POLYCYCLIC, METABELIAN OR SOLUBLE OF TYPE $(FP)_\infty$ GROUPS WITH BOOLEAN ALGEBRA OF RATIONAL SETS AND BIAUTOMATIC SOLUBLE GROUPS ARE VIRTUALLY ABELIAN

VITALY ROMAN'KOV

Omsk State University n.a. Dostoevskii and Omsk State Technical University,
644077, Omsk, Russia
e-mail: romankov48@mail.ru

Abstract. Let G be a polycyclic, metabelian or soluble of type $(FP)_\infty$ group such that the class $Rat(G)$ of all rational subsets of G is a Boolean algebra. Then, G is virtually abelian. Every soluble biautomatic group is virtually abelian.

2010 *Mathematics Subject Classification.* 20F16, 20J06, 03G05, 20F65.

1. Introduction. The topic of this paper is two important concepts: rational sets and biautomatic structure. We study finitely generated soluble groups G such that the class $Rat(G)$ of all rational subsets of G is a Boolean algebra. We conjecture that every such group is virtually abelian. Note that every finitely generated virtually abelian group satisfies to this property. We confirm this conjecture in the case where G is a polycyclic, metabelian or soluble group of type FP_∞. This conjecture remains open in a general case. It appeared that the notion FP_∞ helps to prove by the way that every soluble biautomatic group is virtually abelian. Thus, we give answer to known question posed in [**1**].

We provide full proofs of four theorems attributed to Bazhenova, a former student of the author, stating that some natural assumptions imply that a soluble group is virtually abelian. The original proofs were given by her in collaboration with the author more than 14 years ago and had never been published. Now we fill this gap by presenting improved versions of these proofs.

Recall that a Boolean algebra is a set B together with operations $\neg : B \rightarrow B$, $\wedge : B \times B \rightarrow B$, and $\vee : B \times B \rightarrow B$, and special elements $\mathbf{0} \in B$ and $\mathbf{1} \in B$, which satisfies the following properties for all $a, b, c \in B$: 1) $a \wedge \mathbf{1} = a \vee \mathbf{0} = a$, 2) $a \wedge \neg a = \mathbf{0}$, $a \vee \neg a = \mathbf{1}$, 3) $a \wedge a = a \vee a = a$, 4) $\neg(a \wedge b) = \neg a \vee \neg b$, $\neg(a \vee b) = \neg a \wedge \neg b$, 5) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$, 6) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$, $(a \vee b) \vee c = a \vee (b \vee c)$, 7) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

Let $C \subseteq B$ be a subset of Boolean algebra B containing $\mathbf{0}$ and $\mathbf{1}$ and closed under the Boolean operations. Then, C is a Boolean algebra, and we say C is a subalgebra of B.

Let X be any set and $B = \mathcal{P}(X)$ be the set of all subsets of X. Then, B is a Boolean algebra with $\wedge = \cap$, $\vee = \cup$, $\mathbf{0} = \emptyset$, $\mathbf{1} = X$, and $\neg A = X \setminus A$.

Given a monoid M, a *rational set* is an element of the minimal class $Rat(M)$ of subsets of this monoid that contains all singleton subsets and is closed under union, product and Kleeny star operations. When M is finitely generated monoid, the class

$Rat(M)$ under these operations is a Boolean algebra with $\mathbf{1} = M$ and $\mathbf{0} = \emptyset$ if and only if $Rat(M)$ is closed under intersection.

Rational sets are useful in automata theory, formal languages and algebra. An excellent introduction to rational sets is [4], where the reader can find out basic definitions and fundamental results in this area.

Much of the basic theory of automatic and biautomatic groups is presented by Epstein et al. in [1]. One of the major open questions in group theory is whether or not an automatic group is necessarily biautomatic. The answer is not known even in the class of soluble groups. Note that some known results about biautomatic groups remain open questions for automatic groups. Gersten and Short initiated in [3] the study of the subgroup structure of biautomatic groups. Among other results they established that a polycyclic subgroup of a biautomatic group is virtually abelian. Also they proved that if a linear group is biautomatic, then every soluble subgroup is (finitely generated) virtually abelian.

2. Preliminaries. Given an alphabet Σ, certain subsets of the free semigroup Σ^* on Σ are called *regular languages*. A recursive definition can be used to identify these. The empty subset and the singleton sets are considered to be regular languages. Given two regular languages A and B, their union $A \cup B$ and their concatenation $A \cdot B$ are deemed to be regular languages. Also, if A is a regular language, then so is the monoid generated by A: This is denoted by A^* and is called the *Kleene star* of A. Denote by $R = Rat(\Sigma^*)$ is the set of regular languages of Σ^*.

The construction of a set like R is still possible when Σ^* is changed by any monoid M. Let S be the set of all singleton subsets of M. Consider the closure $Rat(M)$ of S under the rational operations of union, product, and the formation of a submonoid of M. In other words, $Rat(M)$ is the smallest subset of M such that

- $\emptyset \in Rat(M)$,
- $A, B \in Rat(M)$ imply $A \cup B \in Rat(M)$,
- $A, B \in Rat(M)$ imply $AB \in Rat(M)$, where $AB = \{ab | a \in A, b \in B\}$,
- $A \in Rat(M)$ implies $A^* \in Rat(M)$, where A^* is the submonoid of M generated by A.

A *rational set* of M is an element of $Rat(M)$.

If Σ^* is a finite generated free monoid, then a rational set of Σ^* is also called *regular language*.

The rational sets of a monoid M are precisely the subsets accepted by finite automata over M. A *finite automaton* Γ over M is a finite directed graph with a distinguished initial vertex, some distinguished terminal vertices, and with edges labelled by elements from M. The set *accepted* by Γ is the collection of labels of paths from the initial vertex to a terminal vertex, where *label* $\mu(p)$ *of a path* p is the product of labels of sequential edges in p

Recall two auxiliary assertions.

LEMMA 2.1 (The Pumping Lemma (see [4])). *Let M be a monoid, $R \in Rat(M)$. Then, either R is finite, or it contains a set of the form $aq^*b = \{aq^n b | n \geq 0\}$, $a, q, b \in M$, $q \neq 1$. Moreover, if M is a group, then the subset aq^*b of M can be written in the form $abb^{-1}q^*b = ab(b^{-1}qb)^*$. So we might assume $b = 1$.*

LEMMA 2.2 ([2]). *Let G be a group, $H \leq G$ be a subgroup. If $R \subseteq H$ is rational subset of G, then R is a rational subset of H.*

As we have 2.2, we do not have to care if a set is rational as a subset of a larger group or of a smaller one. Note that with monoids that are not groups the situation may be different.

LEMMA 2.3 (see [**4**]). *Let G be a group and $H \leq G$. Then, $H \in Rat(G)$ if and only if H is finitely generated. If $R \in Rat(G)$, then, $gp(R) \in Rat(G)$, and so finitely generated.*

LEMMA 2.4 (see [**2**]). *Let G be a group, $T \lhd G$, and $\varphi : G \to G/T$ is the standard homomorphism. Then, for every $R \in Rat(G)$, we have $\varphi(R) \in Rat(G/T)$. If T is finitely generated, then for every $S \in Rat(G/T)$, we have $\varphi^{-1}(S) \in Rat(G)$.*

3. Polycyclic groups.

The goal of this section is to prove the following theorem.

THEOREM 3.1. *Let G be a polycyclic group. If $Rat(G)$ is a Boolean algebra, then G is virtually abelian.*

Proof. Let us first consider a special case that is the 'core' of the problem, in the sense concentrates all the difficult points of it. So, let a group G be a semidirect product $A \lambda H$, where $A \simeq \mathbb{Z}^r$ is a normal free abelian of rank r subgroup of G, $H = gp(h)$ is a cyclic subgroup of G. Suppose that $Rat(G)$ is a Boolean algebra. We are to prove that G is virtually abelian.

We may assume h to have infinite order. First prove that some nontrivial element $g \in A$ and some exponent $h^m, m > 0$, commute. Take an arbitrary nontrivial element $x \in A$. Consider the set

$$R = \{h^{-n}xh^n | n \in \mathbb{Z}\}.$$

If R is finite, then we have $h^{-n}xh^n = h^{-t}xh^t$ for some $n > t$, which implies $[x, h^{n-t}] = 1$; hence, we get what we need. Now assume that R is infinite. Note that R is rational, because it is equal to intersection $(h^{-1})^*xh^* \cap A$ of two rational subsets of G and $Rat(G)$ is closed under intersection by our assumption that it is Boolean algebra. Then, by Lemma 2.1, R contains a subset $P = aq^*, a, q \in A, q \neq 1$. Let I be the set of indices such that $P = \{h^{-i}xh^i | i \in I\}$. Then, $S = \{h^i | i \in I\} = h^* \cap (x^{-1}h^*P)$ is the infinite rational set. So it contains a subset $T = h^k(h^l)^*, k, l \in \mathbb{Z}, l > 0$. The set $Q = \{f^{-1}xf | f \in T\} = (T^{-1}xT) \cap A$ is rational and subset of P. We can assume that $k = 0$. In other case, we change R, P, a and q to $h^{-k}Rh^k, h^{-k}Ph^k, h^{-k}ah^k$ and $h^{-k}qh^k$, respectively. Now $T = (h^l)^*$.

Since A is isomorphic to the free abelian group of rank r, we may regard it as a lattice of $\mathbb{R}^r$. Let $|\cdot|$ be any standard norm on $\mathbb{R}^r$. Since now we will use additive notation for the operation on A as well as multiplicative one.

Take an arbitrary real $\varepsilon > 0$. Pick a positive integer m such that aq^m (or, in additive notation, $a + mq$) belongs to Q and the inequality

$$1/m|(h^{-l}ah^l) - a| < \varepsilon$$

holds. This is possible, because Q is infinite. Since $aq^m \in Q$, the element $h^{-l}(aq^m)h^l$ is in Q, so it can be written in the form $aq^p, p \in \mathbb{Z}, p \geq 0$. Then,

$$h^{-l}ah^l + m(h^{-l}qh^l) = a + pq,$$
$$m(h^{-l}qh^l) - pq = a - h^{-l}ah^l,$$
$$|h^{-l}qh^l - (p/m)q| = (1/m)|a - h^{-l}ah^l| < \varepsilon.$$

It follows that $h^{-l}qh^l$ is a limit of elements of the form $sq, s \in \mathbb{R}, s \geq 0$, so it has this form too. Clearly, s is nonzero and rational, because $sq \in \mathbb{Z}^r \setminus \{0\}$. Let u be the greatest positive integer with the property: q has the form $sq', q' \in A$. Then, $v = su$ is the greatest positive integer with the property: sq has the form $vq', q' \in A$. Since there exists an automorphism of A which takes q to sq, we get $u = sm$, so $s = 1$. Then, we have $h^{-l}qh^l = q$, so we get what we need.

Let us finish the proof. Let $f \in A, f \neq 1$. Let g, h^u $(g \neq 1, g \in A, u > 0)$ commute. If r, s are nonzero integers such that $g^r = f^s$, then $(h^{-u}fh^u)^{rs} = (h^{-u}f^sh^u)^r = g^{r^2} = f^{rs}$. As A is abelian and torsion-free, $h^{-u}fh^u = f$, so f and h^u commute.

Now suppose that $g^r = f^s$ cannot hold unless $r = s = 0$, or, equivalently, $g^r f^s = g^n f^l$ cannot hold unless $r = n, s = l$. Let $w = h^u$. Consider the set

$$R = (wg)^* f^* \cap w^*(gf)^* = \{w^n g^n f^n | n \geq 0\}.$$

Let

$$S = R(g^* \cup (g^{-1})^*) \cap w^* f^* =$$

$$\{w^n f^n | n \geq 0\}.$$

Take a subset $aq^* \subseteq S, q \neq 1$. Let $a = w^n f^n, aq = w^m f^m, n \neq m$. Then,

$$aq^2 = (aq)a^{-1}(aq) = w^{2m-n}(w^{n-m}f^{m-n}w^{m-n})f^m$$

also has the form $w^t f^t, t \geq 0$. Then, $t = 2m - n$. Hence,

$$f^{2m-n} = f^m(w^{n-m}f^{m-n}w^{m-n}),$$

so

$$f^{m-n} = w^{n-m}f^{m-n}w^{m-n},$$

then,

$$f = w^{n-m}fw^{m-n}.$$

Then, f and h^{m-n} commute.

We proved that for each $f \in A$ there exists $t \in \mathbb{Z}, t > 0$ such that f and h^t commute. Let $f_1, \ldots, f_r$ be free generators for A. Pick some nonzero $t_1, \ldots, t_r$ such that for each i the elements f_i and h^{t_i} commute. Let $N = t_1 \ldots t_r$. Then, h^N and every f_i commute. Hence, the group M generated by A and h^N is abelian. Clearly, M has finite index in G. Thus, G is virtually abelian.

Now we are ready to prove the theorem by using induction on solubility length. If G is abelian, the statement is trivial. Let G be nonabelian. The derived subgroup G' has smaller solubility length, and, as it is finitely generated, its rational subsets are a Boolean algebra. Then, G' is virtually abelian. Let $G = \mathrm{gp}(G', g_1, \ldots, g_j)$. Consider the series of subgroups $G' = G_0 \leq G_1 \leq G_j = G$, where $G_{i+1} = \mathrm{gp}(G_i, g_{i+1})$ for $i = 0, \ldots, j-1$. Clearly, all G_i are normal in G. Prove by induction on i that all G_i are virtually abelian. Suppose that H_i is a finite index normal abelian subgroup of G_i. We have proved above that $\mathrm{gp}(H_i, g_{i+1})$ is virtually abelian. Hence, H_{i+1} is virtually abelian.   □

4. Metabelian groups. Recall that a group G is said to have the *Howson property* (or to be a *Howson group*) if the intersection $H \cap K$ of any two finitely generated subgroups H, K of G is finitely generated subgroup. Let G be a group in which $Rat(G)$ is a Boolean algebra. Then, G has the Howson property. Indeed, a subgroup of arbitrary group is a rational set if and only if it is finitely generated. By our assumption the intersection $H \cap K$ is a rational set. Hence, $H \cap K$ is finitely generated subgroup.

All finitely generated metabelian nonpolycyclic Howson groups are characterized as follows.

THEOREM 4.1 (Kirkinskij [6]). *Let G be a finitely generated metabelian nonpolycyclic group. Then, the following properties are equivalent:*

(1) *G has the Howson property,*
(2) *the finitely generated nonpolycyclic subgroups of G have finite indexes,*
(3) *G has a subgroup H of finite index containing a normal finite subgroup T such that $H/T \simeq gp(x, a | [a, a^{x^i}] = 1, i \in \mathbb{Z}, a^{f(x)} = 1)$ with $f(x)$ being irreducible over $\mathbb{Z}$ polynomial with integral coefficients such that $deg f(x) \geq 1$ and for every $n \in \mathbb{N}$, this polynomial does not divide any polynomial in x^n of degree $deg f(x) - 1$. If $f(x) = q_0 x^m + q_1 x^{m-1} + \cdots + q_m$, then $a^{f(x)}$ means $(a^{x^m})^{q_0}(a^{x^{m-1}})^{q_1} \ldots a^{q_m}$.*

THEOREM 4.2. *Let G be a finitely generated metabelian group such that $Rat(G)$ is a Boolean algebra. Then, G is virtually abelian group.*

Proof. If G is polycyclic, the statement follows by Theorem 3.1. Suppose G is not polycyclic. Then, by Theorem 4.1, G has a series $1 \leq T \lhd H \leq G$. Since H is finitely generated, $Rat(H)$ is a Boolean algebra. Since T is finite, by Lemma 2.4 $Rat(H/T)$ is a Boolean algebra.

Let Theorem 4.1 $H/T \simeq gp(x, a | [a, a^{x^i}] = 1, i \in \mathbb{Z}, a^{f(x)} = 1)$, $f(x) = q_0 x^m + q_1 x^{m-1} + \cdots + q_m$. Note that every element $g \in H/T$ can be expressed as $g = x^k a^{\frac{r(x)}{x^l}}$, where $k, l \in \mathbb{Z}$, $l \geq 0$, and $r(x)$ is a polynomial with integer coefficients. One has $g = 1$ if and only if $k = 0$, and $r(x)$ divides into $f(x)$ in the polynomial ring $\mathbb{Z}[x]$.

Fix some numbers $p, d \in \mathbb{Z}$, $p, d > 0$. Define the following rational sets in H/T:

$$R_1 = ((a^d x^p)^{-1})^* (\{a^d, a^{d+1}\} x^p)^*,$$

$$R_2 = (x^{-p})^* (\{1, a\} x^p)^*,$$

$$R_3 = ((a^d x^{-p})^{-1})^* (\{a^d, a^{d+1}\} x^{-p})^*,$$

$$R_4 = (x^p)^* (\{1, a\} x^{-p})^*.$$

By our assumption, all intersections $R_i \cap R_j$ for $i, j = 1, \ldots, 4$, are rational. Any element of R_1 can be written in the form

$$(a^d x^p)^{-l+k} (a^{\epsilon_1})^{x^{kp}} (a^{\epsilon_2})^{x^{(k-1)p}} \ldots (a^{\epsilon_k})^{x^p},$$

where $l, k \in \mathbb{Z}$, $l, k \geq 0$, and $\epsilon_i = 0$ or $\epsilon_i = 1$.

Any element of R_2 can be written in the form

$$x^{(-l+k)p}(a^{\epsilon_1})^{x^{kp}}(a^{\epsilon_2})^{x^{(k-1)p}}\ldots(a^{\epsilon_k})^{x^p},$$

$l, k \in \mathbb{Z}$, $l, k \geq 0$, $\epsilon_i = 0$ or $\epsilon_i = 1$.

Any element of R_3 can be written in the form

$$(a^d x^{-p})^{-l+k}(a^{\epsilon_k})^{x^{-p}}(a^{\epsilon_{k-1}})^{x^{-2p}}\ldots(a^{\epsilon_1})^{x^{-kp}},$$

$l, k \in \mathbb{Z}$, $l, k \geq 0$, where $\epsilon_i = 0$ or $\epsilon_i = 1$.

Any element of R_4 can be written in the form

$$x^{(l-k)p}(a^{\epsilon_k})^{x^{-p}}(a^{\epsilon_{k-1}})^{x^{-2p}}\ldots(a^{\epsilon_1})^{x^{-kp}},$$

$l, k \in \mathbb{Z}$, $l, k \geq 0$, $\epsilon_i = 0$ or $\epsilon_i = 1$. Also note that for $n > 0$ we have

$$(a^d x^p)^n = x^{np}(a^d)^{x^{np}}\ldots(a^d)^{x^p},$$

$$(a^d x^p)^{-n} = x^{-np}(a^{-d})^{x^{-(n-1)p}}\ldots(a^{-d})^{x^{-p}}a^{-d},$$

$$(a^d x^{-p})^n = x^{-np}(a^d)^{x^{-np}}\ldots(a^d)^{x^{-p}},$$

and

$$(a^d x^{-p})^{-n} = x^{np}(a^{-d})^{x^{(n-1)p}}\ldots(a^{-d})^{x^p}a^{-d}.$$

The sets R_1 and R_2 contain the elements

$$a^{x^{kp}} = ((a^d x^p)^{-1})^k \cdot (a^{d+1}x^p)(a^d x^p)^{k-1} =$$

$$(x^{-p})^k \cdot (ax^p)(x^p)^{k-1}, k = 1, 2, \ldots.$$

Similarly, the sets R_3 and R_4 contain the elements $a^{x^{-kp}}$, $k = 1, 2, \ldots$.

Let $N = \mathrm{ncl}(a)$ be the normal closure of the element a in H/T (that is the minimal normal subgroup of N/T, containing a). If the sets $S_1 = R_1 \cap R_2$ and $S_2 = R_3 \cap R_4$ lie in N, then the subgroup M generated by $S_1 \cup S_2 \cup \{a\}$ is the normal closure of a in the subgroup generated by a and x^p. By Lemma 2.3 every subgroup generated by a rational set is rational and finitely generated. Since N is generated by a finite set of subgroups that are conjugate to M, it is finitely generated too. In the case, H/T and G are polycyclic. We get contradiction to our assumption. Hence, at least one of the subsets S_i, $i = 1, 2$, does not lie in N. Then, one of the following equalities is true:

$$(a^{\epsilon_n})^{x^{np}}\ldots(a^{\epsilon_{k+1}})^{x^{(k+1)p}}(a^{d+\epsilon_k})^{x^{kp}},$$

$$(a^{d+\epsilon_{k-1}})^{x^{(k-1)p}}\ldots(a^{d+\epsilon_1})^{x^p} = 1, \tag{1}$$

$$(a^{\epsilon_n})^{x^{np}}\ldots(a^{\epsilon_1})^{x^p}(a^{-d})(a^{-d})^{x^{-p}}\ldots(a^{-d})^{x^{-lp}} = 1, \tag{2}$$

$$(a^{d+\epsilon_1})^{x^{-p}}\ldots(a^{d+\epsilon_k})^{x^{-kp}}(a^{\epsilon_{k+1}})^{x^{-(k+1)p}}\ldots(a^{\epsilon_n})^{x^{-np}} = 1, \tag{3}$$

$$(a^{-d})^{x^{lp}}\ldots(a^{-d})^{x^p}(a^{-d})(a^{\epsilon_1})^{x^{-p}}(a^{\epsilon_k})^{x^{-kp}} = 1, \tag{4}$$

$l, k, n \in \mathbb{Z}, l \geq 0, n \geq k > 0, \epsilon_i \in \{-1, 0, 1\}$. If the absolute value μ of one of the coefficients q_0, q_m in $f(x)$ is greater than 3, we may assume that chosen number d is such that $d - 1, d$ and $d + 1$ do not divide to μ. It follows that all equalities (1)–(4) failed. Furthermore, both of the coefficients q_0, q_m cannot be ± 1, because in the case H/T is polycyclic. Thus, q_0 or q_m is equal to 2 or 3. We set $d = 2^2 \cdot 3^2 + 2 = 38$. Then, $d - 1, d$, and $d + 1$ do not divide to μ^2. We can assume that $p > m + 1$. Then, each of the equalities (1)–(4) implies that all the coefficients of $f(x)$ divide to μ, and $|q_0| = |q_m| = \mu$. Then, the normal closure $K = \text{ncl}(a^\mu) \lhd H/T$ is finitely generated. By Lemma 2.4, $Rat((H/T)/K)$ is a Boolean algebra. The quotient $(H/T)/K$ is a homomorphic image of the wreath product $Z_\mu wrZ$, where μ is prime. Hence, either $(H/T)/K$ or $Z_\mu wrZ$ is polycyclic. In the first case, $(H/T)/K$ satisfies the ascending chain condition, and since K is finitely generated abelian, H/T satisfies the ascending chain condition too. Every soluble group with the ascending chain condition is polycyclic (see for instance [**5**]). Hence, H/T is polycyclic. The second case is impossible, because $Z_\mu wrZ$ is not the Howson group (see [**6**]).

□

5. Solvable groups of type FP_∞ with Boolean algebras of rational subsets.
DEFINITION 5.1. A group G is said to be of type FP$_\infty$ if and only if there is a projective resolution

$$\ldots \to P_j \to \ldots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0 \tag{5}$$

of finite type: that is, in which every P_j is finitely generated.

DEFINITION 5.2. A group G has finite cohomological dimension if and only if there is a projective resolution

$$\cdots \to P_n \to \cdots \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0, \tag{6}$$

of finite length: that is, in which P_i are zero from some point on.

The following remarkable theorem is a base in our proof of the main result of this section.

THEOREM 5.3 Kropholler [**9**], see also [**8**]). *If G is a soluble group of type FP_∞ then $vcd(G) < \infty$.*

Also we need in a standard statement as follows.

LEMMA 5.4. *If $cd(G) < \infty$, then:*

(1) *G is torsion free,*
(2) *there is $n > 0$ such that: if $A \leq G, A \simeq \mathbb{Z}^k$, then $k \leq n$.*

At this section, we will specialize to soluble groups of type FP$_\infty$ and establish the following algorithm.

THEOREM 5.5. *If G is a finitely generated soluble group of type $(FP)_\infty$ such that $Rat(G)$ is a Boolean algebra. Then, G is virtually abelian group.*

Proof. By Theorem 5.3, there is a subgroup of finite index $H \leq G$ that has finite cohomological dimension. The subgroup H is finitely generated and thus rational in G. Then, $Rat(H)$ is a Boolean algebra. By Lemma 5.4, H is torsion free. Hence,

every finitely generated abelian subgroup of H is a free abelian of bounded rank. By Kargapolov's theorem (see [5] or [10]), H has finite rank, i.e., there is a finite number r such that every finitely generated subgroup of H can be generated by r elements and r is least such integer (*Prüfer* or *Mal'cev* rank).

Thus, H is a soluble torsion-free group of finite rank. By the Robinson-Zaǐcev theorem (see [10]), every finitely generated soluble group with finite rank is a minimax group. It means there is a subnormal series

$$1 = K_0 \le K_1 \le \cdots \le K_n = K, \tag{7}$$

in which every quotient K_{i+1}/K_i satisfies to the ascending or descending condition. Also we know (see [10]) that every soluble torsion-free minimax group is nilpotent-by-(virtually abelian).

Let N be a nilpotent normal subgroup of H such that H/N is virtually abelian. We will prove that N should be abelian. Let $1 \le \zeta_1(N) \le \zeta_2(N) \le \cdots \le \zeta_j(N) = N$ be the upper central series of N. If $u \in N$, $y \in \zeta_2(N) \setminus \zeta_1(N)$ do not commute, then $[u, y] \in \zeta_1(H)$ has an infinite order. We know that the quotient of torsion-free nilpotent group by the center is torsion-free (see [5]). Then, the subgroup $\mathrm{gp}(g, y)$ is isomorphic to the free nilpotent of rank 2 and class 2 group ($UT_3(\mathbb{Z})$ or the *Heizenbergh* group). But $UT_3(\mathbb{Z})$ is obviously non-virtually abelian; hence, $Rat(UT_3(\mathbb{Z}))$ is not Boolean algebra by [2]. It follows that $\zeta_2(N) = \zeta_1(N)$; thus N is abelian. Then, H is extension of the abelian normal subgroup N with virtually abelian group H/N. Then, by Theorem 4.2, G is virtually abelian.

<div style="text-align: right;">□</div>

6. Solvable biautomatic groups. The class of automatic groups is one of the main classes studied by geometric group theory. Different properties of automatic and biautomatic groups are described in the classical monography [1]. See also [3]. We give the main definitions.

Let (A, λ, L) be a rational structure for G. Recall that A is a finite alphabet, λ is a homomorphism of the free monoid A^* onto G, L is a regular language in A^* such that $\lambda(L) = G$. The set A is a generating set for G, considered as a monoid. We assume that A is symmetric: That is, A contains with each of its element a its formal inverse a^{-1}. We assume that homomorphisms of A^* to groups map formally inverse elements to inverse images. The set L is considered as the set of normal forms of expressions of elements of G. We add to A a new symbol \$. Consider alphabet $A^{2\$}$, consisting of the pairs (b, c), where $b, c \in A \cup \$$. Take the corresponding free monoid $(A^{2\$})^*$. The homomorphism λ is naturally extended to the homomorphism of the free monoid $(A^{2\$})^*$ to G^2. One has $\lambda(\$) = 1$.

A rational structure (A, λ, L) called *automatic* for G under the following conditions is satisfied:

$$\{(u, v) \in L^{2\$} : \lambda(u) = \lambda(v)\}, \tag{8}$$

and for every $a \in A$ the language

$$\{(u, v) \in L^{2\$} : \lambda(u) = \lambda(va)\} \tag{9}$$

is regular in $A^{2\$}$.

DEFINITION 6.1. A group G is said to be *automatic* if G has an automatic structure (A, λ, L).

An automatic structure is said to be *biautomatic*, if for every $a \in A$ the language

$$\{(u, v) \in L^{2\$} : \lambda(u) = \lambda(av)\} \tag{10}$$

is regular in $A^{2\$}$.

DEFINITION 6.2. A group G is said to be *biautomatic* if G has a biautomatic structure (A, λ, L).

In [1], a question is formulated: Is every biautomatic group virtually abelian? We give a positive answer to this question by the following theorem. This result has been obtained by Bazhenova, Noskov, Remeslennikov and the author using the information received from Kropholler.

THEOREM 6.3 (Bazhenova, Noskov, Remeslennikov, Roman'kov). *Let G be a finitely generated soluble biautomatic group. Then, G is virtually abelian.*

Proof. By [1], Theorem 10.2.6, every soluble biautomatic group has type FP_∞. Hence, by Theorem 5.3, G has a subgroup of finite index H with $cd(H) < \infty$. By Lemma 5.4, H is torsion free. Moreover, all abelian subgroups of H have bounded rank. Hence, by Kargapolov's theorem (see [5] or [10]), H has finite rank. By the Robinson–Zaĭcev theorem (see [10]), every finitely generated soluble group with a finite rank is a minimax group. Also we know (see [10]) that every soluble torsion-free minimax is nilpotent-by-(virtually abelian). Let N be a nilpotent normal subgroup of H such that H/N is virtually abelian.

We will prove that H is abelian. Let $1 \leq \zeta_1(N) \leq \zeta_2(N) \leq \cdots$ be the upper central series in N. Let $u \in N$, $y \in \zeta_2(N) \setminus \zeta_1(N)$ do not commute. Then, $[u, y] \in \zeta_1(H)$ is a nontrivial element of infinite order. Then, the subgroup $gp(g, y)$ is isomorphic to the free nilpotent of rank 2 and class 2 group $UT_3(\mathbb{Z})$. It cannot happen, since $UT_3(\mathbb{Z})$ is polycyclic but not virtually abelian. Indeed, by [3], every polycyclic subgroup of biautomatic group is virtually abelian. Thus, $\zeta_2(H) = \zeta_1(H)$, and $H = \zeta_1(H)$ is abelian.

The group H is finitely generated and virtually metabelian. Every subgroup of a finite index in a biautomatic group is biautomatic [3]. Thus, H is biautomatic. Then, H satisfies to the minimal condition for centralizers (see [11] or [10]). It means: if $S(H)$ be a class of subgroups of H of type $M = C_H(X) = \{h \in H : \forall x \in X \, [h, x] = 1\}$, where $X \subseteq H$, then descending sequence $M_1 \geq M_2 \geq \cdots \geq M_l \ldots$ of subgroups in $S(H)$ stabilizes on a finite step. Then, there is a number l such that $M_l = M_{l+1} = \cdots$. By [3], each biautomatic group with this property satisfies to the maximal condition on abelian subgroups. By Mal'cev's theorem (see [5]), a soluble group with this condition is polycyclic. Hence, H is polycyclic. Then, by [3] H is virtually abelian. Hence, G is virtually abelian. $\qquad\square$

REFERENCES

1. D. B. A. Epstein, J. W. Cannon, D. F. Holt, S. Levy, M. S. Patterson and W. Thurston, *Word processing in groups* (Jones and Bartlett, Boston, London, 1992).

2. G. Bazhenova, Rational sets in finitely generated nilpotent groups, *Algebra Log.* **39**(4) (2000), 379–394.

3. S. Gersten and H. Short, Rational subgroups of biautomatic groups, *Ann. Math.* **134** (1991), 125–158.

4. R. H. Gilman, Formal languages and infinite groups, in *Geometric and computational perspectives on infinite groups* (Minneapolis, MN and New Brunswick, NJ, 1994), DIMACS Ser. Discrete Math. Theor. Comp. Sci., vol. 25 (American Mathematical Society, Providence, RI, 1996), 27–51.

5. M. I. Kargapolov and Yu. I. Merzlyakov, *Foundations of the theory of groups* (Springer, New York, 1979).

6. A. S. Kirkinskij, Intersections of finitely generated subgroups in metabelian groups, *Algebra Log.* **20**(1) (1981), 24–36.

7. P. H. Kropholler, Soluble groups of type $(FP)_\infty$ have finite torsion-free rank, *Bull. London Math. Soc.* **25**(6) (1993), 558–566.

8. P. H. Kropholler, Hierarchical decompositions, generalized Tate cohomology, and groups of type $(FP)_\infty$, in *Combinatorial and geometric group theory, Edinburgh, 1993* (Duncan, A.J., Gilbert, N.D. and Howie, J., Editors), London Mathematical Society Lecture Note Series, vol. 204 (Cambridge University Press, Cambridge, UK, 1995), 190–216.

9. P. H. Kropholler, On groups of type $(FP)_\infty$, *J. Pure Appl. Algebra* **90**(1) (1993), 55–67.

10. J. C. Lennox and D. J. S. Robinson, *The theory of infinite soluble groups* (Clarendon Press, Oxford, 2004).

11. D. J. S. Robinson, *Finiteness conditions and generalized soluble groups* (Springer-Verlag, Berlin, 1972).