rho(\tilde{S}) = K_S^2 + \rho(S) + (d-1).$$

So, $\tilde{D}^2 + r = D^2 + (d-1)$. Now $\tilde{D}^2 = D'^2 + \sum E_i^2 + 2(r+1)$, so

$$F^2 = D'^2 - D^2 = d - 3r - 3 - \sum E_i^2.$$

Finally, $\sum E_i^2 = d - 3r - 2$ by the inductive description of resolutions of T_d-singularities (see Proposition 2.9), so $F^2 = -1$, as claimed. □

Proof of Theorem 1.3. Let X denote the special fibre of $f\colon V \to T$. Thus, X is a del Pezzo surface with quotient singularities which admits a $\mathbb{Q}$-Gorenstein smoothing. Since $H^1(\mathcal{O}_X) = H^2(\mathcal{O}_X) = 0$ the restriction map $\operatorname{Pic} V \to \operatorname{Pic} X$ is an isomorphism. Hence, $\rho(X) = \rho(V/T) = 1$.

By Theorem 7.1 there exists a (reduced) curve $D \in |-K_X|$ with only nodal singularities. We have $H^1(-K_X) = 0$ by Kawamata–Viehweg vanishing. So, there exists a lift $S \in |-K_V|$ of $D \in |-K_X|$ such that the general fibre of S/T is smooth. The surface S is normal, so the special fibre of S/T equals D (there are no embedded points). Hence, S/T is a smoothing of a nodal curve, and S has only Du Val singularities of type A. □

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References

Ale94 V. Alexeev, *General elephants of $\mathbb{Q}$-Fano 3-folds*, Compositio Math. **91** (1994), 91–116.

AN06 V. Alexeev and V. Nikulin, *Del Pezzo and K3 surfaces*, MSJ Mem. **15** (2006), Preprint (2004), arXiv:math/0406536v6 [math.AG].

Bla95 R. Blache, *Riemann–Roch theorem for normal surfaces and applications*, Abh. Math. Sem. Univ. Hamburg **65** (1995), 307–340.

Dem80 M. Demazure, *Surfaces de del Pezzo II–V*, in *Séminaire sur les Singularités des Surfaces*, Lecture Notes in Mathematics, vol. 777 (Springer, Berlin, 1980), 23–69.

Ful93 W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, vol. 131 (Princeton University Press, Princeton, NJ, 1993).

GS83 G.-M. Greuel and J. Steenbrink, *On the topology of smoothable singularities*, in *Singularities (Arcata, CA, 1981)*, Proceedings of Symposia in Pure Mathematics, vol. 40 (American Mathematical Society, Providence, RI, 1983), 535–545 (Part 1).

GKK08 D. Greb, S. Kebekus and S. Kovács, *Extension theorems for differential forms and Bogomolov–Sommese vanishing on log canonical varieties*, Preprint (2008), arXiv:0808.3647v2 [math.AG].

Hac04 P. Hacking, *Compact moduli of plane curves*, Duke Math. J. **124** (2004), 213–257.

Ill71 L. Illusie, *Complexe cotangent et déformations I*, Lecture Notes in Mathematics, vol. 239 (Springer, Berlin, 1971).

KN98 B. Karpov and D. Nogin, *Three-block exceptional sets on del Pezzo surfaces (in Russian)*, Izv. Ross. Akad. Nauk Ser. Mat. **62** (1998), 3–38. Translation in Izv. Math. 62 (1998), 429–463.

KM99 S. Keel and J. McKernan, *Rational curves on quasi-projective surfaces*, Mem. Amer. Math. Soc. **140** (1999).

Kol91 J. Kollár, *Flips, flops, minimal models, etc*, in *Surveys in differential geometry (Cambridge, MA, 1990)* (Lehigh University, Bethlehem, PA, 1991), 113–199.

Kol08 J. Kollár, *Is there a topological Bogomolov–Miyaoka–Yau inequality?*, Pure Appl. Math. Q. **4** (2008), 203–236.

Kol92 J. Kollár *et al.*, *Flips and abundance for algebraic threefolds*, Astérisque **211** (1992).

KM98 J. Kollár and S. Mori, *Birational geometry of algebraic varieties* (Cambridge University Press, Cambridge, 1998).

KS88 J. Kollár and N. Shepherd-Barron, *Threefolds and deformations of surface singularities*, Invent. Math. **91** (1988), 299–338.

Loo85 E. Looijenga, *Riemann–Roch and smoothings of singularities*, Topology **25** (1985), 292–302.

LW86 E. Looijenga and J. Wahl, *Quadratic functions and smoothing surface singularities*, Topology **25** (1986), 261–291.

Man91 M. Manetti, *Normal degenerations of the complex plane*, J. Reine Angew. Math. **419** (1991), 89–118.

Man93 M. Manetti, *Normal projective surfaces with* $\rho = 1$, $P_{-1} \geqslant 5$, Rend. Sem. Mat. Univ. Padova **89** (1993), 195–205.

Man86 Y. Manin, *Cubic forms—algebra, geometry, arithmetic* (North-Holland, Amsterdam, 1986).

Pro01 Y. Prokhorov, *Lectures on complements on log surfaces*, MSJ Mem. **10** (2001), Preprint (1999), arXiv:math/9912111v2 [math.AG].

Wah81 J. Wahl, *Smoothings of normal surface singularities*, Topology **20** (1981), 219–246.

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