Compositio Math. 140 (2004) 818–826
DOI: 10.1112/S0010437X03000745

# Complete manifolds with non-negative Ricci curvature and the Caffarelli–Kohn–Nirenberg inequalities

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Abstract

In this paper, we prove that complete open Riemannian manifolds with non-negative Ricci curvature of dimension greater than or equal to three in which some Caffarelli–Kohn–Nirenberg type inequalities are satisfied are *close* to the Euclidean space.

## 1. Introduction

Let $n \geqslant 3$ be an integer and let $a, b$, and $p$ be constants satisfying

$$-\infty < a < \frac{n-2}{2}, \quad a \leqslant b \leqslant a+1, \quad \text{and} \quad p = \frac{2n}{n-2+2(b-a)}. \tag{1.1}$$

Denote by $C_0^\infty(\mathbb{R}^n)$ the space of smooth functions with compact support in the $n$-dimensional Euclidean space $\mathbb{R}^n$. In [CKN84], among a much more general family of inequalities, Caffarelli, Kohn, and Nirenberg proved the following result. There exists a positive constant $C$ depending only on $a, b$ and $n$ such that

$$\left(\int_{\mathbb{R}^n} |x|^{-bp}|u|^p\,dx\right)^{1/p} \leqslant C\left(\int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2\,dx\right)^{1/2}, \tag{1.2}$$

for all $u \in C_0^\infty(\mathbb{R}^n)$, where $|x|$ is the Euclidean length of $x \in \mathbb{R}^n$. Note that the Caffarelli–Kohn–Nirenberg inequalities contain the classical Sobolev inequality ($a = b = 0$) and the Hardy inequality ($a = 0,\ b = 1$) as special cases, which have many important applications (see e.g. [Aub82, Aub98, CKN84, HLP52, Heb96, Heb99, Lie83] and references therein).

Let $K_{a,b}$ be the best constant for the Caffarelli, Kohn, and Nirenberg inequality (1.1), that is

$$K_{a,b}^{-1} = \inf_{u\in C_0^\infty(\mathbb{R}^n)-\{0\}} \frac{(\int_{\mathbb{R}^n} |x|^{-2a}|\nabla u|^2\,dx)^{1/2}}{(\int_{\mathbb{R}^n} |x|^{-bp}|u|^p\,dx)^{1/p}}. \tag{1.3}$$

For the Sobolev inequality ($a = b = 0$), it has been proved by Aubin [Aub76] and Talent [Tal76] that

$$K_{0,0} = \left(\frac{1}{n(n-2)}\right)^{1/2}\left(\frac{2\Gamma(n)}{n\omega_n\Gamma^2(n/2)}\right)^{1/n},$$

where $\omega_n$ is the volume of the unit ball in $\mathbb{R}^n$, and that a family of minimizers of (1.3) is given by

$$u(x) = (\lambda + |x|^2)^{1-n/2},\ \lambda > 0.$$

In [Lie83], Lieb considered the case $a = 0,\ 0 < b < 1$, and proved that the best constant is

$$K_{0,b} = \left(\frac{1}{(n-2)(n-bp)}\right)^{1/2}\left(\frac{(2-bp)\Gamma((2n-2bp)/(2-bp))}{n\omega_n\Gamma^2((n-bp)/(2-bp))}\right)^{2(n-bp)/(2-bp)},$$

Received 12 March 2002, accepted in final form 18 June 2002.
*2000 Mathematics Subject Classification* 53C20, 53C21, 57R70, 31C12.
*Keywords:* manifolds, Ricci curvature, Sobolev inequalities.

and a family of minimizers is

$$u(x) = \frac{1}{(\lambda + |x|^{2-bp})^{(n-2)/(2-bp)}}, \quad \lambda > 0.$$

Chou and Chu [CC93] studied the case $a \geqslant 0,\ a \leqslant b < a+1$, and proved that the best constant is

$$K_{a,b} = \bigg(\frac{1}{(n-2a-2)(n-bp)}\bigg)^{1/2} \bigg(\frac{(2-bp+2a)\Gamma((2n-2bp)/(2-bp+2a))}{n\omega_n\Gamma^2((n-bp)/(2-bp+2a))}\bigg)^{2(n-bp)/(2-bp+2a)},$$

and that, for $a > 0$, all minimizers are non-zero constant multiples of the function

$$u(x) = \frac{1}{(\lambda + |x|^{2-bp+2a})^{(n-2-2a)/(2-bp+2a)}}, \quad \lambda > 0.$$

For the remaining case, the best constant $K_{a,b}$ and the existence or non-existence of the minimizers have been studied recently in [CW01].

In this paper, we study complete manifolds with non-negative Ricci curvature in which some Caffarelli–Kohn–Nirenberg inequalities are satisfied. Now we fix some notation. For an integer $n \geqslant 3$, we will from now on let $a$ and $b$ be constants satisfying

$$0 \leqslant a < \frac{n-2}{2}, \quad a \leqslant b < a+1, \tag{1.4}$$

and set

$$p = \frac{2n}{n-2+2(b-a)}. \tag{1.5}$$

For a Riemannian manifold $M$, we let $dv$ be the Riemannian volume element on $M$, denote by $\nabla$ the gradient operator, $C_0^\infty(M)$ the space of smooth functions on $M$ with compact support, $B(x,r)$ the geodesic ball with center $x \in M$ and radius $r$, and $\mathrm{vol}[B(p,r)]$ the volume of $B(p,r)$.

Our purpose is to prove the following result.

Theorem 1.1. *Let $C \geqslant K_{a,b}$ be a constant and $M$ be an $n$-dimensional ($n \geqslant 3$) complete open Riemannian manifold with non-negative Ricci curvature. Fix a point $x_0 \in M$ and denote by $\rho$ the distance function on $M$ from $x_0$. Assume that, for any $u \in C_0^\infty(M)$, we have*

$$\bigg(\int_M \rho^{-bp}|u|^p\,dv\bigg)^{1/p} \leqslant C\bigg(\int_M \rho^{-2a}|\nabla u|^2\,dv\bigg)^{1/2}. \tag{1.6}$$

*Then for any $x \in M$, we have*

$$\mathrm{vol}[B(x,r)] \geqslant (C^{-1}K_{a,b})^{n/(1+a-b)}V_0(r), \quad \forall r > 0, \tag{1.7}$$

*where $V_0(r)$ is the volume of the $r$-ball in $\mathbb{R}^n$.*

In the special case that $a = b = 0$, the above theorem has been proved in [Xia01].

The theorem has several consequences for manifolds with non-negative Ricci curvature.

The Bishop–Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) implies that, if $M$ is an $n$-dimensional complete Riemannian manifold with non-negative Ricci curvature, then for any $x \in M, \mathrm{vol}[B(x,r)] \leqslant V_0(r)$, with equality holding if and only if $B(x,r)$ is isometric to an $r$-ball in $\mathbb{R}^n$. Combining this fact and Theorem 1.1, one immediately gets the following rigidity theorem.

Corollary 1.2. *An $n$-dimensional ($n \geqslant 3$) complete open Riemannian manifold $M$ with non-negative Ricci curvature in which the inequality*

$$\bigg(\int_M \rho^{-bp}|u|^p\,dv\bigg)^{1/p} \leqslant K_{a,b}\bigg(\int_M \rho^{-2a}|\nabla u|^2\,dv\bigg)^{1/2}, \quad \forall u \in C_0^\infty(M),$$

*is satisfied, is isometric to $\mathbb{R}^n$.*



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When $a = b = 0$, Corollary 1.2 is the main theorem in [Led99].

A theorem of Cheeger and Colding [CC97] states that given integer $n \geqslant 2$ there exists a constant $\delta(n) > 0$ such that any $n$-dimensional complete Riemannian manifold with non-negative Ricci curvature and $\mathrm{vol}[B(x,r)] \geqslant (1-\delta(n))V_0(r)$ for some $p \in M$ and all $r > 0$ is diffeomorphic to $\mathbb{R}^n$. Thus combining the Cheeger–Colding theorem and Theorem 1.1, one deduces the following topological rigidity for manifolds with non-negative Ricci curvature.

Corollary 1.3. *Given integer $n \geqslant 3$, there exists a positive constant $\epsilon = \epsilon(n,a,b)$ such that any $n$-dimensional ($n \geqslant 3$) complete non-compact Riemannian manifold $M$ with non-negative Ricci curvature in which the inequality*

$$\left(\int_M \rho^{-bp}|u|^p\,dv\right)^{1/p} \leqslant (K_{a,b}+\epsilon)\left(\int_M \rho^{-2a}|\nabla u|^2\,dv\right)^{1/2}, \quad \forall u \in C_0^\infty(M),$$

*is satisfied, is diffeomorphic to $\mathbb{R}^n$.*

A theorem due to Li [Li86] and Anderson [And90] states that, if $M$ is an $n$-dimensional complete manifold with non-negative Ricci curvature in which the inequality $\mathrm{vol}[B(p,r)] \geqslant \alpha V_0(r)$ holds for some constant $\alpha > 0$ and all $r > 0$, the fundamental group $\pi_1(M)$ is finite and ${}^{\#}\pi_1(M) \leqslant 1/\alpha$. Thus from the Li–Anderson theorem and Theorem 1.1 we have the following corollary.

Corollary 1.4. *Let $C \geqslant K_{a,b}$ be a constant and $M$ be an $n$-dimensional ($n \geqslant 3$) complete open Riemannian manifold with non-negative Ricci curvature. Assume that, for any $u \in C_0^\infty(M)$, we have*

$$\left(\int_M \rho^{-bp}|u|^p\,dv\right)^{1/p} \leqslant C\left(\int_M \rho^{-2a}|\nabla u|^2\,dv\right)^{1/2}. \tag{1.8}$$

*Then $M$ has finite fundamental group and the order of $\pi_1(M)$ is bounded above by $(K_{a,b}^{-1}C)^{n/(1+a-b)}$.*

One can find some related results about the topology of complete manifolds with non-negative Ricci curvature, for example, in [AG90, And90, CX00, Col98, Li86, OSY00, Ots89, SS97, She93, She96, SS01, Sor00, Xia99].

## 2. A Proof of Theorem 1.1

First notice the following fact. The Bishop–Gromov comparison theorem (cf. [BC64, Cha93, GLP81]) tells us that for any $p \in M$ the function $\mathrm{vol}[B(p,r)]/V_0(r)$ is decreasing and so the limit

$$\lim_{r\to+\infty} \frac{\mathrm{vol}[B(p,r)]}{V_0(r)}$$

exists. Also one can easily check that the above limit does not depend on the choice of $p$. It then follows that if (1.7) holds for some point $p_0 \in M$, then it is satisfied for all $x \in M$. Now we are going to show that (1.7) holds at the point $x_0$.

Set

$$w = 2a - bp + 2, \quad q = \frac{(n-2a-2)p}{2a-bp+2} = \frac{2p}{p-2}, \tag{2.1}$$

and, for any $\lambda > 0$, let

$$F(\lambda) = \frac{p-2}{p+2}\int_M \frac{dv}{\rho^{bp}(\lambda+\rho^w)^{q-1}}. \tag{2.2}$$

Then, for $\lambda > 0$, we have from the Fubini theorem (cf. [SY94]) that

$$F(\lambda) = \frac{p-2}{p+2}\int_0^{+\infty} \mathrm{vol}\left\{x : \frac{1}{\rho^{bp}(\lambda+\rho^w)^{q-1}} > s\right\} ds.$$



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Making the variable change $s = 1(h^{bp}(\lambda + h^w)^{q-1})$ in the above equality, one concludes that

$$F(\lambda) = \frac{p-2}{p+2}\int_0^{+\infty} \mathrm{vol}\{x : \rho(x) < h\}\frac{(bp\lambda + (bp + (q-1)w)h^w)}{h^{bp+1}(\lambda + h^w)^q}\,dh$$
$$= \frac{p-2}{p+2}\int_0^{+\infty} \mathrm{vol}[B(x_0, h)]\frac{(bp\lambda + (bp + (q-1)w)h^w)}{h^{bp+1}(\lambda + h^w)^q}\,dh. \tag{2.3}$$

Since the Bishop–Gromov comparison theorem implies that $\mathrm{vol}[B(x_0, h)] \leqslant \omega_n h^n$, we have

$$F(\lambda) \leqslant \frac{\omega_n(p-2)}{p+2}\int_0^{+\infty}(bp\lambda + (bp + (q-1)w)h^w)h^{n-bp-1}(\lambda + h^w)^{-q}\,dh.$$

On the other hand, one can deduce from (1.4), (1.5), and (2.1) that

$$n - bp - 1 > -1, \quad n - bp - 1 + w(1-q) < -1.$$

It then follows that $0 \leqslant F(\lambda) < +\infty$, $\forall\lambda > 0$, and that $F$ is differentiable. Also, we have

$$F'(\lambda) = -\int_M \frac{dv}{\rho^{bp}(\lambda + \rho^w)^q}. \tag{2.4}$$

Consider the function $H_0 : (0, +\infty) \to \mathbb{R}$ defined by

$$H_0(\lambda) = \frac{p-2}{p+2}\int_{\mathbb{R}^n}\frac{dx}{|x|^{bp}(\lambda + |x|^w)^{q-1}}.$$

Recall that when $M = \mathbb{R}^n$ and $C = K_{a,b}$, the extremal functions in the inequality (1.6) are the functions $u_\lambda := (\lambda + |x|^w)^{-q/p}$, $\lambda > 0$. That is, we have

$$(-H_0'(\lambda))^{2/p} = \left(\int_{\mathbb{R}^n}\frac{dx}{|x|^{bp}(\lambda + |x|^w)^q}\right)^{2/p}$$
$$= \left(\frac{K_{a,b}qw}{p}\right)^2\int_{\mathbb{R}^n}\frac{dx}{|x|^{2(1+a-w)}(\lambda + |x|^w)^{2+2q/p}}$$
$$= \left(\frac{K_{a,b}qw}{p}\right)^2\int_{\mathbb{R}^n}\frac{dx}{|x|^{bp-w}(\lambda + |x|^w)^q}$$
$$= \left(\frac{K_{a,b}qw}{p}\right)^2\left(H_0'(\lambda) + \frac{p+2}{p-2}H_0(\lambda)\right).$$

Substituting $H_0(\lambda) = H_0(1)\lambda^{-2/(p-2)}$ into the above equation, one gets

$$H_0(1) = \frac{p-2}{p+2}\int_{\mathbb{R}^n}\frac{dx}{|x|^{bp}(1 + |x|^w)^{q-1}}$$
$$= 2^{2/(p-2)}(p-2)((n-2a-2)^2K_{a,b}^2)^{-p/(p-2)}. \tag{2.5}$$

By a simple approximation procedure, we can apply (1.6) to $(\lambda + \rho^w)^{-q/p}$ for every $\lambda > 0$ to get

$$\left(\int_M \frac{dv}{\rho^{bp}(\lambda + \rho^w)^q}\right)^{2/p} \leqslant \left(\frac{qwC}{p}\right)^2\int_M\frac{dv}{\rho^{2(1+a-w)}(\lambda + \rho^w)^{2+2q/p}}$$
$$= \left(\frac{qwC}{p}\right)^2\int_M\frac{dv}{\rho^{bp-w}(\lambda + \rho^w)^q}.$$

Let $l = (p/qwC)^2$; then the above inequality becomes

$$l(-F'(\lambda))^{2/p} - \lambda F'(\lambda) \leqslant \frac{p+2}{p-2}F(\lambda). \tag{2.6}$$



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The idea now is to compare the solutions of (2.6) to the solutions $H$ of the following differential equality:

$$l(-H'(\lambda))^{2/p} - \lambda H'(\lambda) = \frac{p+2}{p-2} H(\lambda). \tag{2.7}$$

One can easily check that $H_1(\lambda)$ given by

$$H_1(\lambda) := A\lambda^{-2/(p-2)} \tag{2.8}$$

is a particular solution of (2.7), where

$$\begin{aligned} A &= 2^{2/(p-2)}(p-2)\left(\frac{l}{p}\right)^{p/(p-2)} \\ &= 2^{2/(p-2)}(p-2)((n-2a-2)^2 pC^2)^{-p/(p-2)} \\ &= (C^{-1}K_{a,b})^{2p/(p-2)} \cdot 2^{2/(p-2)}(p-2) \cdot ((n-2a-2)^2 pK_{a,b}^2)^{-p/(p-2)} \\ &= (C^{-1}K_{a,b})^{2p/(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(1+|x|^w)^{q-1}} \\ &= (C^{-1}K_{a,b})^{n/(1+a-b)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(1+|x|^w)^{q-1}}. \end{aligned} \tag{2.9}$$

Observe that

$$\begin{aligned} H_1(\lambda) &= (C^{-1}K_{a,b})^{n/(1+a-b)} \cdot \lambda^{-2/(p-2)} \cdot \frac{p-2}{p+2} \int_{\mathbb{R}^n} \frac{dx}{|x|^{bp}(1+|x|^w)^{q-1}} \\ &= (C^{-1}K_{a,b})^{n/(1+a-b)} H_0(\lambda). \end{aligned} \tag{2.10}$$

Before we can conclude the proof of Theorem 1.1, we shall need the following two lemmas.

Lemma 2.1. *If for some $\lambda_0 > 0$, $F(\lambda_0) < H_1(\lambda_0)$, then $F(\lambda) < H_1(\lambda)$ $\forall \lambda \in (0, \lambda_0]$.*

*Proof.* Suppose that Lemma 2.1 is false. Set

$$\lambda_1 = \sup\{\lambda < \lambda_0; F(\lambda) = H_1(\lambda)\}.$$

For each $\lambda > 0$, the function $\phi_\lambda : [0, +\infty) \to \mathbb{R}$ defined by

$$\phi_\lambda(s) = ls^{2/p} + \lambda s$$

is increasing. By (2.6), we have

$$\phi_\lambda(-F'(\lambda)) \leqslant \frac{p+2}{p-2} F(\lambda),$$

which gives

$$-F'(\lambda) \leqslant \phi_\lambda^{-1}\left(\frac{p+2}{p-2} F(\lambda)\right).$$

On the other hand, (2.7) implies that

$$-H_1'(\lambda) = \phi_\lambda^{-1}\left(\frac{p+2}{p-2} H_1(\lambda)\right).$$

Thus, on the subset $\{s \mid F(s) \leqslant H_1(s)\}$, we have

$$F'(\lambda) - H_1'(\lambda) \geqslant \phi_\lambda^{-1}\left(\frac{p+2}{p-2} H_1(\lambda)\right) - \phi_\lambda^{-1}\left(\frac{p+2}{p-2} F(\lambda)\right).$$

Since $(F - H_1)|_{[\lambda_1, \lambda_0]} \leqslant 0$, we conclude therefore that $(F - H_1)' \leqslant 0$ on $[\lambda_1, \lambda_0]$. Consequently, one gets

$$0 = (F - H_1)(\lambda_1) \leqslant (F - H_1)(\lambda_0) < 0.$$

This is a contradiction and completes the proof of Lemma 2.1. □

Lemma 2.2. *We have*

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_0(\lambda)}\geqslant 1. \tag{2.11}$$

*Proof.* Fix a small $\epsilon>0$. Since

$$\lim_{u\to 0}\frac{\mathrm{vol}[B(x_0,u)]}{V_0(u)}=1,$$

there exists a $\delta>0$ such that $\mathrm{vol}[B(x_0,h)]\geqslant(1-\epsilon)V_0(h)$, $\forall h\leqslant\delta$.

It then follows from (2.3) that

$$\begin{aligned}F(\lambda)&\geqslant\frac{p-2}{p+2}(1-\epsilon)\int_0^{\delta}V_0(h)\frac{(bp\lambda+(bp+(q-1)w)h^w)}{h^{bp+1}(\lambda+h^w)^q}\,dh\\&=\frac{p-2}{p+2}(1-\epsilon)\lambda^{[(n+bp)/w]+1-q}\int_0^{\delta/\lambda^{1/w}}V_0(s)\frac{(bp+(bp+(q-1)w)s^w)}{s^{bp+1}(1+s^w)^q}\,ds\\&=\frac{p-2}{p+2}(1-\epsilon)\lambda^{-2/(p-2)}\int_0^{\delta/\lambda^{1/w}}V_0(s)\frac{(bp+(bp+(q-1)w)s^w)}{s^{bp+1}(1+s^w)^q}\,ds.\end{aligned}$$

On the other hand, it is easy to see that

$$H_0(\lambda)=\frac{p-2}{p+2}\lambda^{-2/(p-2)}\int_0^{+\infty}V_0(s)\frac{(bp+(bp+(q-1)w)s^w)}{s^{bp+1}(1+s^w)^q}\,ds.$$

We conclude therefore that

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_0(\lambda)}\geqslant 1-\epsilon.$$

Letting $\epsilon\to 0$, one gets

$$\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_0(\lambda)}\geqslant 1. \tag{2.12}$$

This completes the proof of Lemma 2.2. □

Now we continue on the proof of Theorem 1.1. We separate the proof into two cases.

*Case 1: $C>K_{a,b}$.* In this case, it follows from (2.10) and Lemma 2.2 that

$$\begin{aligned}\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_1(\lambda)}&=\left(\frac{C}{K_{a,b}}\right)^{n/(1+a-b)}\liminf_{\lambda\to 0}\frac{F(\lambda)}{H_0(\lambda)}\\&\geqslant\left(\frac{C}{K_{a,b}}\right)^{n/(1+a-b)}>1,\end{aligned} \tag{2.13}$$

which, combining with Lemma 2.1, implies that

$$F(\lambda)\geqslant H_1(\lambda),\quad\forall\lambda>0. \tag{2.14}$$

That is, for any $\lambda>0$, we have

$$\int_0^{+\infty}(\mathrm{vol}[B(x_0,s)]-(C^{-1}K_{a,b})^{n/(1+a-b)}V_0(s))\frac{bp\lambda+(bp+(q-1)w)s^w}{s^{bp+1}(\lambda+s^w)^q}\,ds\geqslant 0. \tag{2.15}$$

Recall that the Bishop–Gromov comparison theorem says that the function $|B(x_0,s)|/V_0(s)$ is decreasing. Set $d=(C^{-1}K(n,q))^{n/(1+a-b)}$ and assume that

$$\lim_{s\to+\infty}\frac{|B(x_0,s)|}{V_0(s)}=d_0.$$



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The proof of Theorem 1.1 will be completed if we can show that $d_0 \geqslant d$. We prove this fact by contradiction. Thus suppose that $d_0 = d - \epsilon_0$, for some $\epsilon_0 > 0$. Then there exists an $N_0 > 0$ such that

$$\frac{\operatorname{vol}[B(x_0, s)]}{V_0(s)} \leqslant d - \frac{\epsilon_0}{2}, \quad \forall s \geqslant N_0. \tag{2.16}$$

By introducing (2.16) into (2.15), one derives for every $\lambda > 0$ that

$$\begin{aligned}
0 &\leqslant \int_0^{+\infty} \left( \frac{\operatorname{vol}[B(x_0, s)]}{V_0(s)} - d \right) \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\leqslant \int_0^{N_0} \frac{\operatorname{vol}[B(x_0, s)]}{V_0(s)} \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad + \int_{N_0}^{+\infty} \left( d - \frac{\epsilon_0}{2} \right) \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad - d \int_0^{+\infty} \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\leqslant \int_0^{N_0} \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad + \int_{N_0}^{+\infty} \left( d - \frac{\epsilon_0}{2} \right) \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad - d \int_0^{+\infty} \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&= \int_0^{N_0} \left( 1 - d + \frac{\epsilon_0}{2} \right) \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad - \frac{\epsilon_0}{2\omega_n} \int_0^{+\infty} \frac{V_0(s)(bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&= \int_0^{N_0} \left( 1 - d + \frac{\epsilon_0}{2} \right) \frac{s^n (bp\lambda + (bp + (q-1)w)s^w)}{s^{bp+1}(\lambda + s^w)^q} \, ds \\
&\quad - \frac{\epsilon_0}{2\omega_n} \cdot \frac{p+2}{p-2} \cdot H_0(\lambda) \\
&\leqslant \left( 1 - d + \frac{\epsilon_0}{2} \right) \lambda^{-q} \int_0^{N_0} (bp\lambda s^{n-bp-1} + (bp + (q-1)w)s^{n+w-bp-1}) \, ds \\
&\quad - \frac{\epsilon_0}{2\omega_n} \cdot \frac{p+2}{p-2} \cdot \lambda^{-2/(p-2)} \cdot H_0(1) \\
&= \left( 1 - d + \frac{\epsilon_0}{2} \right) \lambda^{-q} \left( \frac{\lambda bp N_0^{n-bp}}{n - bp} + \frac{(bp + (q-1)w)N_0^{n+w-bp}}{n + w - bp} \right) \\
&\quad - \frac{\epsilon_0 (p+2) H_0(1)}{2\omega_n (p-2)} \cdot \lambda^{-2/(p-2)},
\end{aligned}$$

which implies for any $\lambda > 0$ that

$$\frac{\epsilon_0 (p+2) H_0(1)}{2\omega_n (p-2)(1 - d + \epsilon_0/2)} \leqslant \lambda^{2/(p-2)-q} \left( \frac{\lambda bp N_0^{n-bp}}{n - bp} + \frac{(bp + (q-1)w)N_0^{n+w-bp}}{n + w - bp} \right).$$

Letting $\lambda \to +\infty$ in the above inequality and observing that $2/(p-2) - q + 1 < 0$, one obtains the desired contradiction. Thus $d_0 \geqslant d$. This completes the proof of Theorem 1.1 in the case that $C > K_{a,b}$.



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*Case 2:* $C = K_{a,b}$. In this case, we have for any fixed $\delta > 0$ that

$$\left(\int_M \rho^{-bp}|u|^p\,dv\right)^{1/p} \leqslant (K_{a,b}+\delta)\left(\int_M \rho^{-2a}|\nabla u|^2\,dv\right)^{1/2}.$$

Thus for any $x \in M$ we have from *case 1* that

$$\mathrm{vol}[B(x,r)] \geqslant \left(\frac{K_{a,b}}{K_{a,b}+\delta}\right)^{n/(1+a-b)} V_0(r), \quad \forall r > 0.$$

Letting $\delta \to 0$, one obtains that

$$\mathrm{vol}[B(x,r)] \geqslant V_0(r), \quad \forall r > 0.$$

This completes the proof of Theorem 1.1 for the case that $C = K_{a,b}$.

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