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EXISTENCE OF POSITIVE SOLUTIONS FOR BOUNDARY-VALUE PROBLEMS WITH SINGULARITIES IN PHASE VARIABLES

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Abstract The singular boundary-value problem $(g(x'))' = \mu f(t, x, x')$, $x'(0) = 0$, $x(T) = b > 0$ is considered. Here μ is the parameter and $f(t, x, y)$, which satisfies local Carathéodory conditions on $[0, T] \times (\mathbb{R} \setminus \{b\}) \times (\mathbb{R} \setminus \{0\})$, may be singular at the values $x = b$ and $y = 0$ of the phase variables x and y, respectively. Conditions guaranteeing the existence of a positive solution to the above problem for suitable positive values of μ are given. The proofs are based on regularization and sequential techniques and use the topological transversality theorem.

Keywords: singular boundary-value problem; mixed condition; positive solution; topological transversality theorem

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1. Introduction

Let b, T be positive numbers, $J = [0, T]$ and $\mathbb{R}_a = \mathbb{R} \setminus \{a\}$ for $a \in [0, \infty)$. Consider the boundary-value problem (BVP)

$$(g(x'(t)))' = \mu f(t, x(t), x'(t)), \tag{1.1}$$

$$x'(0) = 0, \qquad x(T) = b. \tag{1.2}$$

Here μ is the parameter, $g \in C^0(\mathbb{R})$, f satisfies local Carathéodory conditions on $J \times \mathbb{R}_b \times \mathbb{R}_0$ ($f \in \mathrm{Car}(J \times \mathbb{R}_b \times \mathbb{R}_0)$), and $f(t, x, y)$ may be singular at the values $x = b$ and $y = 0$ of the phase variable x and y, respectively.

Together with (1.2) we also discuss the boundary conditions

$$x'(0) = 0, \qquad x(T) = c, \quad 0 < c < b. \tag{1.3}$$

A function $x \in C^1(J)$ is said to be a *solution of the BVP* (1.1), (j), $j = 1.2, 1.3$, if $g(x') \in \mathrm{AC}(J)$ (absolutely continuous functions on J), x satisfies the boundary conditions (j) and (1.1) holds a.e. on J.

The aim of the paper is to give conditions guaranteeing the existence of positive solutions to BVPs (1.1), (j), $j = 1.2, 1.3$, for suitable positive values of the parameter μ in (1.1).

The study of BVPs (1.1), (1.2) and (1.1), (1.3) was motivated by the paper [**4**] where the results in [**5**] were extended and generalized. In [**4**] the authors discussed the existence of a non-negative solution to the singular (at the time variable t) mixed BVP

$$\left.\begin{aligned} \frac{1}{p(t)}(p(t)x'(t))' + \mu q(t) f_1(t, x(t), p(t)x'(t)) = 0, \\ \lim_{t \to 0^+} p(t)x'(t) = 0, \quad x(1) = b, \end{aligned}\right\} \tag{1.4}$$

with $f_1 \in C^0([0,1] \times \mathbb{R}^2)$ satisfying $f_1(t, x, r_0(t)) = 0$ for $t \in [0,1]$ and $x \in [0, \beta]$, where $r_0 \in C^0([0,1])$ is positive and non-decreasing and $\beta > 0$. Our Equation (1.1) is, on the one hand, a special case of (1.4) with $p = 1$, $q = 1$, but, on the other hand, it is a generalization of (1.4), since on the left-hand side of (1.1) we have $(g(x'))'$ and the nonlinearity $f(t, x, y)$ may be singular at the values $x = b$ and $y = 0$ of the phase variables x and y, respectively.

Note that positive solutions to BVP (1.1), (1.2) with $g(u) \equiv u$ have been considered in [**2**]. Here f may be singular at the value 0 in both its phase variables and f satisfies sign conditions. Conditions guaranteeing the existence of two positive solutions for the regular differential equation $(|x'(t)|^{p-2}x'(t))' + h(t)f_2(x'(t)) = 0$, $p > 1$, satisfying (1.2) are given in [**3**].

Throughout this paper the following assumptions are satisfied.

(H_1) $g \in C^0(\mathbb{R})$ is increasing on $[0, \infty)$, $g(0) = 0$ and $\lim_{u\to\infty} g(u) = \infty$.

(H_2) $f \in \mathrm{Car}(J \times \mathbb{R}_b \times \mathbb{R}_0)$,

$$f(t, x, \alpha(t)) = 0 \quad \text{for a.e. } t \in J \text{ and each } x \in [0, b),$$

where $\alpha \in C^0(J)$ is positive and non-decreasing on J and also there are positive numbers ε, ε_0 and $\nu \in (0, T]$ such that

$$\varepsilon \leqslant f(t, x, y) \quad \text{for a.e. } t \in [0, \nu] \text{ and each } (x, y) \in [0, b) \times (0, \varepsilon_0].$$

(H_3) For a.e. $t \in J$ and each $(x, y) \in [0, b) \times (0, \alpha(t)]$,

$$0 \leqslant f(t, x, y) \leqslant (q_1(x) + q_2(x))(\omega_1(y) + \omega_2(y)),$$

where $q_2 \in C^0([0, b))$, $\omega_1 \in C^0([0, \infty))$ are non-decreasing, $q_2 > 0$, $\omega_1 \geqslant 0$, $q_1 \in C^0([0, b])$, $\omega_2 \in C^0((0, \infty))$ are non-increasing, $q_1 \geqslant 0$, $\omega_2 > 0$, $\omega_1 + \omega_2$ is non-increasing on $(0, \delta]$ with $\delta \in (0, T]$, and

$$\int_0^1 \omega_2(g^{-1}(s))\,\mathrm{d}s < \infty.$$

Remark 1.1. Since $\omega_2(g^{-1}(t))$ is non-increasing positive on $(0,\infty)$ by (H_1) and (H_3), the condition $\int_0^1 \omega_2(g^{-1}(s))\,\mathrm{d}s < \infty$ in (H_3) implies $\int_0^c \omega_2(g^{-1}(s))\,\mathrm{d}s < \infty$ for each $c \in (0,\infty)$.

We denote by $\|x\| = \max\{|x(t)| : t \in J\}$ and $\|x\|_{L_1} = \int_0^T |x(t)|\,\mathrm{d}t$ the norm in the Banach spaces $C^0(J)$ and $L_1(J)$, respectively.

To prove existence results for BVPs (1.1), (1.2) and (1.1), (1.3) we use regularization and sequential techniques. First, we define a family of auxiliary regular differential equations depending on $n \in \mathbb{N}$ and using the topological transversality theorem (see, for example, [**1**]) given in Lemma 2.1 we obtain a general existence principle (Lemma 2.2). Using this principle we show that the sequence of auxiliary BVPs has a sequence $\{x_n\}$ of positive solutions (Lemmas 2.3 and 2.4). Applying the Arzelà–Ascoli Theorem we can select a convergent subsequence of $\{x_n\}$ and then the Lebesgue-dominated convergence theorem shows that its limit is a solution of our BVP (Theorems 3.1 and 3.2). Example 3.3 demonstrates the application of our existence results.

2. Auxiliary BVPs

Let

$$\mathbb{N}_0 = \left\{ n : n \in \mathbb{N},\ n > \max\left\{ \frac{2}{b}, \frac{1}{\alpha(0)}, \frac{1}{\varepsilon_0}, \frac{1}{\delta} \right\} \right\},$$

where the positive numbers b, ε_0, δ and the function α are given by (H_2) and (H_3). For $n \in \mathbb{N}_0$, define $p_n \in C^0(\mathbb{R})$, $\tilde{f}_n \in \mathrm{Car}(J \times \mathbb{R}_b \times \mathbb{R})$ and $f_n \in \mathrm{Car}(J \times \mathbb{R}^2)$ by

$$p_n(x) = \begin{cases} 0 & \text{if } x > b, \\ n(b-x) & \text{if } b - (1/n) < x \leqslant b, \\ 1 & \text{if } (1/n) < x \leqslant b - (1/n), \\ nx & \text{if } 0 < x \leqslant (1/n) \\ 0 & \text{if } x \leqslant 0, \end{cases}$$

$$\tilde{f}_n(t,x,y) = \begin{cases} f(t,x,\alpha(t)) & \text{for } (t,x,y) \in J \times \mathbb{R}_b \times [\alpha(t),\infty), \\ f(t,x,y) & \text{for } (t,x,y) \in J \times \mathbb{R}_b \times [(1/n),\alpha(t)), \\ f(t,x,(1/n)) & \text{for } (t,x,y) \in J \times \mathbb{R}_b \times (-\infty,(1/n)), \end{cases}$$

$$f_n(t,x,y) = \begin{cases} p_n(x)\tilde{f}_n(t,x,y) & \text{for } (t,x,y) \in J \times \mathbb{R}_b \times \mathbb{R}, \\ 0 & \text{for } (t,y) \in J \times \mathbb{R},\ x = b. \end{cases}$$

Then (H_2) and (H_3) give

$$0 \leqslant f_n(t,x,y) \leqslant (q_1(x) + q_2(x))(\omega_1(y) + \omega_2(y)) \tag{2.1}$$

for a.e. $t \in J$ and each $(x,y) \in [0,b) \times (0,\alpha(t)]$.

Finally, define $g_* \in C^0(\mathbb{R})$ by

$$g_*(u) = \begin{cases} g(u) & \text{for } u \in [0,\infty), \\ -g(-u) & \text{for } u \in (-\infty,0). \end{cases} \tag{2.2}$$

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Consider the family of regular differential equations

$$(g_*(x'(t)))' = \lambda\mu f_n(t, x(t), x'(t)) \tag{R$_n^\lambda$}$$

depending on the parameters $n \in \mathbb{N}_0$ and $\lambda \in [0,1]$ together with the auxiliary boundary conditions

$$x'(0) = 0, \qquad x(T) = b - (1/n). \tag{B$_n$}$$

For the solvability of BVPs $(\mathrm{R})_n^1$, $(\mathrm{B})_n$ and $(\mathrm{R})_n^1$, (1.3) we use the existence principle, whose proof is based on a well-known result in the literature (Lemma 2.1).

Lemma 2.1 (see [1]). *Let Ω be a relatively open set of a convex set Φ in a Banach space E and $p \in \Omega$. If*

(a) *$\mathcal{K} : [0,1] \times \bar{\Omega} \to \Phi$ is a compact operator,*

(b) *$\mathcal{K}(0, x) = p$ for $x \in \bar{\Omega}$, and*

(c) *$\mathcal{K}(\lambda, x) \neq x$ for $\lambda \in (0,1)$ and $x \in \partial\Omega$,*

then $\mathcal{K}(1, \cdot)$ has a fixed point in $\bar{\Omega}$.

Lemma 2.2. *Let $F \in \mathrm{Car}(J \times \mathbb{R}^2)$, g_* be given by (2.2), $d \in \mathbb{R}$ and let there exist positive constants P_0, P_1 independent of λ, $P_0 > |d|$, such that*

$$\|x\| \neq P_0, \qquad \|x'\| \neq P_1$$

for any solution x to BVP

$$(g_*(x'(t)))' = \lambda F(t, x(t), x'(t)), \tag{2.3$^\lambda$}$$

$$x'(0) = 0, \qquad x(T) = d \tag{2.4}$$

with $\lambda \in (0,1)$. Then there exists a solution of BVP $(2.3)^1$, (2.4).

Proof. Set

$$\Omega = \{x : x \in C^1(J),\ \|x\| < P_0,\ \|x'\| < P_1\}$$

and let $\mathcal{K} : [0,1] \times \bar{\Omega} \to C^1(J)$,

$$\mathcal{K}(\lambda, x)(t) = d - \int_t^T g_*^{-1}\biggl(\lambda \int_0^s F(v, x(v), x'(v))\,\mathrm{d}v\biggr)\,\mathrm{d}s.$$

Then x is a fixed point of the operator $\mathcal{K}(\lambda, \cdot)$ if and only if x is a solution of BVP $(2.3)^\lambda$, (2.4). To prove the solvability of BVP $(2.3)^1$, (2.4) we use Lemma 2.1 (with $E = \Phi = C^1(J)$ and $p = d$).

First we see that $\mathcal{K}(0, x) = d$ for $x \in \bar{\Omega}$. If $\mathcal{K}(\lambda_0, x_0) = x_0$ for some $(\lambda_0, x_0) \in (0,1) \times \partial\Omega$, then x_0 is a solution of BVP $(2.3)^{\lambda_0}$, (2.4) and $\|x_0\| \neq P_0$, $\|x_0'\| \neq P_1$ by our assumption, that is $x_0 \notin \partial\Omega$, contrary to $x_0 \in \partial\Omega$.

It remains to show that $\mathcal{K}$ is a compact operator. For this let $\{(\lambda_n, x_n)\} \subset [0,1] \times \bar{\Omega}$ be a convergent sequence, $\lim_{n\to\infty} \lambda_n = \lambda_0$, $\lim_{n\to\infty} x_n = x_0$. Then there is a positive constant L such that $\|x_n\| \leqslant L$, $\|x_n'\| \leqslant L$ for $n \in \mathbb{N}$, and since $F \in \mathrm{Car}(J \times \mathbb{R}^2)$,

$$|F(t, x_n(t), x_n'(t))| \leqslant \gamma(t) \quad \text{for a.e. } t \in J \text{ and each } n \in \mathbb{N}, \tag{2.5}$$

where $\gamma \in L_1(J)$. Hence

$$\lim_{n\to\infty} \lambda_n \int_0^t F(s, x_n(s), x_n'(s))\,\mathrm{d}s = \lambda_0 \int_0^t F(s, x(s), x'(s))\,\mathrm{d}s$$

uniformly on J by the Lebesgue-dominated convergence theorem, and so the sequence

$$\{g_*((\mathcal{K}(\lambda_n, x_n))'(t))\} = \left\{\lambda_n \int_0^t F(s, x_n(s), x_n'(s))\,\mathrm{d}s\right\}$$

is uniformly convergent on J to $\lambda_0 \int_0^t F(s, x_0(s), x_0'(s))\,\mathrm{d}s$. Now from the equalities (for $n \in \mathbb{N}$ and $t \in J$)

$$|(\mathcal{K}(\lambda_n, x_n))'(t) - (\mathcal{K}(\lambda_0, x_0))'(t)| = |g_*^{-1}[g_*((\mathcal{K}(\lambda_n, x_n))'(t))] - g_*^{-1}[g_*((\mathcal{K}(\lambda_0, x_0))'(t))]|$$

and g_* being continuous increasing on $\mathbb{R}$, we deduce that

$$\lim_{n\to\infty} (\mathcal{K}(\lambda_n, x_n))'(t) = (\mathcal{K}(\lambda_0, x_0))'(t) \quad \text{uniformly on } J.$$

Thus $\lim_{n\to\infty} \mathcal{K}(\lambda_n, x_n) = \mathcal{K}(\lambda_0, x_0)$ in $C^1(J)$ since $\mathcal{K}(\lambda_n, x_n)(T) = \mathcal{K}(\lambda_0, x_0)(T) = d$ for $n \in \mathbb{N}$. We have proved that $\mathcal{K}$ is a continuous operator. To prove the relative compactness of $\mathcal{K}([0,1] \times \bar{\Omega})$ in $C^1(J)$, let $\{(\lambda_m, x_m)\} \subset [0,1] \times \bar{\Omega}$. Then (2.5) (with m instead of n) is satisfied with a $\gamma \in L_1(J)$ and we can verify that

$$|g_*((\mathcal{K}(\lambda_m, x_m))'(t))| \leqslant \|\gamma\|_{L_1}, \qquad |\mathcal{K}(\lambda_m, x_m)(t)| \leqslant |d| + T g_*^{-1}(\|\gamma\|_{L_1})$$

and

$$|g_*((\mathcal{K}(\lambda_m, x_m))'(t_2)) - g_*((\mathcal{K}(\lambda_m, x_m))'(t_1))| \leqslant \left|\int_{t_1}^{t_2} \gamma(t)\,\mathrm{d}t\right| \tag{2.6}$$

for $t, t_1, t_2 \in J$ and $m \in \mathbb{N}$. Therefore, $\{\mathcal{K}(\lambda_m, x_m)\}$ is bounded in $C^1(J)$ and from (2.6) and g_* being increasing on $\mathbb{R}$ it follows that $\{(\mathcal{K}(\lambda_m, x_m))'(t)\}$ is equicontinuous on J. Consequently, there is a subsequence of $\{\mathcal{K}(\lambda_m, x_m)\}$ converging in $C^1(J)$ by the Arzelà–Ascoli Theorem. This completes the proof. □

Let

$$Q(u) = \int_0^u (q_1(s) + q_2(s))\,\mathrm{d}s, \qquad u \in [0, b),$$

and

$$H(u) = \int_0^{g(u)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\,\mathrm{d}s, \quad u \in [0, \infty).$$

Then, by (H_3), $Q \in C^0([0,b))$ and $H \in C^0([0,\infty))$ are increasing. In addition, if

$$\int_0^b q_2(t)\,\mathrm{d}t < \infty, \tag{2.7}$$

then $Q \in C^0([0,b])$.

Lemma 2.3. *Let (2.7) be satisfied and let there exist $\mu_0 > 0$ such that*

$$\int_0^b \frac{1}{H^{-1}(\mu_0 Q(s))}\,\mathrm{d}s < \infty. \tag{2.8}$$

Then the function

$$r(\mu) = \int_0^b \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \tag{2.9}$$

is continuous and decreasing on $(0,\mu_0]$.

Proof. We first see that (2.8) shows that the interval $[0,\mu_0 Q(b))$ belongs to the domain of H^{-1}. By (H_3), there are positive constants k_1, k_2, $k_1 < k_2$, and $b_0 \in (0,b)$ such that $k_1 \leqslant q_1(u) + q_2(u) \leqslant k_2$ for $u \in [0,b_0]$, and so

$$k_1 u \leqslant Q(u) \leqslant k_2 u \quad \text{for } u \in [0,b_0]. \tag{2.10}$$

Then $H^{-1}(\mu_0 Q(u)) \leqslant H^{-1}(\mu_0 k_2 u)$ for $u \in [0,b_1]$, where $0 < b_1 < \min\{b_0, Q(b)/k_2\}$ and (see (2.8))

$$\int_0^{b_1} \frac{1}{H^{-1}(\mu_0 k_2 s)}\,\mathrm{d}s \leqslant \int_0^{b_1} \frac{1}{H^{-1}(\mu_0 Q(s))}\,\mathrm{d}s < \int_0^b \frac{1}{H^{-1}(\mu_0 Q(s))}\,\mathrm{d}s < \infty,$$

which shows that

$$\int_0^{\mu_0 k_2 b_1} \frac{1}{H^{-1}(s)}\,\mathrm{d}s < \infty. \tag{2.11}$$

Let $\mu \in (0,\mu_0)$. Now from the inequalities (see (2.10))

$$\begin{aligned} H^{-1}(\mu Q(u)) &\geqslant H^{-1}(\mu k_1 u) && \text{for } u \in [0,b_0], \\ H^{-1}(\mu Q(u)) &\geqslant H^{-1}(\mu Q(b_0)) && \text{for } u \in [b_0,b], \end{aligned}$$

we have

$$\int_0^b \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \leqslant \int_0^{b_0} \frac{1}{H^{-1}(\mu k_1 s)}\,\mathrm{d}s + \frac{b-b_0}{H^{-1}(\mu Q(b_0))},$$

and then using (2.11) we deduce that the function r given by (2.9) is defined on $(0,\mu_0]$. Since the function $v(u,\mu) = H^{-1}(\mu Q(u))$ is continuous on $[0,b] \times [0,\mu_0]$ and $v(u,\cdot)$ is increasing on $[0,\mu_0]$ for each $u \in (0,b]$, it follows that r is continuous and decreasing on $(0,\mu_0]$. □

Lemma 2.4. *Let (2.7) be satisfied and let μ_* be a positive number such that*

$$\int_0^b \frac{1}{H^{-1}(\mu_* Q(s))}\,\mathrm{d}s = T. \tag{2.12}$$

Then for each $\mu \in (0, \mu_*)$ there exists $n_\mu \in \mathbb{N}$ such that BVP $(\mathrm{R})_n^1$, $(\mathrm{B})_n$ with $n \in \mathbb{N}_\mu = \{n : n \in \mathbb{N}_0,\ n \geqslant n_\mu\}$ has a solution x_n and

$$\frac{1}{n_\mu} \leqslant x_n(t) \leqslant b - \frac{1}{n}, \quad 0 \leqslant x_n'(t) \leqslant \alpha(t) \quad \textit{for } t \in J. \tag{2.13}$$

Proof. Fix $\mu \in (0, \mu_*)$. By Lemma 2.3, there exist $n_\mu \in \mathbb{N}$ such that

$$\int_{1/n_\mu}^{b-1/n_\mu} \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \geqslant T. \tag{2.14}$$

Fix $n \in \mathbb{N}_\mu$. Consider the family of BVPs $(\mathrm{R})_n^\lambda$, $(\mathrm{B})_n$ for $0 < \lambda < 1$. First, we show that any solution x to $(\mathrm{R})_n^\lambda$, $(\mathrm{B})_n$ satisfies

$$x(t) \geqslant 0 \quad \text{for } t \in J. \tag{2.15}$$

To see this suppose that $\min\{x(t) : t \in J\} = x(t_0) < 0$. Then $t_0 \in [0, T)$, $x(t_1) = 0$ for a $t_1 \in (t_0, T)$ and $x < 0$ on $[t_0, t_1)$ since $x(T) = b - 1/n > 0$ and also $x'(t_0) = 0$, which is clear for $t_0 \in (0, T)$ and it follows from $(\mathrm{B})_n$ for $t_0 = 0$. Moreover,

$$(g_*(x'(t)))' = \lambda\mu f_n(t, x(t), x'(t)) = 0 \quad \text{for a.e. } t \in [t_0, t_1],$$

and so $x(t) = x(t_0) < 0$ for $t \in [t_0, t_1)$, which yields $x(t_1) = x(t_0) < 0$, contrary to $x(t_1) = 0$. Now, from $(g_*(x'(t)))' = \lambda\mu f_n(t, x(t), x'(t)) \geqslant 0$ for a.e. $t \in J$ we have $x' \geqslant 0$ on J, and consequently (2.15) is true and $x(t) \leqslant x(T) = b - (1/n)$ for $t \in J$. Next we will show that

$$(0 \leqslant) x'(t) \leqslant \alpha(t) \quad \text{for } t \in J. \tag{2.16}$$

If not, $x'(t_1) > \alpha(t_1)$ for some $t_1 \in (0, T)$ and then from $x'(0) = 0$ and $\alpha(0) > 0$ we deduce that $x'(t_*) = \alpha(t_*)$ and $x' > \alpha$ on $(t_*, t_1]$ with a $t_* \in (0, t_1)$. Therefore,

$$(g_*(x'(t)))' = \lambda\mu f_n(t, x(t), \alpha(t)) = 0 \quad \text{for a.e. } t \in [t_*, t_1],$$

hence $g_*(x'(t)) = g(\alpha(t_*))$ for $t \in [t_*, t_1]$, and so $\alpha(t_*) = x'(t_*) = x'(t_1) > \alpha(t_1)$, which contradicts α being non-decreasing on J by (H_2). Now $(\mathrm{B})_n$, (2.15), (2.16) together with Lemma 2.2 (with $F(t, x, y) = \mu f_n(t, x, y)$, $P_0 = b$ and $P_1 = \|\alpha\| + 1$) imply that $(\mathrm{R})_n^1$, $(\mathrm{B})_n$ has a solution x_n. In addition (argue as above)

$$0 \leqslant x_n(t) \leqslant b - (1/n), \quad 0 \leqslant x_n'(t) \leqslant \alpha(t) \quad \text{for } t \in J.$$

To prove (2.13) it remains to verify that $x_n \geqslant 1/n_\mu$ on J. By (2.1),

$$(g_*(x_n'(t)))' = (g(x_n'(t)))' \leqslant \mu[q_1(x_n(t)) + q_2(x_n(t))][\omega(x_n'(t)) + \omega_2(x_n'(t))]$$

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for a.e. $t \in J$. Then, integrating the inequality

$$\frac{(g(x'_n(t)))' x'_n(t)}{\omega_1(x'_n(t)) + \omega_2(x'_n(t))} \leqslant \mu[q_1(x_n(t)) + q_2(x_n(t))]x'_n(t)$$

from 0 to $t \in (0, T]$, we get

$$\begin{aligned}\int_0^{g(x'_n(t))} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s)) + \omega_2(g^{-1}(s))}\,\mathrm{d}s &\leqslant \mu \int_{x_n(0)}^{x_n(t)} (q_1(s) + q_2(s))\,\mathrm{d}s \\ &\leqslant \mu \int_0^{x_n(t)} (q_1(s) + q_2(s))\,\mathrm{d}s \qquad (2.17)\end{aligned}$$

for $t \in J$. Hence $H(x'_n(t)) \leqslant \mu Q(x_n(t))$ for $t \in J$ and integrating from $\bar{t} = \min\{t : t \in J,\ x_n(t) > 0\}$ to T the inequality

$$\frac{x'_n(t)}{H^{-1}(\mu Q(x_n(t)))} \leqslant 1 \quad \text{for } t \in (\bar{t}, T], \qquad (2.18)$$

we have

$$\int_{x_n(\bar{t})}^{b-1/n} \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \leqslant T.$$

Since $1/n \leqslant 1/n_\mu$, (2.14) shows that $x_n(\bar{t}) \geqslant 1/n_\mu$, so $\bar{t} = 0$ and $x_n(0) \geqslant 1/n_\mu$. We have proved the validity of (2.13). □

Lemma 2.5. *Let $0 < c < b$ and let μ_0 be a positive number such that*

$$\int_0^c \frac{1}{H^{-1}(\mu_0 Q(s))}\,\mathrm{d}s = T. \qquad (2.19)$$

Then for each $\mu \in (0, \mu_0)$ there exists $n_\mu \in \mathbb{N}$ such that BVP $(\mathrm{R})^1_n$, (1.3) with $n \in \mathbb{N}_\mu = \{n : n \in \mathbb{N}_0,\ n \geqslant n_\mu\}$ has a solution x_n and

$$\frac{1}{n_\mu} \leqslant x_n(t) \leqslant c \leqslant b - \frac{1}{n}, \quad 0 \leqslant x'_n(t) \leqslant \alpha(t) \quad \text{for } t \in J. \qquad (2.20)$$

Proof. Arguing as in the proof of Lemma 2.3 we can prove that the function

$$r_*(\mu) = \int_0^c [1/H^{-1}(\mu Q(s))]\,\mathrm{d}s$$

is continuous decreasing on $(0, \mu_0]$. Fix $\mu \in (0, \mu_0)$. Then there exists $n_\mu \in \mathbb{N}$, $n_\mu \geqslant 1/(b - c)$ such that

$$\int_{1/n_\mu}^c \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \geqslant T. \qquad (2.21)$$

Let $n \in \mathbb{N}_\mu$. Consider the family of BVPs $(\mathrm{R})^\lambda_n$, (1.3) for $0 < \lambda < 1$. An analysis similar to that in the proof of Lemma 2.4 shows that any solution x to $(\mathrm{R})^\lambda_n$, (1.3) satisfies the inequality

$$0 \leqslant x(t) \leqslant c, \quad 0 \leqslant x'(t) \leqslant \alpha(t), \quad t \in J,$$

and so Lemma 2.2 (with $P_0 = c+1$ and $P_1 = \|\alpha\| + 1$) guarantees that there exists a solution x_n of BVP $(\mathrm{R})^1_n$, (1.3) for $n \in \mathbb{N}_\mu$. Then

$$0 \leqslant x_n(t) \leqslant c \leqslant b - (1/n), \quad 0 \leqslant x_n'(t) \leqslant \alpha(t), \quad t \in J,$$

and integrating (2.18) from $\bar{t} = \min\{t : t \in J,\ x_n(t) > 0\}$ to T, we get

$$\int_{x_n(\bar{t})}^{c} \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s \leqslant T.$$

Now (2.21) shows that $x_n(\bar{t}) \geqslant 1/n_\mu$, so $\bar{t} = 0$ and $x_n(0) \geqslant 1/n_\mu$, which completes the proof of (2.20). □

3. Existence results and an example

Theorem 3.1. *Let assumptions (H_1)–(H_3), (2.7) be satisfied and let (2.12) be true with a positive number μ_*. Then BVP (1.1), (1.2) has a solution x for each $\mu \in (0, \mu_*)$ and*

$$0 < x(t) \leqslant b, \quad 0 \leqslant x'(t) \leqslant \alpha(t) \quad \text{for } t \in J. \tag{3.1}$$

Proof. Fix $\mu \in (0, \mu_*)$. By Lemma 2.4, there exists $n_\mu \in \mathbb{N}$ such that BVP $(\mathrm{R})^1_n$, $(\mathrm{B})_n$ has a solution x_n satisfying (2.13) for each $n \in \mathbb{N}_\mu$ with $\mathbb{N}_\mu$ given in Lemma 2.4. Then (see (2.1) and (2.13))

$$(g(x_n'(t)))' \leqslant \mu[q_1(x_n(t)) + q_2(x_n(t))][\omega_1(x_n'(t)) + \omega_2(x_n'(t))] \tag{3.2}$$

for a.e $t \in J$ and each $n \in \mathbb{N}_\mu$. Arguing as in the proof of Lemma 2.4 we can see that (see (2.17))

$$H(x_n'(t)) \leqslant \mu[Q(x_n(t)) - Q(x_n(0))] \quad \text{for } t \in J,\ n \in \mathbb{N}_\mu. \tag{3.3}$$

Now let ε, ε_0 and ν be given by (H_2). If there exists a subsequence $\{x_{k_n}\}$ of $\{x_n\}_{n\in\mathbb{N}_\mu}$ such that $x_{k_n}'(\tau_n) = \varepsilon_0$ with $\lim_{n\to\infty} \tau_n = 0$, then letting $n \to \infty$ in (see (3.3))

$$H(x_{k_n}'(\tau_n)) \leqslant \mu[Q(x_{k_n}(\tau_n)) - Q(x_{k_n}(0))],$$

we get $H(\varepsilon_0) \leqslant 0$ since $0 \leqslant x_{k_n}(\tau_n) - x_{k_n}(0) \leqslant \|\alpha\|\tau_n$, contrary to $H(\varepsilon_0) > 0$. Consequently, there is a $\nu_0 \in (0, \nu]$ such that $x_n'(t) \leqslant \varepsilon_0$ for $t \in [0, \nu_0]$ and $n \in \mathbb{N}_\mu$, and then (H_2) and (2.13) imply that $f_n(t, x_n(t), x_n'(t)) \geqslant \varepsilon$ for a.e. $t \in [0, \nu_0]$ and each $n \in \mathbb{N}_\mu$. Hence $g(x_n'(t)) \geqslant \varepsilon t$ for $t \in [0, \nu_0]$ and since x_n' is non-decreasing on J, we have

$$x_n'(t) \geqslant \chi(t) \quad \text{for } t \in J \text{ and } n \in \mathbb{N}_\mu, \tag{3.4}$$

with

$$\chi(t) = \begin{cases} g^{-1}(\varepsilon t) & \text{for } t \in [0, \nu_0], \\ g^{-1}(\varepsilon \nu_0) & \text{for } t \in (\nu_0, T]. \end{cases}$$

Then

$$\omega_2(x_n'(t)) \leqslant \omega_2(\chi(t)) \quad \text{for } t \in (0, T],\ n \in \mathbb{N}_\mu. \tag{3.5}$$

Using the assumption (see (H_3) and Remark 1.1) $\int_0^1 \omega_2(g^{-1}(s))\,\mathrm{d}s < \infty$ we see that $\omega_2(\chi(t)) \in L_1(J)$. In addition, from $(\mathrm{B})_n$ and (3.4) we have

$$x_n(t) = x_n(b) - \int_t^b x_n'(s)\,\mathrm{d}s < b - \int_t^b \chi(s)\,\mathrm{d}s,$$

and so

$$x_n(t) < \eta(t) \quad \text{for } t \in J,\ n \in \mathbb{N}_\mu,$$

where

$$\eta(t) = \begin{cases} b - g^{-1}(\varepsilon\nu_0)(b-\nu_0) & \text{for } t \in [0,\nu_0], \\ b - g^{-1}(\varepsilon\nu_0)(b-t) & \text{for } t \in (\nu_0,T]. \end{cases}$$

Hence

$$q_2(x_n(t)) \leqslant q_2(\eta(t)) \quad \text{for } t \in J,\ n \in \mathbb{N}_\mu \tag{3.6}$$

and $q_2(\eta(t)) \in L_1(J)$, which follows from assumption (2.7) and the inequalities

$$\begin{aligned} \int_0^b q_2(\eta(t))\,\mathrm{d}t &= \nu_0 q_2(b - g^{-1}(\varepsilon\nu_0)(b-\nu_0)) + \int_{\nu_0}^b q_2(b - g^{-1}(\varepsilon\nu_0)(b-t))\,\mathrm{d}t \\ &= \nu_0 q_2(b - g^{-1}(\varepsilon\nu_0)(b-\nu_0)) + \frac{1}{g^{-1}(\varepsilon\nu_0)} \int_{b-g^{-1}(\varepsilon\nu_0)(b-\nu_0)}^b q_2(t)\,\mathrm{d}t \\ &\leqslant \nu_0 q_2(b - g^{-1}(\varepsilon\nu_0)(b-\nu_0)) + \frac{1}{g^{-1}(\varepsilon\nu_0)} \int_0^b q_2(t)\,\mathrm{d}t. \end{aligned}$$

Now, from (2.13), (3.2), (3.5) and (3.6) we have

$$(0 \leqslant)\ (g(x_n'(t)))' \leqslant \mu[q_1(0) + q_2(\eta(t))][\omega_1(\alpha(t)) + \omega_2(\chi(t))]$$

for a.e. $t \in J$ and each $n \in \mathbb{N}_\mu$. Hence

$$0 \leqslant g(x_n'(t_2)) - g(x_n'(t_1)) \leqslant \mu \int_{t_1}^{t_2} [q_1(0) + q_2(\eta(t))][\omega_1(\alpha(t)) + \omega_2(\chi(t))]\,\mathrm{d}t$$

for $0 \leqslant t_1 \leqslant t_2 \leqslant T$ and $n \in \mathbb{N}_\mu$, and since $[q_1(0)+q_2(\eta(t))][\omega_1(\alpha(t))+\omega_2(\chi(t))] \in L_1(J)$, $\{g(x_n'(t))\}_{n\in\mathbb{N}_\mu}$ is equicontinuous on J and then $\{x_n'(t)\}_{n\in\mathbb{N}_\mu}$ is equicontinuous on J too since g is continuous increasing on $[0,\infty)$ by (H_1). Applying the Arzelà–Ascoli Theorem, going if necessary to a subsequence, we can assume that $\{x_n'\}_{n\in\mathbb{N}_\mu}$ is convergent in $C^1(J)$ and let $\lim_{n\to\infty} x_n = x$. Then $x \in C^1(J)$, $1/n_\mu \leqslant x(t) \leqslant \eta(t)$, $\chi(t) \leqslant x'(t) \leqslant \alpha(t)$ for $t \in J$, $x'(0) = 0$ and $x(T) = b$. As

$$\lim_{n\to\infty} f_n(t, x_n(t), x_n'(t)) = f(t, x(t), x'(t)) \quad \text{for a.e. } t \in J$$

and

$$0 \leqslant f_n(t, x_n(t), x_n'(t)) \leqslant \mu[q_1(0) + q_2(\eta(t))][\omega_1(\alpha(t)) + \omega_2(\chi(t))]\ (\in L_1(J))$$

for a.e. $t \in J$ and each $n \in \mathbb{N}_\mu$, taking the limit as $n \to \infty$ in

$$g(x_n'(t)) = \mu \int_0^t f_n(s, x_n(s), x_n'(s))\,\mathrm{d}s, \quad t \in J,\ n \in \mathbb{N}_\mu,$$

we get

$$g(x'(t)) = \mu \int_0^t f(s, x(s), x'(s))\,\mathrm{d}s, \qquad t \in J$$

by the Lebesgue-dominated convergence theorem. Therefore, $g(x') \in \mathrm{AC}(J)$ and x is a solution of BVP (1.1), (1.2). □

Theorem 3.2. *Let assumptions (H_1)–(H_3) be satisfied, $0 < c < b$ and let (2.19) be true with a positive number μ_0. Then BVP (1.1), (1.3) has a solution x for each $\mu \in (0, \mu_0)$ and*

$$0 < x(t) \leqslant c, \quad 0 \leqslant x'(t) \leqslant \alpha(t) \quad \text{for } t \in J. \tag{3.7}$$

Proof. Fix $\mu \in (0, \mu_0)$. By Lemma 2.5, there exists $n_\mu \in \mathbb{N}$ such that BVP $(\mathrm{R})_n^1$, (1.3) has a solution x_n satisfying (2.20) for each $n \in \mathbb{N}_\mu$ with $\mathbb{N}_\mu$ given in Lemma 2.5. Then (see (2.1) and (2.20))

$$(g(x_n'(t)))' \leqslant \mu[q_1(0) + q_2(c)][\omega_1(\alpha(t)) + \omega_2(x_n'(t))]$$

for a.e. $t \in J$ and each $n \in \mathbb{N}_\mu$. We can now proceed analogously to the proof of Theorem 3.1 to show that

$$(g(x_n'(t)))' \leqslant \mu[q_1(0) + q_2(c)][\omega_1(\alpha(t)) + \omega_2(\chi(t))]\ (\in L_1(J))$$

for a.e. $t \in J$ and each $n \in \mathbb{N}_\mu$, and (the details are left to the reader) we can also show that there is a convergent in $(C^1(J))$ subsequence of $\{x_n\}_{n \in \mathbb{N}_\mu}$ and its limit is a solution of BVP (1.1), (1.3). □

It is easy to construct examples so that Theorems 3.1 and 3.2 can be applied in practice. To illustrate this we consider the following two examples.

Example 3.3. Let b, r, γ and η be positive constants, $\gamma \neq 1$, $\eta < r$. Consider the differential equation

$$(|x'(t)|^r)' = \mu \frac{\operatorname{sgn}(b - x(t))}{|x(t) - b|^\gamma} \left(\frac{\operatorname{sgn} x'(t)}{|x'(t)|^\eta} - 1 \right), \tag{3.8}$$

which is the special case of (1.1) with

$$g(u) = |u|^r, \qquad f(t, x, y) = \frac{\operatorname{sgn}(b - x)}{|x - b|^\gamma} \left(\frac{\operatorname{sgn} y}{|y|^\eta} - 1 \right).$$

Then g satisfies assumption (H_1). Assumption (H_2) is satisfied with $\alpha(t) = 1$, $\nu = T$ and for instance $\varepsilon = 1/b^\gamma$, $\varepsilon_0 = 1/\sqrt[\eta]{2}$ and assumption (H_3) is satisfied with $q_1(u) = 0$, $q_2(u) = 1/(b - u)^\gamma$, $\omega_1(u) = 0$ and $\omega_2(u) = 1/u^\eta$. Since $\eta < r$, we see that

$$\int_0^1 \omega_2(g^{-1}(s))\,\mathrm{d}s = \int_0^1 (1/u^{\eta/r})\,\mathrm{d}u < \infty.$$

Then

$$\begin{aligned}Q(u) &= \int_0^u (q_1(s)+q_2(s))\,\mathrm{d}s\\ &= \int_0^u \frac{1}{(b-s)^\gamma}\,\mathrm{d}s\\ &= \frac{b^{1-\gamma}-(b-u)^{1-\gamma}}{1-\gamma}\end{aligned}$$

and

$$\begin{aligned}H(u) &= \int_0^{g(u)} \frac{g^{-1}(s)}{\omega_1(g^{-1}(s))+\omega_2(g^{-1}(s))}\,\mathrm{d}s\\ &= \int_0^{u^r} s^{(\eta+1)/r}\,\mathrm{d}s\\ &= \frac{r}{\eta+r+1}u^{\eta+r+1}.\end{aligned}$$

Hence

$$H^{-1}(u) = \sqrt[\eta+r+1]{\frac{\eta+r+1}{r}u}$$

and

$$H^{-1}(\mu Q(u)) = K\sqrt[\eta+r+1]{b^{1-\gamma}-(b-u)^{1-\gamma}},$$

where

$$K = \sqrt[\eta+r+1]{\mu\frac{\eta+r+1}{r(1-\gamma)}}.$$

By a routine calculation, one can show that (for $0<c\leqslant b$)

$$\begin{aligned}\int_0^c \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s &= \frac{1}{K}\int_0^c \frac{1}{\sqrt[\eta+r+1]{b^{1-\gamma}-(b-s)^{1-\gamma}}}\,\mathrm{d}s\\ &= \frac{\sqrt[\eta+r+1]{b^{\eta+r+\gamma}}}{K}\int_0^{c/b} \frac{1}{\sqrt[\eta+r+1]{1-(1-s)^{1-\gamma}}}\,\mathrm{d}s\\ &= \frac{\sqrt[\eta+r+1]{b^{\eta+r+\gamma}}}{(1-\gamma)K}\int_0^{1-(1-(c/b))^{1-\gamma}} \frac{\sqrt[1-\gamma]{(1-s)^\gamma}}{\sqrt[\eta+r+1]{s}}\,\mathrm{d}s.\end{aligned}$$

Now

$$\int_0^b \frac{1}{H^{-1}(\mu Q(s))}\,\mathrm{d}s = \sqrt[\eta+r+1]{\frac{rb^{\eta+r+\gamma}}{\mu(\eta+r+1)(1-\gamma)^{\eta+r}}}\,\mathsf{B}\biggl(\frac{\eta+r}{\eta+r+1},\frac{1}{1-\gamma}\biggr),$$

where $\mathsf{B}(\cdot,\cdot)$ is the beta function, and so Theorem 3.1 guarantees that the BVP (3.8), (1.2) has a solution x satisfying (3.1) if $0<\gamma<1$, $r>\eta$ and $\mu\in(0,\mu_*)$ with

$$\mu_* = \frac{rb^{\eta+r+\gamma}}{(\eta+r+1)(1-\gamma)^{\eta+r}}\biggl(\frac{1}{T}\,\mathsf{B}\biggl(\frac{\eta+r}{\eta+r+1},\frac{1}{1-\gamma}\biggr)\biggr)^{\eta+r+1}.$$

Applying Theorem 3.2, BVP (3.8), (1.3) has a solution x satisfying (3.7) if $\gamma \in (0, \infty)$, $r > \eta$ and $\mu \in (0, \mu_0)$ with

$$\mu_0 = \frac{rb^{\eta+r+\gamma}}{(\eta + r + 1)(1-\gamma)^{\eta+r}} \left(\frac{1}{T} \int_0^{1-(1-(c/b))^{1-\gamma}} \frac{\sqrt[1-\gamma]{(1-s)^\gamma}}{\sqrt[\eta+r+1]{s}} \, \mathrm{d}s \right)^{\eta+r+1}.$$

Example 3.4. Consider the BVP

$$x''(t) = \mu(1 - |x(t)|^\gamma) \left(\frac{1}{|x'(t)|^\eta} - 1 \right), \tag{3.9}$$

$$x'(0) = 0, \qquad x(1) = \tfrac{1}{2}, \tag{3.10}$$

where $\gamma \in (0, \infty)$ and $\eta \in (0, 1)$. Assumptions (H_1)–(H_3) are satisfied with $T = 1$, $b = 1/2$, $g(u) = u$, $\alpha(t) = 1$, $\varepsilon = 1 - 1/2^\gamma$, $\varepsilon_0 = 1/\sqrt[\eta]{2}$, $\nu = 1$, $q_1(x) = 0$, $q_2(x) = 1$, $\omega_1(y) = 0$ and $\omega_2(y) = 1/y^\eta$. Hence

$$Q(u) = \int_0^u \mathrm{d}s = u, \qquad H(u) = \int_0^u s^{\eta+1} \, \mathrm{d}u = \frac{u^{\eta+2}}{\eta + 2}$$

and

$$H^{-1}(\mu Q(u)) = \sqrt[\eta+2]{(\eta + 2)\mu u}.$$

Since

$$\int_0^{1/2} \frac{1}{H^{-1}(\mu Q(s))} \, \mathrm{d}s = \int_0^{1/2} \frac{1}{\sqrt[\eta+2]{(\eta + 2)\mu u}} = \frac{1}{\eta + 1} \sqrt[\eta+2]{\frac{1}{\mu} \left(1 + \frac{\eta}{2}\right)^{\eta+1}},$$

Equation (2.12) (with $b = 1/2$ and $T = 1$) has the unique solution

$$\mu_* = \frac{(\eta/2 + 1)^{\eta+1}}{(\eta + 1)^{\eta+2}}.$$

Now Theorem 3.1 guarantees that BVP (3.9), (3.10) with $\mu \in (0, \mu_*)$ has a solution x satisfying the inequalities $0 < x(t) \leqslant 1/2$ and $0 \leqslant x'(t) \leqslant 1$ for $t \in [0, 1]$.

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