Canad. Math. Bull. Vol. **54** (3), 2011 pp. 506–518
doi:10.4153/CMB-2011-069-7

On the Canonical Solution of the Sturm–Liouville Problem with Singularity and Turning Point of Even Order

A. Neamaty and S. Mosazadeh

Abstract. In this paper, we are going to investigate the canonical property of solutions of systems of differential equations having a singularity and turning point of even order. First, by a replacement, we transform the system to the Sturm–Liouville equation with turning point. Using of the asymptotic estimates provided by Eberhard, Freiling, and Schneider for a special fundamental system of solutions of the Sturm–Liouville equation, we study the infinite product representation of solutions of the systems. Then we transform the Sturm–Liouville equation with turning point to the equation with singularity, then we study the asymptotic behavior of its solutions. Such representations are relevant to the inverse spectral problem.

1 Introduction

We consider the following system of differential equations

$$\frac{dy}{dt} = i\rho\frac{1}{R_1(t)}x, \quad \frac{dx}{dt} = \left(i\rho R_2(t) + \frac{p(t)}{i\rho R_1(t)}\right)y, \quad t \in [0,1], \tag{1.1}$$

with initial conditions $x(0,\rho) = 0,\ y(0,\rho) = 1$, where ρ is the spectral parameter, R_1, R_2 and $p(t)$ are bounded and integrable in $I = [0,1]$, and $R_2(t)$ has one zero inside the interval I of even order.

System (1.1) is a canonical form for many problems in natural sciences. For example, for a wide class of problems describing the propagation of electromagnetic waves in a stratified medium, Maxwell's equations can be reduced to the canonical form (1.1) (see [20]). System (1.1) often appears in optics, spectroscopy, and acoustic problems. System (1.1) also appears for the design of directional couplers for heterogeneous electronic lines, which constitutes one of the important classes of radio physical synthesis problems (see [16, 20]). Some aspects of synthesis problems for system (1.1) with $R_1 = R_2 = R > 0$ were studied in [14] and other works. Inverse problems for system (1.1) with the initial conditions $x(0,\rho) = 1,\ y(0,\rho) = -1$ and with $R > 0$ were studied in [5, 6]. In [18], the authors studied the eigenvalues and derived a formula for the asymptotic distribution of the eigenvalues in the case when the system (1.1) has arbitrary order singularities and turning points inside the interval [0,T].

Received by the editors July 24, 2008; revised September 22, 2008.
Published electronically April 14, 2011.
AMS subject classification: **34B05**, 34Lxx, 47E05.
Keywords: turning point, singularity, Sturm–Liouville, infinite products, Hadamard's theorem, eigenvalues.

The importance of asymptotic analysis in obtaining information on the solution of a Sturm–Liouville equation with multiple turning points was realized by Olver [19] and Eberhard, Freiling, and Schneider [3]. Also, the inverse problems for Sturm–Liouville equations with turning points were studied by Freiling and Yurko in [6]. In [4], the asymptotic estimates for a special fundamental system of solutions of the corresponding differential equation studied by Eberhard, Freiling, and Wilchen, and determined the asymptotic distribution of the eigenvalues with several Singularities or/and turning points inside the interval [0,1]. The results of Kazarinoff [10], Langer [12], Olver [19] and Neamaty [17] bring important innovations to the asymptotic approximation of solution of Sturm–Liouville equations with two turning points. Also in [11], the infinite product representation of solution of the equation with one turning point of odd order was obtained. In [15], the infinite product representation of solutions of Sturm–Liouville equations with a finite number of turning points was derived.

It is necessary to point out that applying asymptotic solutions for studying inverse problem in turning points cases is more complicated and impractical. Especially in deriving the asymptotic formulas, one should apply the Bessel function type. In addition, a more difficult and challenging task is shaping the asymptotic behavior of the solutions and corresponding eigenvalues. So the inverse problem of reconstructing the potential function from the given spectral information and corresponding dual equation cannot be studied by using the asymptotic forms. In fact, one cannot generally express the exact solution in closed form using asymptotic methods. Indeed, the closed form of the solution is needed in methods connected with dual equations. The representing solution of the infinite product form plays an important role in investigating the corresponding dual equations. We mention that some aspects of the inverse problem with a singularity were studied in [21]. Also some aspects of the inverse problem with turning points were studied in [9] as well as in other works connected with ideas of the dual equation method.

In the previous article ([11]), the authors considered the following Sturm–Liouville equation

$$y'' + (\lambda\phi^2(z) - q(z))y = 0, \quad 0 \le z \le 1. \tag{1.2}$$

It is assumed that $q(z)$ is a real function that is Lebesgue integrable on the interval $[0,1]$, $\lambda = \rho^2$ is the spectral parameter, and

$$\phi^2(z) = (z - z_1)^{4m+1}\phi_0(z),$$

where $0 < z_1 < 1$, $m \in N$, $\phi_0 > 0$, for $z \in [0,1]$, ϕ_0 is a twice continuously differentiable on $[0,1]$, and $\phi^2(z)$ has one zero in $[0,1]$, the so called turning point. The infinite product form of solution $y(z,\lambda)$ for (1.2) with initial condition $y(0,\lambda) = 0$, $\frac{\partial y}{\partial z}(0,\lambda) = 1$ is of the form

$$y(z,\lambda) = \begin{cases} \frac{1}{2}|\phi(0)\phi(z)|^{\frac{-1}{2}} p(z) \prod_{n\ge 1} \frac{(\lambda-\lambda_n(z))}{\zeta_n^2}, & 0 \le z < z_1, \\ \frac{1}{2}\csc(\frac{\pi\mu}{2})|\phi(0)\phi(z)|^{\frac{-1}{2}} \pi(f(z)p(z_1))^{\frac{1}{2}} & \\ \qquad \times \prod_{n\ge 1} \frac{(\lambda-r_n(z))p^2(z_1)}{\widetilde{j}_n^2} \prod_{n\ge 1} \frac{(\widetilde{r}_n(z)-\lambda)f^2(z)}{\widetilde{j}_n^2}, & z_1 < z \le 1, \end{cases}$$

https://doi.org/10.4153/CMB-2011-069-7 Published online by Cambridge University Press

where the sequence $\{r_n(z)\}_{n\geq 1}$ represents the sequence of negative eigenvalues and $\{\widetilde{r}_n(z)\}_{n\geq 1}$ the sequence of positive eigenvalues of the Dirichlet problem associated with (1.2) on $[0, z]$, for each z in $(0, z_1]$. The sequence $\{\lambda_n(z)\}_{n\geq 1}$, for each fixed z, $0 < z \leq z_1$, represents the sequence of negative eigenvalues of the Dirichlet problem for equation (1.2) on the closed interval $[0, z]$, where

$$p(z) = \int_0^z |\phi(\zeta)| d\zeta, \quad 0 \leq z < z_1,$$

$$f(z) = \int_{z_1}^z |\phi(\zeta)| d\zeta, \quad z_1 < z \leq 1,$$

$$\zeta_n(z) = \frac{n\pi}{p(z)},$$

and $\widetilde{j}_n$, $n = 1, 2, \dots$, are the positive zeros of $J_1'(z)$, respectively.

In this paper, we first transform (1.1) to the Sturm–Liouville equation with turning point of even order, then we define a fundamental system of solutions (FSS) of equation when $|\rho| \to \infty$ (see Section 2). Using these asymptotic solutions we derive a formula for the asymptotic distribution of the eigenvalues (in Section 4); further, we obtain the infinite product representation of solutions (see Section 5). In Section 6, by a replacement, we transform the Sturm–Liouville equation with turning point to an equation with singularity and we determine the asymptotic behavior of the solution. Also, using the infinite representation of solutions of Section 5, we obtain the canonical representation of solutions of equation with singularity (see Section 7). Therefore, we define singularity's and turning's relation by upper replacements. The other missing cases will be treated in a future paper as they require different techniques.

2 Notations and Preliminary Results

Let us consider the system of differential equations (1.1), where the function $R_1(t) = A_1 > 0$ is a constant function and

$$R_2(t) = A_2(t - t_1)^{4\ell},$$

where the coefficient A_2 is a positive constant, $\ell \in N$ and $t_1 \in (0, 1)$.

System (1.1), after the elimination of x, reduces to the linear second-order Sturm–Liouville equation,

$$-y'' + p(t)y = \lambda\phi^2(t)y, \tag{2.1}$$

with initial conditions

$$y(0, \rho) = 1, \ y'(0, \rho) = 0,$$

where $\lambda = \rho^2$ is a real parameter and $\phi^2(t) = \frac{A_2}{A_1}(t - t_1)^{4\ell}$, has one zero t_1 in $(0, 1)$, the so called turning point. In the terminology of [3], t_1 is of Type II.

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Notations 2.1 (i) We introduce

$$\mu := \frac{1}{4\ell + 2}, \quad [1] := 1 + O\Big(\frac{1}{\rho}\Big), \quad \text{as } \rho \to \infty.$$

(ii) For $k \in Z$ we consider the sectors

$$S_k := \Big\{ \rho \mid \frac{k\pi}{4} \le \arg \rho \le \frac{(k+1)\pi}{4} \Big\}.$$

Now let $C(t, \lambda)$ be the solution of (2.1) corresponding to the initial conditions $C(0, \lambda) = 1, C'(0, \lambda) = 0$. In order to represent the solution $C(t, \lambda)$ as an infinite product we use a suitable fundamental system of solutions (FSS) for equation (2.1) as constructed in [3].

According to the type of t_1 we know from [3, Theorem 3.2] that an FSS of (2.1) $\{w_1(t, \rho), w_2(t, \rho)\}$ exists in the sector S_{-1} and that

(2.2) $w_1(t, \rho) =$

$$\begin{cases} \phi^{-\frac{1}{2}}(t)\, e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1], & 0 \le t < t_1, \\ \csc \pi\mu\, \phi^{-\frac{1}{2}}(t)\, \{\, e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] + i \cos \pi\mu\, e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] \,\}, & t_1 < t \le 1, \end{cases}$$

(2.3)

$$w_2(t, \rho) = \begin{cases} \phi^{-\frac{1}{2}}(t) \{ e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] + i \cos \pi\mu\, e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] \}, & 0 \le t < t_1, \\ \sin \pi\mu\, \phi^{-\frac{1}{2}}(t)\, e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1], & t_1 < t \le 1. \end{cases}$$

This leads to the following:

(2.4)

$$w_1'(t, \rho) = \begin{cases} i\rho \phi^{\frac{1}{2}}(t)\, e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1], & 0 \le t < t_1, \\ \rho \csc \pi\mu\, \phi^{\frac{1}{2}}(t)\, \{ i e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] + \cos \pi\mu\, e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] \}, & t_1 < t \le 1, \end{cases}$$

(2.5)

$$w_2'(t, \rho) = \begin{cases} \rho \phi^{\frac{1}{2}}(t)\, \{ -i e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] - \cos \pi\mu\, e^{i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1] \}, & 0 \le t < t_1, \\ -i\rho \sin \pi\mu\, \phi^{\frac{1}{2}}(t)\, e^{-i\rho \int_{t_1}^{t} \phi(\zeta) d\zeta}[1], & t_1 < t \le 1. \end{cases}$$

We also need $\{w_1(t_1, \lambda), w_2(t_1, \lambda)\}$. Similarly, for $t = t_1$ from [3] we have

$$w_1(t_1, \rho) = \frac{\sqrt{2\pi}}{2} \rho^{\frac{1}{2} - \mu} \csc \pi\mu \{ e^{i\pi(\frac{1}{4} - \frac{\mu}{2})}\, u_1(t_1, \rho)[1] + e^{i\pi(\frac{1}{4} + \frac{\mu}{2})}\, u_2(t_1, \rho)[1] \},$$

$$w_2(t_1, \rho) = \frac{\sqrt{2\pi}}{2} \rho^{\frac{1}{2} - \mu} \{ e^{i\pi(\frac{1}{4} - \frac{\mu}{2})}\, u_1(t_1, \rho)[1] - e^{i\pi(\frac{1}{4} + \frac{\mu}{2})}\, u_2(t_1, \rho)[1] \},$$

where

$$u_1(t_1,\rho)=\frac{2^{\mu}\,\psi(t_1)}{\Gamma(1-\mu)},\quad u_2(t_1,\rho)=0,$$

where $\psi(t_1)=\lim_{t\to t_1}\ \phi^{\frac{-1}{2}}(t)\{\int_{t_1}^{t}\phi(\zeta)d\zeta\}^{\frac{1}{2}-\mu}$.

Consequently

$$w_1(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}\csc\pi\mu\, e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\,\frac{2^{\mu}\,\psi(t_1)}{\Gamma(1-\mu)}[1], \tag{2.6}$$

$$w_2(t_1,\rho)=\frac{\sqrt{2\pi}}{2}\rho^{\frac{1}{2}-\mu}\, e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\,\frac{2^{\mu}\,\psi(t_1)}{\Gamma(1-\mu)}[1]. \tag{2.7}$$

It follows that the wronskian of FSS satisfies

$$W(\rho)\equiv W(w_1(t,\rho),\quad w_2(t,\rho))=-2i\rho[1],\quad \text{as } \rho\to\infty.$$

3 Asymptotic Form of the Solution

We consider the differential equation (2.1) with the following conditions:

$$C(0,\lambda)=1,\quad C'(0,\lambda)=0.$$

Applying the FSS $\{w_1(t,\rho),w_2(t,\rho)\}$ for $t\in[0,1]$, we have

$$C(t,\rho)=c_1w_1(t,\rho)+c_2w_2(t,\rho),$$

and using of Cramer's rule leads to the equation

$$C(t,\rho)=\frac{1}{W(\rho)}(w_2'(0,\rho)w_1(t,\rho)-w_1'(0,\rho)w_2(t,\rho)), \tag{3.1}$$

where

$$W(\rho)=W(w_1,w_2)=-2i\rho[1].$$

Taking (2.2)–(2.5) into account we derive

$$C(t,\rho)= \tag{3.2}$$

$$\begin{cases}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\,\{\cosh(i\rho\int_0^t\phi(\zeta)d\zeta)+O(\frac{1}{\rho})\}, & 0\le t<t_1,\\ \frac{1}{2}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\,\{M_1(\rho)\,e^{i\rho\int_{t_1}^{t}\phi(\zeta)d\zeta}[1]+M_2(\rho)\,e^{-i\rho\int_{t_1}^{t}\phi(\zeta)d\zeta}[1]\,\}, & t_1<t\le 1,\end{cases}$$

where

$$\begin{cases}M_1(\rho)=\csc\pi\mu\,e^{i\rho\int_0^{t_1}\phi(\zeta)d\zeta}-i\cot\pi\mu\,e^{-i\rho\int_0^{t_1}\phi(\zeta)d\zeta},\\ M_2(\rho)=i\cot\pi\mu\,e^{i\rho\int_0^{t_1}\phi(\zeta)d\zeta}+\csc\pi\mu\,e^{-i\rho\int_0^{t_1}\phi(\zeta)d\zeta}.\end{cases} \tag{3.3}$$

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By virtue of (3.2) and (3.3), the following estimates are also valid:

$$C(t,\rho)=\begin{cases}\frac{1}{2}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\,e^{i\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & 0\le t<t_1,\\ \frac{1}{2}\csc\pi\mu\,\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\,e^{i\rho\int_0^t|\phi(\zeta)|d\zeta}E_k(t,\rho), & t_1<t\le 1,\end{cases}\tag{3.4}$$

where

$$E_k(t,\rho)=\sum_{n=1}^{\nu(t)}e^{\rho\alpha_k\beta_{kn}(t)}b_{kn}(t),$$

and

$$\alpha_{-2}=\alpha_1=-1\,,\ \alpha_0=-\alpha_{-1}=i,$$
$$\beta_{k\nu}(t)\not=0, 0<\delta\le\beta_{k1}(t)<\beta_{k2}(t)<\cdots\le\beta_{k\nu(t)}(t)\le 2R_+(1),$$

where the integer-valued functions ν and b_{kn} are constant in every interval $[0,t_1-\varepsilon]$ and $[t_1+\varepsilon,1]$ for ε sufficiently small, and

$$R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(\zeta)\}}\,d\zeta.\tag{3.5}$$

Similarly, by using (2.6), (2.7), and (3.1) for $t=t_1$ we find that

$$C(t_1,\rho)=\frac{\sqrt{2\pi}\phi^{\frac{1}{2}}(0)\rho^{\frac{1}{2}-\mu}\,e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\,2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\,e^{i\rho\int_0^{t_1}\phi(\zeta)d\zeta}\,E_k(t_1,\rho).\tag{3.6}$$

In addition, differentiating (3.2) we calculate

$$C'(t,\rho)=\begin{cases}i\rho(\phi(0)\phi(t))^{\frac{1}{2}}\,\{\sinh(i\rho\int_0^t\phi(\zeta)d\zeta)+O(\frac{1}{\rho^{\sigma_0}})\}, & 0\le t<t_1,\\ \frac{1}{2}i\rho(\phi(0)\phi(t))^{\frac{1}{2}} & \\ \quad\times\{M_1(\rho)\,e^{i\rho\int_{t_1}^t\phi(\zeta)d\zeta}[1]-M_2(\rho)\,e^{-i\rho\int_{t_1}^t\phi(\zeta)d\zeta}[1]\,\}, & t_1<t\le 1.\end{cases}$$

Thus, we deduce the following theorem:

Theorem 3.1 *Let $C(t,\rho)$ be the solution of (2.1) under the initial conditions* $(0,\lambda)=1, C'(0,\lambda)=0$, *then the following estimates hold:*

$$C(t,\rho)=\frac{1}{2}\{\csc\pi\mu\}^{\nu}\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\,e^{i\rho\int_0^t\phi(\zeta)d\zeta}E_k(t,\rho), t\in D_\nu,\ \nu=0,1,$$

where $D_0=[0,t_1)$ *and* $D_1=(t_1,1]$, *also*

$$C(t_1,\rho)=\frac{\sqrt{2\pi}\phi^{\frac{1}{2}}(0)\rho^{\frac{1}{2}-\mu}\,e^{i\pi(\frac{1}{4}-\frac{\mu}{2})}\,2^{\mu}\psi(t_1)\csc\pi\mu}{4\Gamma(1-\mu)}\,e^{i\rho\int_0^{t_1}\phi(\zeta)d\zeta}\,E_k(t_1,\rho).$$

4 Distribution of the Eigenvalues

We consider the boundary value problem $L_1 = L_1(p(t),\ \phi^2(t),\ s)$ for equation (2.1) with boundary condition

$$y(0,\lambda) = 1, \quad y'(0,\lambda) = 0, \quad y(s,\lambda) = 0.$$

The boundary value problem L_1 for $s \in (0,1)\backslash\{t_1\}$ has a countable set of positive eigenvalues $\{\lambda_n(s)\}_{n\geq 1}$. From (3.4), we have the following asymptotic distribution for each $\{\lambda_n(s)\}$:

$$\sqrt{\lambda_n(s)} = \frac{n\pi - \frac{\pi}{2}}{\int_0^s \phi(\zeta)d\zeta} + O\Big(\frac{1}{n}\Big). \tag{4.1}$$

Similarly, according to (3.6), the spectrum $\{\lambda_n\}_{n\geq 1}$ of boundary value problem L_1 for $s = t_1$ consists of positive eigenvalues

$$\sqrt{\lambda_n(t_1)} = \frac{n\pi + (\frac{\pi\mu}{2} - \frac{3\pi}{4})}{\int_0^{t_1} \phi(\zeta)d\zeta} + O\Big(\frac{1}{n}\Big).$$

5 Main Results

Since the solution $C(t,\rho)$ of the Sturm–Liouville equation defined by a fixed set of initial conditions is an entire function of ρ for each fixed $t \in [0,1]$, it follows from the classical Hadamard's factorization theorem (see [13, p. 24]) that such a solution is expressible as an infinite product.

For fixed $s \in (0,1)\backslash\{t_1\}$ by Halvorsen's result [7], $C(s,\rho)$ is an entire function of order $\frac{1}{2}$. Therefore, we can use Hadamard's theorem to represent the solution in the form

$$C(s,\lambda) = h(s)\prod_{n\geq 1}\Big(1 - \frac{\lambda}{\lambda_n(s)}\Big),$$

where $h(s)$ is a function independent of λ but may depend on s and the infinite number of positive eigenvalues, $\{\lambda_n(s)\}_{n=1}^{\infty}$ form the zero set of $C(s,\lambda)$ for each s. Let $\zeta_n,\ n \geq 1$, be the sequence of positive zeros of $J'_{\frac{1}{2}}(t)$. Then (see [1, § 9.5.11])

$$\frac{\zeta_n^2}{R_+^2(t)\lambda_n(t)} = 1 + O\Big(\frac{1}{n^2}\Big).$$

Consequently, the infinite product $\prod_{n\geq 1}\frac{\zeta_n^2}{R_+^2(t)\lambda_n(t)}$ is absolutely convergent for each $s \in (0,1)\backslash\{t_1\}$. Therefore we have that

$$C(s,\lambda) = h(s)\prod_{n\geq 1}(1 - \frac{\lambda}{\lambda_n(s)}) = h_1(s)\prod_{n\geq 1}\frac{(\lambda_n(t) - \lambda)R_+^2(t)}{\zeta_n^2}, \tag{5.1}$$

with $h_1(s) := h(s)\ \prod_{n\geq 1}\frac{\zeta_n^2}{R_+^2(s)\lambda_n(s)}$.

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Theorem 5.1 *Let $C(t,\lambda)$ be the solution of (2.1) satisfying the initial conditions $C(0,\lambda)=1$, $C'(0,\lambda)=0$. Then for $t\in B_\nu,\ \nu=0,1$,*

$$C(t,\lambda)=\frac{1}{2}\{\csc\pi\mu\}^{\nu}\,\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t)\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2},$$

where $B_0=(0,t_1),\ B_1=(t_1,1)$, $R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(s)\}}\,ds$, $\zeta_n,\ n\geq 1$, is the sequence of positive zeros of $J'_{\frac{1}{2}}$, the sequence $\lambda_n(t),\ n\geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t]$.

Proof Let $\lambda_n(s)$ be the eigenvalues of the boundary value problem L_1 on $[0,s]$, for fixed $s=t,\ t\in B_\nu$. Then according to [1, §9.5.11, §10.1.1, §10.1.11] we have

$$\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2}=2\cos(\sqrt{\lambda}R_+(t))[1],$$

as $\lambda\to\infty$. Thus from (3.4) and (5.1), we obtain

$$h_1(t)=\frac{C(t,\lambda)}{\prod_{n\geq 1}\frac{(\lambda_n(t)-\lambda)R_+^2(t)}{\zeta_n^2}}=\frac{1}{2}\{\csc\pi\mu\}^{\nu}\,\phi^{\frac{1}{2}}(0)\phi^{-\frac{1}{2}}(t).\qquad\blacksquare$$

We can proceed similarly for $s=t_1$, using Hadamard's theorem to obtain

$$C(t_1,\lambda)=A\prod_{n\geq 1}\Big(1-\frac{\lambda}{\lambda_n(t_1)}\Big),$$

where A is constant. Let $j_n,\ n\geq 1$, be the sequence of positive zeros of J'_μ, then (see [1, § 9.5.11])

$$\frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)}=1+O\Big(\frac{1}{n^2}\Big),$$

and so the infinite product $\prod\frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)}=1+O(\frac{1}{n^2})$ is absolutely convergent. Consequently we may write as before,

$$C(t_1,\lambda)=A_1\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},\tag{5.2}$$

where $A_1=A\prod\frac{j_n^2}{R_+^2(t_1)\lambda_n(t_1)}$.

Theorem 5.2 *For $s=t_1$,*

$$C(t_1,\lambda)=\frac{1}{2}\phi^{\frac{1}{2}}(0)R_+^{\mu-\frac{1}{2}}(t_1)\psi(t_1)\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},$$

where $R_+(t)=\int_0^t\sqrt{\max\{0,\phi^2(s)\}}\,ds$, $j_n,\ n=1,2,\ldots$, is the sequence of positive zeros of J'_μ. The sequence $\lambda_n(t_1)$ represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,t_1]$ and $\psi(t_1)=\lim_{t\to t_1}\ \phi^{\frac{-1}{2}}(t)\{\int_{t_1}^t\phi(s)ds\}^{\frac{1}{2}-\mu}$.

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Proof According to [7] the infinite product

$$\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2},$$

is an entire function of λ, whose roots are precisely $\lambda_n(t_1),\ n\geq 1$. From [1, §9.2.11] we have

$$J'_\mu(z)=\sqrt{\frac{2\pi}{z}}\,\{\,-R(\mu,z)\sin\chi-S(\mu,z)\cos\chi\,\},$$

where ν is fixed and

$$\begin{aligned}\chi&=z-(\frac{\mu}{2}+\frac{1}{4})\pi,\\ R(\mu,z)&\sim\sum_{k=0}^{\infty}(-1)^k\frac{4\mu^2+16k^2-1}{4\mu^2-(4k+1)^2}\left\{\frac{(\mu,2k)}{(2z)^{2k}}\right\}\\ &=1-\frac{(\alpha-1)(\alpha+15)}{2!(8z)^2}+\cdots,\\ S(\mu,z)&\sim\sum_{k=0}^{\infty}(-1)^k\frac{4\mu^2+4(2k+1)^2-1}{4\mu^2-(4k+1)^2}\left\{\frac{(\mu,2k)}{(2z)^{2k+1}}\right\}\\ &=\frac{\alpha+3}{8z}-\frac{(\alpha-1)(\alpha-9)(\alpha+35)}{3!(8z)^3}+\cdots,\end{aligned}$$

as $|z|\to\infty$, where $\alpha=4\mu^2$. Now, by inserting $z=R_+(t_1)\sqrt{\lambda}$, and from [1, § 9.5.11], we get

$$\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2}=$$
$$\sqrt{\frac{2}{\pi}}\,\Gamma(\mu)\{R_+(t_1)\sqrt{\lambda}\}^{\frac{1}{2}-\mu}\,2^\mu\cos\Big(R_+(t_1)\sqrt{\lambda}+\frac{\pi}{4}-\frac{\pi\mu}{2}\Big)\Big(1+O(\frac{1}{\sqrt{\lambda}})\Big),$$

Thus it follows from (3.6) and (5.2):

$$A_1=\frac{C(t_1,\lambda)}{\prod_{n\geq 1}\frac{(\lambda_n(t_1)-\lambda)R_+^2(t_1)}{j_n^2}}=\frac{1}{2}\phi^{\frac{1}{2}}(0)R_+^{\mu-\frac{1}{2}}(t_1)\psi(t_1).\qquad\blacksquare$$

6 Solution of Differential Equations with Singularity

In this section we transform (2.1) by a replacement to the differential equation with a singular point, and we study the asymptotic behavior of the solutions.

Denote $T=\int_0^1\phi(\zeta)d\zeta$.

We transform (2.1) by means of the replacement

$$z = \int_0^t \phi(\zeta)d\zeta, \quad u(z) = \phi^{\frac{1}{2}}(t)y(t) \tag{6.1}$$

to the differential equation

$$-u''(z) + q(z)u(z) = \lambda u(z), \quad z \in [0, T], \tag{6.2}$$

with initial conditions

$$u(0, \lambda) = r_1, \quad u'(0, \lambda) = r_2, \tag{6.3}$$

where $r_1 = \phi^{\frac{1}{2}}(0)$, $r_2 = \frac{1}{2}\phi^{\frac{-3}{2}}(0)\phi'(0)$, and $q(z)$ has quadratic singularity in the interval $(0, T)$ and has the form:

$$q(z) = \frac{F}{(z - z_1)^2} + q_0(z),$$

where $F = \mu^2 - \frac{1}{4}$, $\mu = \frac{1}{4\ell+2}$ and $z_1 = \int_0^{t_1} \phi(\zeta)d\zeta$. We also assume that

$$q_0(z)(z - z_1)^{1-2\mu} \in L(0, T).$$

Since the solutions of equation (6.2) have singularity at $z = z_1$, therefore, in general, the values of the solutions and their derivatives at $z = z_1$ are not defined.

Remark 6.1 In [21], fundamental system of solutions $\{S_k(z, \lambda)\}$, $k = 1, 2$, of equation (6.2) were constructed with the following properties:

(i) For each fixed $z \in [0, T]$, the functions $S_k^{(\nu)}(z, \lambda)$, $\nu = 0, 1$, are entire in λ of order $\frac{1}{2}$.

(ii) Denote $\mu_k = (-1)^k\mu + \frac{1}{2}$, $k = 1, 2$. Then

$$S_k(z, \lambda) \leq C|\rho(z - z_1)^{\mu_k}|,$$

for $|\rho(z - z_1)| \leq 1$, where C is a positive constant in estimate not depending on z and ρ.

(iii) The following relation holds

$$< S_1(z, \lambda), S_2(z, \lambda) >\equiv 1,$$

where $< y(z), y(\tilde{z}) >:= y(z)\tilde{y}'(z) - y'(z)\tilde{y}(z)$ is the wronskian of y and $\tilde{y}$.

Let $\omega_0 = [0, z_1)$, $\omega_1 = (z_1, T]$, from [2] for $z \in \omega_0 \cup \omega_1$,

$$S_k(z, \lambda) = (z - z_1)^{\mu_k} \sum_{m=0}^{\infty} S_{km}(\rho(z - z_1))^{2m}, \quad k = 1, 2,$$

https://doi.org/10.4153/CMB-2011-069-7 Published online by Cambridge University Press

where

$$S_{10}S_{20} = (2\mu)^{-1}, S_{km} = (-1)^m S_{k0}\Big(\prod_{s=1}^{m}\big((2s+\mu_k)(2s+\mu_k-1)-F\big)\Big)^{-1}.$$

Denote

$$\varphi_k(z,\lambda) = (-1)^{k-1}(S_2^{(2-k)}(0,\lambda)S_1(z,\lambda) - S_1^{(2-k)}(0,\lambda)S_2(z,\lambda)), \quad k=1,2.$$

The functions $\varphi_k(z,\lambda)$ are solutions of (6.2) and

$$\varphi_k^{(m-1)}(0,\lambda) = \delta_{k,m}, \;\; k,m=1,2, \tag{6.4}$$

($\delta_{k,m}$ is the Kronecker delta). Moreover,

$$< \varphi_1(z,\lambda), \varphi_2(z,\lambda) >= 1.$$

From [6], we have the following lemma.

Lemma 6.2 *For $(\rho,z)\in\Omega := \{(\rho,z) : |\rho(z-z_1)| \geq 1\}$, $z\in\omega_s$, $s=0,1$:*

$$\begin{aligned}\varphi_k^{(m-1)}(z,\lambda) = \frac{1}{2}(i\rho)^{m-k}\{&\exp(i\rho z)[1]_\gamma + (-1)^{m-k}\exp(-i\rho z)[1]_\gamma \\ &+ (-1)^k 2si\cos\pi\mu\;\exp(i\rho(z-2z_1))[1]_\gamma\}, \;\; |\rho|\to\infty, \;\; k,m=1,2,\end{aligned} \tag{6.5}$$

where $[1]_\gamma = 1 + O((\rho(z-z_1))^{-1})$.

Using the preceding results, from (6.3) and (6.4), we have

$$u(z,\rho) = r_1\varphi_1(z,\lambda) + r_2\varphi_2(z,\lambda). \tag{6.6}$$

Now, from (6.5) and (6.6) we obtain the asymptotic solution of equation (6.2) in the following theorem.

Theorem 6.3 *For $z\in\omega_s$, $s=0,1$, $(\rho,z)\in\Omega$, $|\rho|\to\infty$, $Im\rho\geq 0$, $m=0,1$:*

$$\begin{aligned}u^{(m)}(z,\rho) = &\frac{1}{2}(i\rho)^{m-1}(i\rho r_1 + r_2)\exp(i\rho z)[1]_\gamma \\ &+ \frac{1}{2}(-i\rho)^{m-1}(-i\rho r_1 + r_2)\exp(-i\rho z)[1]_\gamma \\ &+ s(i\rho)^{m-1}(\rho r_1 + ir_2)\cos\pi\mu\;\exp\big(i\rho(z-2z_1)\big)[1]_\gamma.\end{aligned}$$

https://doi.org/10.4153/CMB-2011-069-7 Published online by Cambridge University Press

7 Canonical Product Representation of Solution

According to (6.1), the boundary value problem $L_1 = L_1(p(t),\ \phi^2(t),\ s)$ defined by equation (2.1) with boundary conditions $y(0,\lambda) = 1,\ \ y'(0,\lambda) = 0,\ \ y(s,\lambda) = 0$, transforms to the boundary value problem $L_2 = L_2(q(z),\ b)$ with boundary conditions

$$u(0,\lambda) = r_1, \quad u'(0,\lambda) = r_2, \quad u(b,\lambda) = 0, \tag{7.1}$$

where $b = \int_0^s \phi(\varsigma)d\varsigma,\ s \in (0,,1)\backslash\{t_1\}$, $r_1 = \phi^{\frac{1}{2}}(0)$ and $r_2 = \frac{1}{2}\phi^{\frac{-3}{2}}(0)\phi'(0)$.

Thus, according to (4.1) and (6.1) for $b \in (0,,T)\backslash\{z_1\}$, the boundary value problem L_2 has a countable set of positive eigenvalues $\{\lambda_{1n}\}_{n\geq 1}$:

$$\sqrt{\lambda_{1n}(b)} = \frac{n\pi - \frac{\pi}{2}}{b} + O\Big(\frac{1}{n}\Big). \tag{7.2}$$

According to Remark 6.1, the solution $u(z,\rho)$ of Sturm–Liouville equation (6.2) defined by initial conditions (7.1) is an entire function of ρ for each fixed $z \in [0,T]$. Thus it follows from the Hadamard's theorem (see [13, p. 24]) that such a solution is expressible as an infinite product.

To complete the investigation of the last sections, we want to prove the following theorem. Theorem 5.1 is a useful tool for the proof of this result.

Theorem 7.1 *Let $u(z,\lambda)$ be the solution of (6.2) satisfying the initial conditions $u(0,\lambda) = r_1$, $u'(0,\lambda) = r_2$. Then for $z \in C_\nu,\ \nu = 0,1$,*

$$u(z,\lambda) = \frac{1}{2}r_1\,\{\csc\pi\mu\}^\nu \prod_{n\geq 1}\frac{(\lambda_{1n}(z)-\lambda)z^2}{\zeta_n^2}, \tag{7.3}$$

where $C_0 = (0,z_1),\ C_1 = (z_1,T)$. The sequence $\lambda_{1n}(z),\ n \geq 1$, represents the sequence of positive eigenvalues of the boundary value problem L_1 on $[0,z]$, and $\zeta_n,\ n = 1,2,\ldots,$ is the sequence of positive zeros of $J'_{\frac{1}{2}}$.

Proof From (3.5) and (6.1) we obtain $R_+(t) = z$. Thus, according to (6.1), (7.2), $r_1 = \phi^{\frac{1}{2}}(0)$, and Theorem 5.1, we arrive at (7.3). ■

This completes the representation of the solution of (6.2) with initial conditions (6.3) as an infinite product.

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Department of Mathematics, University of Mazandaran, Babolsar, IRAN
e-mail: namaty@umz.ac.ir s.mosazadeh@umz.ac.ir