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ASPECTS OF RECURRENCE AND TRANSIENCE FOR LÉVY PROCESSES IN TRANSFORMATION GROUPS AND NONCOMPACT RIEMANNIAN SYMMETRIC PAIRS

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Abstract

We study recurrence and transience for Lévy processes induced by topological transformation groups acting on complete Riemannian manifolds. In particular the transience–recurrence dichotomy in terms of potential measures is established and transience is shown to be equivalent to the potential measure having finite mass on compact sets when the group acts transitively. It is known that all bi-invariant Lévy processes acting in irreducible Riemannian symmetric pairs of noncompact type are transient. We show that we also have 'harmonic transience', that is, local integrability of the inverse of the real part of the characteristic exponent which is associated to the process by means of Gangolli's Lévy–Khinchine formula.

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1. Introduction

Lévy processes in Lie groups have recently attracted considerable interest and the monograph [28] is dedicated to their investigation. The purpose of this paper is to develop some theoretical insight into the recurrence and transience of such processes. Lévy processes are often considered as natural continuous-time generalisations of random walks. It is well known that recurrence and transience of random walks in groups is intimately related to volume growth in the group (see [5, 6, 18]). From another point of view, a path continuous Lévy process on a group is a Brownian motion with drift and there has been a great deal of work on transience/recurrence of Brownian motion in the more general context of processes on Riemannian manifolds. For a nice survey, see [17].

The classic analysis of transience and recurrence for Lévy processes on locally compact abelian groups involves a subtle blend of harmonic analytic and probabilistic techniques. This work was carried out by Port and Stone [30] who showed that a necessary and sufficient condition for transience is local integrability of the real part of the inverse of the characteristic exponent with respect to the Plancherel measure on the dual group. A nonprobabilistic version of this proof was given by Itô [26]—see also [21, Section 6.2] and [24]. It relies heavily on reduction to the case where the group is of the form $\mathbb{R}^d \times \mathbb{Z}^n$.

Harmonic analytic methods may, at least in principle, be applied to the noncommutative case if we work with Gelfand pairs (G, K) (or more generally, with hypergroups for which we refer readers to [10, Section 6.3]). Here we can take advantage of the existence of spherical functions to develop harmonic analysis of probability measures in the spirit of the abelian case — indeed in the important case where G is a connected Lie group and K is a compact subgroup (so that G/K is a symmetric space), a Lévy–Khintchine type formula which classifies bi-invariant Lévy processes in terms of their characteristic exponent was developed by Gangolli [16]. For further developments of these ideas, see [1, 29] and the survey article [22]. An important approach to establishing the transience of a Markov process is to prove that the associated Dirichlet form gives rise to a transient Dirichlet space. A comprehensive account of this approach can be found in [15, Section 1.5]. Bi-invariant Dirichlet forms associated to Gelfand pairs (G, K) were first studied in a beautiful paper by Berg [7]. He was able to establish that if the Dirichlet space is transient then the inverse of the characteristic exponent is locally integrable with respect to Plancherel measure on the space of positive definite spherical functions. However he was only able to establish the converse to this result in the case where the group was compact or of rank-one. By using different techniques, Berg and Faraut [8] (see also the survey [23]) were able to show that all bi-invariant Lévy processes associated to noncompact irreducible Riemannian symmetric pairs are transient as are the associated Dirichlet spaces in the symmetric case. One of the goals of the current paper is to show that all such processes are *harmonically transient* in the sense that the inverse of the real part of the characteristic exponent is locally integrable with respect to Plancherel measure. Note that this is a slightly stronger result than is obtained in the Euclidean case.

The organisation of this paper is as follows. First we study Lévy processes in a quite abstract context, namely topological transformation groups G acting on a complete metric space M. This allows us to work with processes on the group and then study the induced action on the space of interest. For the main part of the paper, M will be a complete Riemannian manifold. There are three key results here. First we establish a recurrence/transience dichotomy for the induced process in terms of potential measures of open balls. This part of the paper closely follows the development given in Sato [31, Section 7.35] for Euclidean spaces. Secondly we show that such processes are always recurrent when the space is compact, and thirdly, when the group acts transitively, we establish that transience of the process is equivalent to the finiteness of the potential measure on compact sets. This last result (well known in the abelian case) is important for bridging the gaps between probabilistic and analytic approaches to transience.

In the second half of the paper we specialise by taking the group G to be a noncompact semisimple group having a finite centre. We fix a compact subgroup K so that we can consider the action on the symmetric space G/K. We consider symmetric K-bi-invariant Lévy processes within this context. Using the spherical transform, we establish pseudo-differential operator representations of the Markov semigroup, its generator and the associated Dirichlet form which may be of interest in their own right (cf. [3]). We then establish the result on harmonic transience as described above.

Notation. If M is a topological space, $\mathcal{B}(M)$ is the σ-algebra of Borel measurable subsets of M, $B_b(M)$ is the Banach space of all bounded Borel measurable real-valued functions on M (equipped with the supremum norm $\|\cdot\|$). When M is locally compact and Hausdorff then $C_0(M)$ is the closed subspace of $B_b(M)$ comprising continuous functions on M which vanish at infinity and $C_c(M)$ is the dense linear manifold in $C_0(M)$ of continuous functions on M which have compact support. If $\mathcal{F}(M)$ is any real linear space of real-valued functions on M, then $\mathcal{F}_+(M)$ always denotes the cone therein of nonnegative elements. Throughout this article, G is a topological group with neutral element e. For each $\sigma \in G$, l_σ denotes left translation on G and l_σ^* is its differential. If m is a Haar measure on a locally compact group G, we write $m(d\sigma)$ simply as $d\sigma$. When G is compact, we always take m to be normalised. The *reversed measure* $\widetilde{\mu}$ that is associated to each Borel measure μ on a topological group G is defined by $\widetilde{\mu}(A) := \mu(A^{-1})$ for each $A \in \mathcal{B}(G)$. If $f \in L^1(G, \mu)$ we will sometimes write $\mu(f) := \int_G f d\mu$. If X and Y are G-valued random variables defined on some probability spaces, with laws P_X and P_Y, respectively, then $X \stackrel{d}{=} Y$ means $P_X = P_Y$. $\mathbb{R}^+ := [0, \infty)$. We will use Einstein summation convention throughout this paper. The complement of a set A is A^c.

2. Transformation groups and Lévy processes

2.1. Probability on transformation groups. Let G be a topological group with neutral element e, M be a topological space and $\Phi : G \times M \to M$ be continuous. We say that (G, M, Φ) is a *transformation group* if for all $m \in M$:

(T1) $\Phi(e, m) = m$;
(T2) $\Phi(\sigma, \Phi(\tau, m)) = \Phi(\sigma\tau, m)$, for all $\sigma, \tau \in G$;

in other words, Φ is a left action of G on M.

For fixed $m \in M$, we will often write Φ_m to denote the continuous map from G to M defined by $\Phi_m := \Phi(\cdot, m)$. We will be particularly interested in the case where M is a Riemannian manifold. We then say that (G, M, Φ) is *Riemannian*.

The transformation group (G, M, Φ) is said to be *transitive* if for all $m, p \in M$ there exists $\sigma \in G$ such that $p = \Phi(\sigma, m)$. In this case each mapping Φ_m is surjective. A rich class of transitive transformation groups is obtained by choosing a closed subgroup K

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of G and taking M to be the homogeneous space G/K of left cosets. In this case, we will write

$$\Phi_p(\sigma) := \Phi(\sigma, p) = \sigma\tau K,$$

where $p = \tau K$ for some $\tau \in G$. The canonical surjection from G to G/K will be denoted by π.

Now let $(\Omega, \mathcal{F}, P)$ be a probability space and X be a G-valued random variable with law p_X. For each $m \in M$ we obtain an M-valued random variable $X^{\Phi,m}$ by the prescription $X^{\Phi,m} := \Phi_m(X)$. The law of $X^{\Phi,m}$ is $p_X^{\Phi,m} := p_X \circ \Phi_m^{-1}$ and it is clear that if Y is another G-valued random variable then $X^{\Phi,m}$ and $Y^{\Phi,m}$ are independent if X and Y are. Sometimes we will work with a fixed $m \in M$ and in this case we will write $X^\Phi = X^{\Phi,m}$. In the case where $M = G/K$ we will always take $m = eK$ and write $X^\pi := X^{\Phi,m}$.

2.2. Metrically invariant transformation groups. For much of the work that we will carry out in this paper we will need additional structure on the transformation group (G, M, Φ). Specifically we will require the space M to be metrisable by a complete invariant metric d. In this case we will say that (G, M, Φ) is *metrically invariant*. We emphasise that invariance in this context is the requirement that

$$d(\Phi(\sigma, m_1), \Phi(\sigma, m_2)) = d(m_1, m_2),$$

for all $m_1, m_2 \in M, \sigma \in G$.

Two examples will be of particular relevance for our work:

Example 2.1. Compact Lie groups.

Here $M = G$ is a compact Hausdorff Lie group and Φ is left translation. Fix an inner product $\langle\cdot,\cdot\rangle$ on $\mathbf{g}$ and let $\langle\langle X, Y\rangle\rangle := \int_G \langle l_\sigma^* X, l_\sigma^* Y\rangle d\sigma$ for all $X, Y \in \mathbf{g}$. Then $\langle\langle\cdot,\cdot\rangle\rangle$ induces a left-invariant Riemannian metric on G and the associated distance function inherits left invariance. Since G is compact, it is geodesically complete and so is a complete metric space by the Hopf–Rinow theorem (see, for example, [12, pages 26–27]).

Example 2.2. Symmetric spaces [19, 32].

Let G be a Lie group that is equipped with an involutive automorphism Θ and let K be a compact subgroup of G such that $\Theta(K) \subseteq K$. $M = G/K$ is then a C^∞-manifold and we take $m = \pi(e) = eK$. Let $\mathbf{g}$ be the Lie algebra of G. We may write $\mathbf{g} = \mathbf{l} \oplus \mathbf{p}$, where $\mathbf{l}$ and $\mathbf{p}$ are the $+1$ and -1 eigenspaces of Θ^*, respectively, and we have $\pi^*(\mathbf{p}) = T_m(M)$. Every positive definite $\mathrm{Ad}(K)$-invariant inner product defines a G-invariant Riemannian metric on M under which M becomes a globally Riemannian symmetric space (with geodesic symmetries induced by Θ). Integral curves of left-invariant vector fields on G project to geodesics on M and completeness follows by the same argument as in Example 2.1.

In particular, if G is semi-simple then $\mathbf{l} = \mathbf{k}$ where $\mathbf{k}$ is the Lie algebra of K and $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$ is a Cartan decomposition of $\mathbf{g}$. Using the fact that any tangent

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vector $Y_q \in T_q(M)$ (where $q = \tau K$ for $\tau \in G$) is of the form $l_\tau^* \circ \pi^*(Y)$ for some $Y \in \mathbf{p}$, the required metric is given by

$$\langle Y_q, Z_q \rangle = B(Y, Z),$$

where $Z_q = l_\tau^* \circ \pi^*(Z)$ and B is the Killing form on $\mathbf{g}$.

A third important class of examples is obtained by taking M to be an arbitrary geodesically complete Riemannian manifold and G to be the group of all isometries of M.

3. Lévy processes in groups

Let $Z = (Z(t), t \geq 0)$ be a stochastic process defined on $(\Omega, \mathcal{F}, P)$ and taking values in the topological group G. The right increment of Z between s and t where $s \leq t$ is the random variable $Z(s)^{-1}Z(t)$.

We say that Z is a *Lévy process in* G if it satisfies the following:

1. Z has stationary and independent right increments;
2. $Z(0) = e$ (a.s.);
3. Z is stochastically continuous, that is, $\lim_{s \downarrow t} P(Z(s)^{-1}Z(t) \in A) = 0$ for all $A \in \mathcal{B}(G)$ with $e \notin \bar{A}$ and all $t \geq 0$.

Note that Z is called a left Lévy process in [28]. The corresponding notion of right Lévy process is obtained by using left instead of right increments, where the left increment of Z between s and t, $s \leq t$, is the random variable $Z(t)Z(s)^{-1}$.

Now let $(\mu_t, t \geq 0)$ be the law of the Lévy process Z, then it follows from the definition that $(\mu_t, t \geq 0)$ is a weakly continuous convolution semigroup of probability measures on G, where the convolution operation is defined for probability measures μ and ν on G to be the unique probability measure $\mu * \nu$ such that

$$\int_G f(\sigma)(\mu * \nu)(d\sigma) = \int_G \int_G f(\sigma\tau)\mu(d\sigma)\nu(d\tau),$$

for each $f \in B_b(G)$. In particular, we have, for all $s, t \geq 0$,

$$\mu_{s+t} = \mu_s * \mu_t \quad \text{and} \quad \lim_{t \to 0} \mu_t = \mu_0 = \delta_e,$$

where δ_e is the Dirac measure concentrated at e, and the limit is taken in the weak topology of measures.

We obtain a contraction semigroup of linear operators $(T_t, t \geq 0)$ on $B_b(G)$ by the prescription

$$(T_t f)(\sigma) = \mathbb{E}(f(\sigma Z(t))) = \int_G f(\sigma\tau)\mu_t(d\tau), \tag{3.1}$$

for each $t \geq 0$, $f \in B_b(G)$, $\sigma \in G$. The semigroup is left-invariant in that $L_\sigma T_t = T_t L_\sigma$ for each $t \geq 0$, $\sigma \in G$, where $L_\sigma f(\tau) = f(\sigma^{-1}\tau)$ for all $f \in B_b(G)$, $\tau \in G$.

Conversely, given any weakly continuous convolution semigroup of probability measures $(\mu_t, t \geq 0)$ on G, we can always construct a Lévy process $Z = (Z(t), t \geq 0)$ such that each $Z(t)$ has law μ_t by taking Ω to be the space of all mappings from $\mathbb{R}^+$ to G and $\mathcal{F}$ to be the σ-algebra generated by cylinder sets. The existence of P then follows by Kolmogorov's construction, and the time-ordered finite-dimensional distributions have the form

$$\begin{aligned} &P(Z(t_1) \in A_1, Z(t_2) \in A_2, \ldots, Z(t_n) \in A_n) \\ &\quad = \int_G \int_G \cdots \int_G 1_{A_1}(\sigma_1) 1_{A_2}(\sigma_1\sigma_2) \cdots 1_{A_n}(\sigma_1\sigma_2 \cdots \sigma_n) \mu_{t_1}(d\sigma_1) \\ &\quad \times \mu_{t_2 - t_1}(d\sigma_2) \cdots \mu_{t_n - t_{n-1}}(d\sigma_n), \end{aligned}$$

for all $A_1, A_2, \ldots, A_n \in \mathcal{B}(G)$, $0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$. For a proof in the case $G = \mathbb{R}^d$, see [31, Theorem 10.4 on pages 55–57]. The extension to arbitrary G is straightforward. The Lévy process Z that is constructed by these means is called a canonical Lévy process.

If G is a locally compact Hausdorff group then $(T_t, t \geq 0)$ is a left-invariant Feller semigroup in that

$$T_t(C_0(G)) \subseteq C_0(G) \quad \text{and} \quad \lim_{t \downarrow 0} \|T_t f - f\| = 0$$

for each $f \in C_0(G)$. The infinitesimal generator of $(T_t, t \geq 0)$ is denoted by $\mathcal{A}$. A characterisation of $\mathcal{A}$ can be found in [11] (see [25, 28] for the Lie group case.) It follows from the argument in [28, page 10] that if $(Z(t), t \geq 0)$ is a G-valued Markov process with left-invariant Feller transition semigroup then it is a Lévy process. Moreover, if G is separable and metrisable as well as being locally compact then by [14, Theorem 2.7 in Ch. 4, page 169], the process has a càdlàg modification (that is one that is almost surely right continuous with left limits.)

Let (G, M, Φ) be a transformation group and $(Z(t), t \geq 0)$ be a Lévy process on G. Then for each $m \in M$, $t \geq 0$ we define $Z^{\Phi,m}(t) := \Phi_m(Z(t))$. The M-valued process $Z^{\Phi,m} = (Z^{\Phi,m}(t), t \geq 0)$ will be called a *Lévy process on M starting at m*. This is to some extent an abuse of terminology as the one-point motion $Z^{\Phi,m}$ is not, in general, a Markov process if Z is (as we have assumed) a *left* Lévy process. However, this will be the case when M is a symmetric space as in Section 5. Note that (as is shown in [28, Proposition 2.1 on page 33]), the left action always produces a Markov process on M when Z is a *right* Lévy process and the reader can check that all the results that we obtain in the next section are also valid in this case.

4. Criteria for recurrence and transience

Throughout this section, (G, M, Φ) will be a metrically invariant transformation group and d will denote the complete metric on M, Z will be a Lévy process on G and $Z^{\Phi,m}$ will be the associated Lévy process on M starting at m.

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We say that Z is *recurrent at m* if

$$\liminf_{t\to\infty} d(Z^{\Phi,m}(t), m) = 0 \quad \text{(a.s.)},$$

and *transient at m* if

$$\lim_{t\to\infty} d(Z^{\Phi,m}(t), m) = \infty \quad \text{(a.s.)}.$$

If (G, M, Φ) is also transitive, it follows easily from the invariance of d that if Z is recurrent (respectively, transient) at any given point of M then it is recurrent (respectively, transient) at every point of M.

Define the *potential measure V* associated to Z by

$$V(A) = \int_0^\infty \mu_s(A)\, ds,$$

so that $V(B) \in [0, \infty]$, for each $B \in \mathcal{B}(G)$. For each $m \in M$, the *induced potential measure* on $\mathcal{B}(M)$ is defined by

$$V^{\Phi,m} := V \circ \Phi_m^{-1}.$$

In the following, we will frequently apply $V^{\Phi,m}$ to open balls of the form $B_r(m) = \{p \in M : d(p, m) < r\}$ for some $r > 0$. When $m \in M$ is fixed, we will write $V^\Phi := V^{\Phi,m}$.

The following transience–recurrence dichotomy gives a characterisation in terms of the behaviour of potential measures in the case where M is a (complete) Riemannian manifold.

Theorem 4.1. *If $Z = (Z(t), t \geq 0)$ is a Lévy process in a group G and (G, M, Φ) is a metrically invariant Riemannian transformation group then for fixed $m \in M$:*

(1) *Z is either recurrent or transient at m;*
(2) *Z is recurrent if and only if $V^\Phi(B_r(m)) = \infty$ for all $r > 0$;*
(3) *Z is recurrent if and only if $\int_0^\infty 1_{B_r(m)}(Z^\Phi(t))\, dt = \infty$ (a.s.), for all $r > 0$;*
(4) *Z is transient if and only if $V^\Phi(B_r(m)) < \infty$, for all $r > 0$;*
(5) *Z is transient if and only if $\int_0^\infty 1_{B_r(m)}(Z^\Phi(t))\, dt < \infty$ (a.s.), for all $r > 0$.*

Proof. We omit the full details, as this proof is carried out in the same way as the analogous proof for the case $G = M = \mathbb{R}^d$, which can be found in [31, pages 237–242]. The main difference is that we systematically replace the Euclidean norm $|\cdot|$ with $d(m, \cdot)$ in all arguments. So analogues of (1), (2) and (3) are first established for G-valued random walks (see also [18, pages 19–20]). Observe that for each $h > 0$, $(Z(nh), n \in \mathbb{Z}_+)$ is a random walk on G since for each $n \in \mathbb{N}$,

$$Z(nh) = Z(h).Z(h)^{-1}Z(2h) \cdots Z((n-2)h)^{-1}Z((n-1)h).Z((n-1)h)^{-1}Z(nh)$$

is the composition of n i.i.d. G-valued random variables.

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Another important step in the proof which we emphasise is the generalisation of the following inequality, due to Kingman [27] in the case $G = \mathbb{R}^d$, that is, there exists $\gamma : \mathbb{R}^+ \to \mathbb{R}$ with $\lim_{\epsilon \downarrow 0} \gamma(\epsilon) = 1$, such that for all $r, \epsilon, t > 0$,

$$P\Big(\int_t^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds > \epsilon\Big) \\ \geq \gamma(\epsilon) P(d(Z^\Phi(s+t), m) < r, \text{ for some } s > 0).$$

To illustrate how the arguments work we present the proof of (4). First observe that by (1) if Z is transient it is not recurrent and so (2) fails to hold, hence (4) holds. Conversely assume that $V^\Phi(B_r(m)) < \infty$, for all $r > 0$ and choose ϵ so that $\gamma(\epsilon) > \frac{1}{2}$. By the Markov and Kingman inequalities we find that for all $t \geq 0$,

$$\mathbb{E}\Big[\int_t^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds\Big] \geq \epsilon P\Big[\int_t^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds > \epsilon\Big] \\ > \frac{\epsilon}{2} P(d(Z^\Phi(s+t), m) < r, \text{ for some } s > 0).$$

By Fubini's theorem,

$$V^\Phi(B_{2r}(m)) = \mathbb{E}\Big[\int_0^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds\Big],$$

and so by our assumption, $\int_t^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds < \infty$ (a.s.). By a similar argument we find that

$$\lim_{t\to\infty} \mathbb{E}\Big[\int_t^\infty 1_{B_{2r}(m)}(Z^\Phi(s))\, ds\Big] = 0,$$

and hence $\lim_{t\to\infty} P(d(Z^\Phi(s+t), m) < r, \text{ for some } s > 0) = 0$. Transience at m then follows from the observation that

$$\{\lim_{t\to\infty} d(Z^\Phi(t), m) = \infty\} = \bigcap_{k=1}^\infty \bigcup_{n=1}^\infty \{d(Z^\Phi(s+n), m) \geq k \text{ for all } s > 0\}.$$

This completes the proof of the theorem. □

We now show that, just as in the case $G = \mathbb{R}^d$, the recurrence or transience of Lévy processes is related to that of certain embedded random walks. Again the proof of this theorem follows along the same lines as that in Sato [31, page 242]. This time, we give more of the details.

Theorem 4.2. *If the G-valued random walk $(Z(nh), n \in \mathbb{Z}_+)$ is recurrent at $m \in M$ for some $h > 0$, then so is the Lévy process Z. Conversely, if Z is a càdlàg G-valued Lévy process that is recurrent at $m \in M$, then there exists $h > 0$ such that the random walk $(Z(nh), n \in \mathbb{Z}_+)$ is recurrent at m.*

Proof. Suppose the random walk is recurrent at m, then $\liminf_{n\to\infty} d(Z^\Phi(nh), m) = 0$ (a.s.). Since

$$0 \leq \liminf_{t\to\infty} d(Z^\Phi(t), m) \leq \liminf_{n\to\infty} d(Z^\Phi(nh), m) = 0,$$

we see that Z is recurrent.

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Conversely, suppose that Z is recurrent at m. First note that $\sup_{s\in[0,h]} d(Z^{\Phi}(s), m) < \infty$ (a.s.) for each $h > 0$ since Z is càdlàg and d is continuous. It follows that there exists $h > 0$, $r > 0$ such that $P(\sup_{s\in[0,h]} d(Z^{\Phi}(s), m) < r) > \frac{1}{2}$. By the argument of [31, page 241] ((3) ⇒ (4) therein), we deduce that

$$P(Z^{\Phi}(nh) \in B_{3r}(m)) \geq \frac{1}{2h} \int_{(n-1)h}^{nh} P(Z^{\Phi}(t) \in B_{2r}(m))\, dt.$$

From this and Theorem 4.1(2), it follows that $\sum_{n=1}^{\infty} P(Z^{\Phi}(nh) \in B_{3r}(m)) = \infty$ and so the random walk is recurrent, as required. □

We complete this section by establishing two straightforward but useful results.

Proposition 4.3. *If M is compact then every G-valued càdlàg Lévy process is recurrent at every point $m \in M$.*

Proof. The mapping $p \to d(m, p)$ from M to $\mathbb{R}$ is continuous and hence its image is compact. Consequently, the mapping $t \to d(m, Z^{\Phi}(t))$ is a.s. bounded for every càdlàg Lévy process $(Z(t), t \geq 0)$ on G, and so such a process cannot be transient. Hence it is recurrent by Theorem 4.1(1). □

Remark. Take $G = M$ in Proposition 4.3 to be a compact Lie group equipped with a left-invariant Riemannian metric. It follows from [4, Lemma 5.4] that (modulo some technical conditions on the characteristics of $(\mu_t, t \geq 0)$) the unique invariant measure for the Markov semigroup $(T_t, t \geq 0)$ is (normalised) Haar measure.

Theorem 4.4. *Let (G, M, Φ) be a metrically invariant Riemannian transformation group and $Z = (Z(t), t \geq 0)$ be a Lévy process in G. If $Z^{\Phi,m}$ is transient at m then $V^{\Phi,m}(K) < \infty$ for every compact set K in M. If (G, M, Φ) is also transitive then the converse statement holds.*

Proof. Fix $m \in M$. If Z is transient at m then $V^{\Phi,m}(B_r(p)) < \infty$ for all $p \in M$, $r > 0$. To see this, observe that we can always find a $u > 0$ such that $B_r(p) \subseteq B_u(m)$. Then by Theorem 4.1(4), we have $V^{\Phi,m}(B_r(p)) \leq V^{\Phi,m}(B_u(m)) < \infty$. Now since M is complete, every compact K in M is totally bounded and so we can find $N \in \mathbb{N}$, $p_1, \ldots, p_N \in M$ and $r_1, \ldots, r_N > 0$ such that $K \subseteq \bigcup_{i=1}^{N} B_{r_i}(p_i)$. Hence

$$V^{\Phi,m}(K) \leq \sum_{i=1}^{N} V^{\Phi,m}(B_{r_i}(p_i)) < \infty,$$

as was required.

Now suppose that (G, M, Φ) is transitive and assume that $V^{\Phi,m}(K) < \infty$ for every compact set K in M. Since M is locally compact and Hausdorff there exists a compact K which has nonempty interior K^0, and we choose $p \in K^0$. Then we can find $r > 0$ such that $B_r(p) \subseteq K^0$. Hence $V^{\Phi,m}(B_r(p)) \leq V^{\Phi,m}(K^0) \leq V^{\Phi,m}(K) < \infty$ and so by transitivity $V^{\Phi,m}(B_r(m)) < \infty$. Hence, by Theorem 4.1 the process cannot be recurrent and so it is transient. □

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5. Transience and harmonic transience for bi-invariant convolution semigroups on noncompact symmetric pairs

In this section we will be concerned with weakly continuous bi-invariant convolution semigroups of probability measures $(\mu_t, t \geq 0)$ defined on noncompact Riemannian symmetric pairs (G, K). We begin with some harmonic analysis of the associated left-invariant Feller semigroup on G and its generator.

5.1. Spherical representation of semigroups and generators. Let (G, K) be a Riemannian symmetric pair of noncompact type so G is a connected semisimple Lie group with finite centre and K is a maximal compact subgroup. We note that G is unimodular and we fix a bi-invariant Haar measure. The Iwasawa decomposition gives a global diffeomorphism between G and a direct product KAN, where A and N are simply connected with A being abelian and N nilpotent wherein each $g \in G$ is mapped onto $k(g)\exp(H(g))n(g)$, where $k(g) \in K, n(g) \in N$ and $H(g) \in \mathfrak{a}$ which is the Lie algebra of A. We recall that the *spherical functions* on (G, K) are the unique mappings $\phi \in C(G, \mathbb{C})$ for which $\phi \neq 0$ and

$$\int_K \phi(\sigma k \tau)\, dk = \phi(\sigma)\phi(\tau), \tag{5.1}$$

for all $\sigma, \tau \in G$. We refer the reader to [20] for general facts about spherical functions. In particular, it is shown therein that every spherical function on G is of the form

$$\phi_\lambda(\sigma) = \int_K e^{(i\lambda+\rho)(H(k\sigma))}\, dk, \tag{5.2}$$

for $\sigma \in G$, where $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, which is the complexification of the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$, and ρ is half the sum of positive roots (relative to a fixed lexicographic ordering). Note that if $\lambda \in \mathfrak{a}^*$ then ϕ_λ is positive definite.

A Borel measure μ on G is said to be K-bi-invariant if

$$\mu(k_1 A k_2) = \mu(A),$$

for all $k_1, k_2 \in K, A \in \mathcal{B}(G)$. The set $\mathcal{M}(K\backslash G/K)$ of all K-bi-invariant Borel probability measures on G forms a *commutative* monoid under convolution, that is,

$$\mu * \nu = \nu * \mu,$$

for all $\mu, \nu \in \mathcal{M}(K\backslash G/K)$. The *spherical transform* of $\mu \in \mathcal{M}(K\backslash G/K)$ is defined by

$$\widehat{\mu}(\lambda) = \int_G \phi_\lambda(\sigma)\mu(d\sigma),$$

for all $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$. Note that

$$\widehat{\mu * \nu}(\lambda) = \widehat{\mu}(\lambda)\widehat{\nu}(\lambda), \tag{5.3}$$

for all $\mu, \nu \in \mathcal{M}(K\backslash G/K), \lambda \in \mathfrak{a}^*_{\mathbb{C}}$.

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Fix a basis $(X_j, 1 \leq j \leq n)$ of the Lie algebra $\mathbf{g}$ of G, where n is the dimension of G. As is shown in [25, 28], there exist functions $x_i \in C_c^\infty(G)$, $1 \leq i \leq n$, so that $x_i(e) = 0$, $X_i x_j(e) = \delta_{ij}$ and $\{x_i, 1 \leq i \leq n\}$ are canonical coordinates for G in a neighbourhood of the identity in G. A measure ν defined on $\mathcal{B}(G - \{e\})$ is called a *Lévy measure* whenever $\int_{G-\{e\}} (\sum_{i=1}^n x_i(\sigma)^2)\nu(d\sigma) < \infty$ and $\nu(U^c) < \infty$ for any Borel neighbourhood U of G.

From now on we will assume that the measures forming the convolution semigroup $(\mu_t, t \geq 0)$ are K-bi-invariant for each $t > 0$. Recall that $(T_t, t \geq 0)$ is a Feller semigroup which extends to a Markovian semigroup on $L^2(G)$. Under the bi-invariance assumption, it follows that for each $t > 0$, T_t preserves the real Hilbert space $L^2(K\backslash G/K)$ of K-bi-invariant square integrable functions on G. Let $\|\cdot\|_2$ denote the norm in $L^2(G)$. From now on we make the following assumption.

Assumption. $\alpha_0 := \lim_{t\to 0}(1/t)\|T_t\|_2 < 0$.

The importance of this assumption is precisely that it excludes the degenerate case that μ_t is equal to normalised Haar measure on K for all $t > 0$.

We have Gangolli's Lévy–Khintchine formula (see [1, 16, 29])

$$\widehat{\mu_t}(\lambda) = \exp\{-t\eta_\lambda\}, \tag{5.4}$$

for all $\lambda \in \mathfrak{a}^*$, $t \geq 0$. Here

$$\eta_\lambda = \beta_\lambda + \int_{G-\{e\}} (1 - \phi_\lambda(\tau))\nu(d\tau), \tag{5.5}$$

where $\beta_\lambda \in \mathbb{C}$ and ν is a K-bi-invariant Lévy measure on G (see [16, 29] for details). We call η_λ the *characteristic exponent* of the convolution semigroup. We will always assume that (G, K) is *irreducible*, that is, that the adjoint action of K leaves no proper subspace of $\mathbf{p}$ invariant. In this case $\beta_\lambda \geq 0$. Note that $\alpha_0 = -\eta_0 < 0$, as required (see [8, page 286].) We can and will equip G with a left-invariant Riemannian metric that is also right K-invariant. Let Δ_G be the associated Laplace-Beltrami operator of G. Then $-\beta_\lambda$ is an eigenvalue of $a\Delta_G$ where $a \geq 0$. Specifically, for each $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we have

$$\beta_\lambda = a(|\lambda|^2 + |\rho|^2),$$

where the norm is that induced on $\mathfrak{a}_{\mathbb{C}}^*$ by the Killing form (see [20, page 427]). We define $\widetilde{\beta_\lambda} = |\lambda|^2 + |\rho|^2$.

Proposition 5.1. *There exists $K > 0$ such that for all $\lambda \in \mathfrak{a}^*$,*

$$|\eta_\lambda| \leq K(1 + |\lambda|^2 + |\rho|^2).$$

Proof. Let U be a coordinate neighbourhood of e in G and write

$$\int_{G-\{e\}} (1 - \phi_\lambda(\tau))\nu(d\tau) = \int_{U-\{e\}} (1 - \phi_\lambda(\tau))\nu(d\tau) + \int_{U^c} (1 - \phi_\lambda(\tau))\nu(d\tau).$$

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Since ϕ_λ is positive definite we have $\sup_{\sigma\in G}|\phi_\lambda(\sigma)| \leq \phi_\lambda(e) = 1$, and so $\int_{U^c}|1-\phi_\lambda(\tau)|\nu(d\tau) \leq 2\nu(U^c) < \infty$. Using a Taylor series expansion as in [28, page 13] and the fact that $X_i\phi_\lambda(e) = 0$ for each $1 \leq i \leq n$, we see that for each $\tau \in U$ there exists $\tau' \in U$ such that

$$1 - \phi_\lambda(\tau) = -\tfrac{1}{2}x^i(\tau)x^j(\tau)X_iX_j\phi_\lambda(\tau').$$

Arguing as in [29, Proof of Lemma 1], we apply a Schauder estimate to the equation $\Delta_G\phi_\lambda = -\widetilde{\beta_\lambda}\phi_\lambda$ to deduce that for each $1 \leq i, j \leq n$ there exists $C_{ij} > 0$ so that

$$\begin{aligned}|X_iX_j\phi_\lambda(\tau')| &\leq C_{ij}(1+\widetilde{\beta_\lambda})\sup_{\sigma\in G}|\phi_\lambda(\sigma)|\\ &\leq C_{ij}(1+\widetilde{\beta_\lambda}).\end{aligned}$$

Hence, we have

$$\begin{aligned}\int_{U-\{e\}}|1-\phi_\lambda(\tau))|\nu(d\tau) &\leq \frac{1+\widetilde{\beta_\lambda}}{2}\sum_{i,j=1}^{n} C_{ij}\int_{U-\{e\}}|x^i(\tau)x^j(\tau)|\nu(d\tau)\\ &\leq \frac{1+\widetilde{\beta_\lambda}}{2}\bigg(\sum_{i,j=1}^{n} C_{ij}^2\bigg)^{\frac{1}{2}}\int_{U-\{e\}}\sum_{i=1}^{n} x^i(\tau)^2\nu(d\tau) < \infty,\end{aligned}$$

and the required result follows easily from here. □

If $\lambda \in \mathfrak{a}^*$ and $(\mu_t, t \geq 0)$ is symmetric, it is easily verified that η_λ is real-valued and nonnegative. Indeed from (5.5) and (5.2) we obtain

$$\eta_\lambda = \beta_\lambda + \int_{G-\{e\}}\int_K\{1-\cos((\lambda+\rho)(H(k\tau)))\}\,dk\nu(d\tau).$$

Let $C_c(K\backslash G/K)$ denote the subspace of $C_c(G)$ which comprises K-bi-invariant functions. If $f \in C_c(K\backslash G/K)$, its spherical transform is the mapping $\widehat{f} : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ defined by

$$\widehat{f}(\lambda) = \int_G f(\sigma)\phi_{-\lambda}(\sigma)\,d\sigma. \tag{5.6}$$

We have the key Paley–Wiener type estimate that for each $N \in \mathbb{Z}_+$ there exists $C_N > 0$ such that

$$|\widehat{f}(\lambda)| \leq C_N(1+|\lambda|)^{-N}e^{R|\mathrm{Im}(\lambda)|}, \tag{5.7}$$

where $R > 0$ (see [20, page 450]).

We define the Plancherel measure ω on $\mathfrak{a}^*$ by the prescription

$$\omega(d\lambda) = \kappa|c(\lambda)|^{-2}\,d\lambda,$$

where $c : \mathfrak{a}^*_{\mathbb{C}} \to \mathbb{C}$ is Harish-Chandra's c-function. We will not require the precise definition of c nor the value of the constant $\kappa > 0$; however, we will find a use for the estimate

$$|c(\lambda)|^{-1} \leq C_1 + C_2|\lambda|^p \tag{5.8}$$

for all $\lambda \in \mathfrak{a}^*$, where $C_1, C_2 > 0$ and $2p = \dim(N)$. This result follows from [20, Proposition 7.2, page 450 and Equation (16) therein on page 451].

By [20, Theorem 7.5, page 454] we have the Fourier inversion formula for $f \in C_c^\infty(K\backslash G/K)$,

$$f(\sigma) = \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)\omega(d\lambda), \tag{5.9}$$

which holds for all $\sigma \in G$, and the Plancherel formula,

$$\int_G |f(\sigma)|^2 d\sigma = \int_{\mathfrak{a}^*} |\widehat{f}(\lambda)|^2 \omega(d\lambda).$$

By polarisation we obtain the Parseval identity for $f, g \in C_c^\infty(K\backslash G/K)$,

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle,$$

where $\langle \widehat{f}, \widehat{g}\rangle := \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\overline{\widehat{g}(\lambda)}\omega(d\lambda)$. Although the spherical transform $f \to \widehat{f}$ extends to a unitary transformation from $L^2(K\backslash G/K)$ to a suitable space of functions on $\mathfrak{a}^*$ we cannot assume that its precise form extends beyond the functions of compact support. We will have more to say about this later in a special case that will be important for us.

The next result is analogous to the pseudo-differential operator representations obtained in Euclidean space in [2, Theorem 3.3.3] (see also [3] for extensions to compact groups).

THEOREM 5.2. *For each $\sigma \in G$, $f \in C_c^\infty(K\backslash G/K)$*

1. $$T_t f(\sigma) = \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)e^{-t\eta_\lambda}\omega(d\lambda), \quad \textit{for each } t \geq 0.$$

2. $$\mathcal{A} f(\sigma) = -\int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)\eta_\lambda\omega(d\lambda).$$

PROOF.

1. Applying Fourier inversion (5.9) in (3.1) and using Fubini's theorem, we obtain

$$\begin{aligned}
T_t f(\sigma) &= \int_{\mathfrak{a}^*}\int_G \widehat{f}(\lambda)\phi_\lambda(\sigma\tau)\mu_t(d\tau)\omega(d\lambda)\\
&= \int_{\mathfrak{a}^*}\int_G \widehat{f}(\lambda)\int_K \phi_\lambda(\sigma k\tau)\, dk\mu_t(d\tau)\omega(d\lambda)\\
&= \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)\Big(\int_G \phi_\lambda(\tau)\mu_t(d\tau)\Big)\omega(d\lambda)\\
&= \int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)e^{-t\eta_\lambda}\omega(d\lambda),
\end{aligned}$$

where we have used the left K-invariance of μ_t, (5.1) and (5.4).

2. We have

$$\begin{aligned}
\frac{1}{\kappa}\mathcal{A} f(\sigma) &= \lim_{t\to 0}\int_{\mathfrak{a}^*}\Big(\frac{e^{-t\eta_\lambda}-1}{t}\Big)\widehat{f}(\lambda)\phi_\lambda(\sigma)|c(\lambda)|^{-2}\, d\lambda\\
&= -\lim_{t\to 0}\int_{\mathfrak{a}^*} e^{-\theta_\lambda t\eta_\lambda}\eta_\lambda\widehat{f}(\lambda)\phi_\lambda(\sigma)|c(\lambda)|^{-2}\, d\lambda,
\end{aligned}$$

where $0 < \theta_\lambda < 1$ for each $\lambda \in \mathfrak{a}^*$, $t > 0$.

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The required result follows by Lebesgue's dominated convergence theorem. To see this, first observe that for all $\lambda \in \mathfrak{a}^*$, $t \geq 0$, since $\mathrm{Re}(\eta_\lambda) \geq 0$, we have $|e^{-t\theta_\lambda \eta_\lambda}| \leq 1$ and so by Proposition 5.1 for each $t \geq 0$

$$\int_{\mathfrak{a}^*} |\eta_\lambda e^{-t\theta_\lambda \eta_\lambda} \widehat{f}(\lambda)\phi_\lambda(\sigma)||c(\lambda)|^{-2}\, d\lambda \leq K \int_{\mathfrak{a}^*} (1 + |\lambda|^2 + |\rho|^2)\widehat{f}(\lambda)\phi_\lambda(\sigma)|c(\lambda)|^{-2}\, d\lambda.$$

The integral on the right-hand side is easily seen to be finite by using (5.8) and taking N to be sufficiently large in (5.7). □

Note. Based on the results of Theorem 5.2 we may write

$$\widehat{T_t f}(\lambda) = e^{-t\eta_\lambda}\widehat{f}(\lambda) \quad \text{and} \quad \widehat{\mathcal{A}f}(\lambda) = -\eta_\lambda \widehat{f}(\lambda),$$

for all $t \geq 0$, $\lambda \in \mathfrak{a}^*$.

We will need to extend the precise form of Parseval's formula to include the range of T_t acting on $C_c(K\backslash G/K)$.

THEOREM 5.3. *For all $f, g \in C_c(K\backslash G/K)$, $t \geq 0$,*

$$\langle T_t f, g\rangle = \langle \widehat{T_t f}, \widehat{g}\rangle. \tag{5.10}$$

PROOF. Using the result of Theorem 5.2(1), Fubini's theorem, the fact that for all $\sigma \in G$, $\lambda \in \mathfrak{a}^*$, $\overline{\phi_\lambda(\sigma)} = \phi_{-\lambda}(\sigma)$ and (5.6), we find that for all $f, g \in C_c(K\backslash G/K)$, $t \geq 0$,

$$\begin{aligned}
\langle T_t f, g\rangle &= \int_G \Big(\int_{\mathfrak{a}^*} \widehat{f}(\lambda)\phi_\lambda(\sigma)e^{-t\eta_\lambda}\omega(d\lambda)\Big) g(\sigma)\, d\sigma \\
&= \int_{\mathfrak{a}^*} e^{-t\eta_\lambda}\widehat{f}(\lambda)\Big(\overline{\int_G g(\sigma)\phi_{-\lambda}(\sigma)\, d\sigma}\Big)\omega(d\lambda) \\
&= \int_{\mathfrak{a}^*} e^{-t\eta_\lambda}\widehat{f}(\lambda)\overline{\widehat{g}(\lambda)}\omega(d\lambda) \\
&= \langle \widehat{T_t f}, \widehat{g}\rangle.
\end{aligned}$$

This completes the proof of the theorem. □

Now assume that the convolution semigroup $(\mu_t, t \geq 0)$ is symmetric, that is $\mu_t = \widetilde{\mu_t}$ for all $t \geq 0$. The space $C_c^\infty(K\backslash G/K)$ is a dense linear subspace of the Hilbert space $L^2(K\backslash G/K)$ of K-bi-invariant square integrable functions on G. We consider the restriction therein of the Dirichlet form $\mathcal{E}$, where $\mathcal{E}(f) := \|(-\mathcal{A})^{\frac{1}{2}} f\|^2$ on $\mathcal{D} := \mathrm{Dom}(\mathcal{A}^{\frac{1}{2}})$.

COROLLARY 5.4. *For each $f, g \in C_c^\infty(K\backslash G/K)$ we have*

$$\mathcal{E}(f, g) = \int_{a^*} \widehat{f}(\lambda)\eta_\lambda \overline{\widehat{g}(\lambda)}\omega(d\lambda).$$

PROOF. This follows from differentiating both sides of (5.10) using Theorem 5.2(1). □

5.2. Harmonic transience. Let $(\mu_t, t \geq 0)$ be an arbitrary K-bi-invariant convolution semigroup in G. As is shown in [28, Theorem 2.2, page 43], we obtain a Feller semigroup $(S(t), t \geq 0)$ on G/K by the prescription:

$$(S(t)f) \circ \pi = T(t)(f \circ \pi)$$

for all $f \in C_0(G/K)$. We define the *potential operator* N on $C_0(G/K)$ by the prescription

$$Nf := \lim_{t\to\infty} \int_0^t S_u f\, du$$

for all

$$f \in \mathrm{Dom}(N) := \Big\{ f \in C_0(G/K); \lim_{t\to\infty} \int_0^t S_u f\, du \text{ exists in } C_0(G/K) \Big\}.$$

We say that $(\mu_t^\pi, t \geq 0)$ is *integrable* if $C_c(G/K) \subseteq \mathrm{Dom}(N)$. By the same arguments as are presented in [9, Lemmas 12.2 and 13.19 and Proposition 13.21], it follows that integrability implies transience. Berg and Faraut [8] have shown that if G/K is irreducible then the projection to G/K of every bi-invariant convolution semigroup on G is integrable and hence all of these semigroups of measures are transient. It then follows that $(\mathcal{E}, D)$ is a transient Dirichlet space. In [23, Theorem 7.3] this result is extended to the case where the symmetric space is no longer required to be irreducible. Furthermore, it is shown that the associated Feller semigroup is *mean ergodic*, that is, $\lim_{T\to\infty}(1/T) \int_0^T S_u f\, du$ exists for all $f \in C_0(G/K)$.

We say that $(\mu_t, t \geq 0)$ is *harmonically transient* if the mapping

$$\lambda \to \frac{1}{\mathrm{Re}(\eta_\lambda)}$$

from $\mathfrak{a}^*$ to $\mathbb{R}$ is locally integrable with respect to the Plancherel measure ω.

Note that the result of (i) in Theorem 5.5 below is well known to be a necessary and sufficent condition for a transient Dirichlet space (see [13]); however, we include a short proof to make the paper more self-contained.

THEOREM 5.5. *If (G, K) is an irreducible Riemannian symmetric pair and $(\mu_t, t \geq 0)$ is a symmetric K-bi-invariant convolution semigroup (with $\alpha_0 < 0$) then*

(i) $\int_0^\infty \langle T_t f, f\rangle\, dt < \infty$ *for all* $f \in C_{c,+}(K\backslash G/K)$;

(ii) $(\mu_t, t \geq 0)$ *is harmonically transient.*

PROOF.

(i) Let $f \in C_{c,+}(K\backslash G/K)$ and let $C = \mathrm{supp}(f)$ then

$$\int_0^\infty \langle T_t f, f\rangle\, dt = \int_C f(\sigma) \int_G 1_{\sigma^{-1}C}(\tau) f(\sigma\tau) V(d\tau)\, d\sigma.$$

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Define $C^{-1}C := \{\sigma^{-1}\tau; \sigma, \tau \in C\}$. Since $C^{-1}C$ is the image of the continuous mapping from $C \times C$ to G which takes (σ, τ) to $\sigma^{-1}\tau$, it is compact. It follows that $\sup_{\sigma \in C} V(\sigma^{-1}C) \leq V(C^{-1}C) < \infty$ as $(\mu_t, t \geq 0)$ is transient, and so the mapping $\sigma \to \int_G 1_{\sigma^{-1}C}(\tau) f(\sigma\tau) V(d\tau)$ is bounded. The required result follows.

(ii) This is proved by a similar method to the Euclidean space case (see [15, Example 1.5.2, pages 42–43]). We include it for the convenience of the reader, noting that it was for this specific purpose that we established Theorem 5.3. We choose $f \in C_{c,+}(K\backslash G/K)$ such that $\widehat{f}(\lambda) \geq 1$ for all λ in some compact neighbourhood A of $\mathfrak{a}^*$. Then using Theorems 5.2(1) and 5.3 we get

$$\int_A \frac{\omega(d\lambda)}{\eta_\lambda} \leq \int_A \frac{|\widehat{f}(\lambda)|^2}{\eta_\lambda} \omega(d\lambda) \leq \int_{\mathfrak{a}^*} \frac{|\widehat{f}(\lambda)|^2}{\eta_\lambda} \omega(d\lambda)$$
$$= \int_0^\infty \langle \widehat{T_t f}, \widehat{f} \rangle \, dt = \int_0^\infty \langle T_t f, f \rangle \, dt < \infty.$$

This completes the proof of the theorem. □

Corollary 5.6. *If (G, K) is an irreducible Riemannian symmetric pair and $(\mu_t, t \geq 0)$ is a K-bi-invariant convolution semigroup then it is harmonically transient.*

Proof. Let $(\mu_t, t \geq 0)$ be an arbitrary bi-invariant convolution semigroup with characteristic exponent η_λ. We associate to it a symmetric bi-invariant convolution semigroup $(\mu_t^{(R)}, t \geq 0)$ by the prescription $\mu_t^{(R)} = \mu_t * \widetilde{\mu_t}$. It follows easily from (5.3) and (5.4) that it has characteristic exponent $2\text{Re}(\eta_\lambda)$ and using this fact we can immediately deduce that $(\mu_t, t \geq 0)$ is harmonically transitive by using Theorem 5.5. □

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