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CONCERNING THE CONE = HYPERSPACE PROPERTY

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Introduction. In this paper it is shown that a sufficient condition for a continuum X to have the cone = hyperspace property is that there exists a selection for $C(X)\backslash\{X\}$ which, for some Whitney map for $C(X)$, maps each nondegenerate Whitney level homeomorphically onto X. Also, we construct an example of a one-dimensional, nonchainable, noncircle-like continuum which has the cone = hyperspace property. The continuum is described by means of inverse limits using only one bonding map. Each factor space in the inverse limit sequence is the quotient space resulting from an upper semi-continuous decomposition of a disjoint union of simple triods. The bonding map is an adaptation of the bonding map defined by W. T. Ingram in his construction of an atriodic, tree-like continuum which is not chainable [**4**].

Definitions, notation, and terminology. By *continuum* we mean a nonempty, compact, connected metric space. If X is a continuum with metric d, the *hyperspace of subcontinua* $C(X)$ is the space of all subcontinua of X metrized by the Hausdorff metric ρ, that is,

$$\rho(A, B) = \inf\{\epsilon > 0 | A \subseteq N_d(\epsilon, B) \text{ and } B \subseteq N_d(\epsilon, A)\},$$

where $N_d(\epsilon, A)$ is the set to which the point x belongs if and only if $d(x, A) < \epsilon$. The subspace of $C(X)$ consisting of the degenerate subcontinua of X is denoted by $F_1(X)$.

The *cone over* X is the decomposition space of the upper semicontinuous decomposition of $X \times [0, 1]$ obtained by identifying $X \times \{1\}$ to a point. The cone over X is denoted by Cone (X), its base $X \times \{0\}$ by $B(X)$, and its vertex $X \times \{1\} \in$ Cone (X) by v. We let p_1 be the projection mapping of Cone $(X)\backslash\{v\}$ onto X, and p_2 the projection mapping of Cone (X) onto $[0, 1]$. If X is a continuum such that Cone (X) is homeomorphic to $C(X)$,

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then X is said to be a C-H continuum. A C-H continuum for which there exists a homeomorphism that maps $v \in \text{Cone}(X)$ to $X \in C(X)$ and also maps $B(X)$ onto $F_1(X)$ is said to have the *cone = hyperspace property*.

The term mapping refers to continuous function. A *Whitney map* for $C(X)$ is a mapping $\mu:C(X) \to [0, \infty)$ satisfying the following:

1) $\mu(\{x\}) = 0$ for each $x \in X$, and
2) if $A \subsetneq B$, then $\mu(A) < \mu(B)$.

Suppose that $X_1, X_2, \dots$ is a sequence of compact metric spaces each having diameter less than a fixed positive number K, and suppose that f_1^2, $f_2^3, \dots$ is a sequence of mappings such that

$$f_n^{n+1}:X_{n+1} \to X_n \quad \text{for } n = 1, 2, \dots .$$

The inverse limit $\varprojlim \{X_n, f_n^{n+1}\}$ of the inverse limit sequence $\{X_n, f_n^{n+1}\}$ is the subset of the product $\prod_{n=1}^{\infty} X_n$ to which $(x_1, x_2, \dots)$ belongs if and only if

$$f_n^{n+1}(x_{n+1}) = x_n \quad \text{for } n = 1, 2, \dots .$$

We consider $\prod_{n=1}^{\infty} X_n$ metrized by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n),$$

where d_n denotes the metric on X_n for each positive integer n. For $i = 1, 2, \dots$, let π_i be the i^{th}-projection mapping of $\varprojlim \{X_n, f_n^{n+1}\}$ into X_i, that is,

$$\pi_i((x_1, x_2, \dots, x_i, \dots)) = x_i \quad \text{for each } (x_1, x_2, \dots) \in \varprojlim \{X_n, f_n^{n+1}\}.$$

For positive integers $i < j$, f_i^j denotes the composite mapping

$$f_i^{i+1} f_{i+1}^{i+2} \dots f_{j-1}^j : X_j \to X_i,$$

with composition of mappings denoted by juxtaposition. Thus if f_1^2, $f_2^3, \dots$ is a constant sequence, where $f_n^{n+1} = f$ for each positive integer n, then $f_i^j = f^{j-i}$.

If $f:X \to Y$ is a mapping of a continuum X into a continuum Y, then $\hat{f}:C(X) \to C(Y)$ defined by $\hat{f}(A) = \{f(a) | a \in A\}$, for each $A \in C(X)$, is the mapping induced by f.

In [**19**] J. Segal proved that the hyperspace operation C commutes with inverse limits. We state here a portion of this theorem as it is given in [**12**, 1.169].

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THEOREM. *Let X be a continuum and assume that*

$$X = \varprojlim \{X_n, f_n^{n+1}\},$$

where each of the spaces X_n is a continuum. Let

$$C_\infty(X) = \varprojlim \{C(X_n), \hat{f}_n^{n+1}\}.$$

Then $C_\infty(X)$ is homeomorphic to $C(X)$.

If M is a connected polyhedron, then a continuum X is said to be *M-like* provided that, for each $\epsilon > 0$, there exists an ϵ-mapping from X onto M. In [**6**] Mardešić and Segal show that a continuum X is M-like if and only if X is homeomorphic to the inverse limit of an inverse limit sequence $\{X_n, f_n^{n+1}\}$ such that $X_n = M$ and f_n^{n+1} maps X_{n+1} onto X_n for $n = 1, 2, \ldots$. The term chainable is synonymous with arc-like.

Selections and the cone = hyperspace property. In [**7**] E. Michael began the study of selections. The following definition of selection is restricted to the class of continua and is as stated by S. B. Nadler [**12**, Chapter V]. Suppose that X is a continuum and Γ is a subset of the hyperspace $C(X)$. A function $f:\Gamma \to X$ is called a *selection for* Γ provided that f is continuous and for each $A \in \Gamma$, $f(A) \in A$.

THEOREM 3.1. *Suppose that $\nu:C(X)\backslash\{X\} \to X$ is a selection for $C(X)\backslash\{X\}$. If there exists a Whitney map $\mu:C(X) \to [0, 1]$ for $C(X)$ such that $\nu|\mu^{-1}(t)$ is a homeomorphism from $\mu^{-1}(t)$ onto X for $0 \leqq t < 1$, then X has the cone = hyperspace property.*

Proof. Let $h:C(X) \to \text{Cone}(X)$ be defined as follows:

$$h(A) = \begin{cases} (\nu(A), \mu(A)) & \text{if } A \neq X, \\ v & \text{if } A = X. \end{cases}$$

Clearly h is a function from $C(X)$ into Cone (X) which is continuous on $C(X)\backslash\{X\}$ and satisfies $h(X) = v$. We wish to show that h is a homeomorphism from $C(X)$ onto Cone (X), and furthermore, that h maps $F_1(X)$ onto $B(X)$.

To establish that the image under h of $C(X)$ is Cone (X), let $(x, t) \in$ Cone $(X)\backslash\{v\}$. Then $0 \leqq t < 1$, and so $\nu|\mu^{-1}(t)$ is a mapping of $\mu^{-1}(t)$ onto X. Hence there exists $A \in \mu^{-1}(t)$ such that $\nu(A) = x$; since $t \neq 1$, then $A \neq X$, and thus

$$h(A) = (\nu(A), \mu(A)) = (x, t).$$

In order to prove that h is continuous at X, let $A_1, A_2, \ldots$ be a sequence in $C(X)$ converging to X. Since μ is continuous on $C(X)$, the sequence

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$\mu(A_1), \mu(A_2), \ldots$ converges to 1, and hence $h(A_1), h(A_2), \ldots$ converges to $v = h(X)$. Therefore h is a mapping from $C(X)$ onto Cone (X).

Suppose now that $h(A) = h(B)$. In case $h(A) = v = h(B)$, then we have that $A = X = B$. Assume then that $h(A) \neq v$. This implies that $\nu(A) = \nu(B)$ and $\mu(A) = \mu(B)$, where $0 \leqq \mu(A) < 1$. But $\nu|\mu^{-1}(\mu(A))$ is a homeomorphism and so $A = B$. Since the spaces are compact metric, this establishes that h is a homeomorphism from $C(X)$ onto Cone (X).

Now $\mu^{-1}(0) = F_1(X)$ and $\nu|\mu^{-1}(0)$ maps $F_1(X)$ onto X, thus $h|F_1(X)$ maps $F_1(X)$ onto $B(X)$. This concludes the proof that X has the cone = hyperspace property.

A nonchainable, noncircle-like continuum with the cone = hyperspace property. In [**15**] J. T. Rogers proved that if X is a finite-dimensional continuum and has the cone = hyperspace property, then X is an arc, a circle, or an indecomposable continuum each nondegenerate proper subcontinuum of which is an arc. Subsequently, in [**11**] Nadler proved that if X is a finite dimensional, indecomposable C-H continuum, then every homeomorphism from the cone over X to the hyperspace $C(X)$ must map the base $B(X)$ onto $F_1(X)$ and map the vertex v to $X \in C(X)$. Hence X has the cone = hyperspace property. As a result of the structure required of a finite-dimensional, indecomposable C-H continuum, Nadler asked the following question: "Must a finite-dimensional, indecomposable C-H continuum be chainable or circle-like?" [**12**, 8.14, p 310]. In this section we answer that question by constructing a finite-dimensional, indecomposable C-H continuum which is neither chainable nor circle-like. In [**3**] A. M. Dilks and J. T. Rogers prove that the plane continuum described by R. H. Bing in [**1**, p 222] answers this question. For completeness and notational convenience we include a description of the nonchainable, atriodic, tree-like continuum defined by Ingram in [**4**].

Example 4.1. (The mapping f and the continuum Y.) Let T denote the simple triod

$$\{(r, \theta) | 0 \leqq r \leqq 1 \text{ and } \theta = 0, \theta = \tfrac{1}{2}\pi, \text{ or } \theta = \pi\}$$

in polar coordinates in the plane. We denote by J the junction point $(0, 0) = (0, \frac{1}{2}\pi) = (0, \pi)$, by A the point $(1, \frac{1}{2}\pi)$, by B the point $(1, \pi)$, and by C the point $(1, 0)$. If $0 \leqq r \leqq 1$, we let rA be the point $(r, \frac{1}{2}\pi)$, while rB denotes (r, π), and rC denotes $(r, 0)$. Thus the triod T is the sum of the three arcs JA, JB, and JC, and the junction point J may also be denoted by $0A$, $0B$, or $0C$. Let d_1 be the metric on T defined as follows:

If each of r_1V_1 and r_2V_2 belongs to T, where $0 \leqq r_i \leqq 1$ and $V_i \in \{A, B, C\}$ for $i = 1, 2$, then

$$d_1(r_1V_1, r_2V_2) = \begin{cases} |r_1 - r_2| & \text{if } V_1 = V_2, \\ r_1 + r_2 & \text{otherwise.} \end{cases}$$

Thus T has diameter 2, and the distance between points in T measures the length of the shortest arc in T which contains those points.

Define $f:T \to T$ as follows:

$$f(rA) = \begin{cases} (1 - 4r)B & \text{if } 0 \leqq r \leqq 1/4, \\ (4r - 1)A & \text{if } 1/4 \leqq r \leqq 1/2, \\ (3 - 4r)A & \text{if } 1/2 \leqq r \leqq 3/4, \\ (4r - 3)C & \text{if } 3/4 \leqq r \leqq 1. \end{cases}$$

$$f(rB) = \begin{cases} (1 - 3r)B & \text{if } 0 \leqq r \leqq 1/3, \\ (3r - 1)A & \text{if } 1/3 \leqq r \leqq 1/2, \\ (2 - 3r)A & \text{if } 1/2 \leqq r \leqq 2/3, \\ (3r - 2)C & \text{if } 2/3 \leqq r \leqq 1. \end{cases}$$

$$f(rC) = \begin{cases} (1 - 2r)B & \text{if } 0 \leqq r \leqq 1/2, \\ (2r - 1)C & \text{if } 1/2 \leqq r \leqq 1. \end{cases}$$

Figure 1 is a schematic representation of the mapping f.

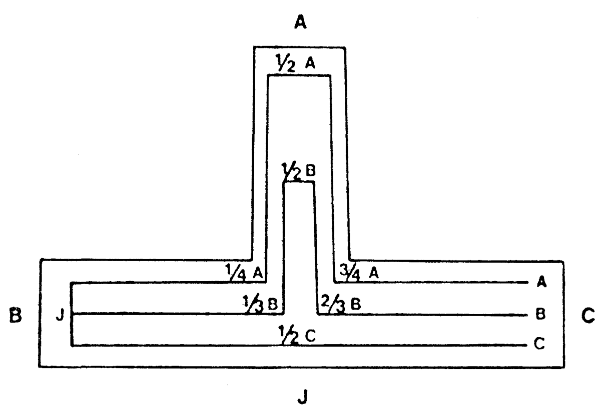

Figure 1

For each positive integer n, put $T_n = T$ and $f_n^{n+1} = f$. Let

$$Y = \varprojlim \{T_n, f_n^{n+1}\},$$

and ρ_1 denote the metric on Y determined by d_1.

Example 4.2. (The mapping g and the continuum X.) Let (T, d_1) and $f:T \to T$ be as defined in Example 4.1. We define an upper semi-continuous decomposition D of the disjoint union $T \times \{1, 2\}$ by listing the only nondegenerate elements of D:$\{ (A, 1), (A, 2) \}$, $\{ (B, 1), (B, 2) \}$, $\{ (C, 1), (C, 2) \}$, $\{ (J, 1), (J, 2) \}$, and $\{ (\frac{1}{2}A, 1), (\frac{1}{2}A, 2) \}$. Let R be the equivalence relation for the decomposition D. Put $S = [T \times \{1, 2\}]/R$ and let $\psi:T \times \{1, 2\} \rightarrow S$ be the quotient map.

If $V \in \{A, B, C, J, \frac{1}{2}A\}$, we shall denote the element $\psi[\{ (V, 1), (V, 2) \}]$ in S by V; for $V \in T \backslash \{A, B, C, J, \frac{1}{2}A\}$, we shall write (V, i) for the element $\psi(\{ (V, i) \})$, $i = 1, 2$. We consider $S = \cup_{i=1}^{8} Z_i$, where Z_i is given as follows:

$$Z_1 = \psi[JB \times \{1\}],\ Z_2 = \psi[JB \times \{2\}],$$
$$Z_3 = \psi[J\tfrac{1}{2}A \times \{1\}],\ Z_4 = \psi[J\tfrac{1}{2}A \times \{2\}],$$
$$Z_5 = \psi[\tfrac{1}{2}AA \times \{1\}],\ Z_6 = \psi[\tfrac{1}{2}AA \times \{2\}],$$
$$Z_7 = \psi[JC \times \{1\}],\ Z_8 = \psi[JC \times \{2\}].$$

Refer to Figure 2 for a diagram of the continuum S.

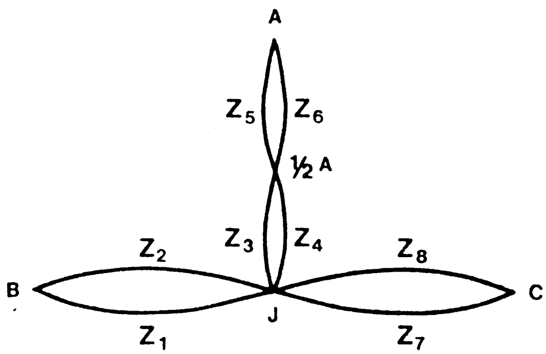

Figure 2

Suppose that M is a subcontinuum of S. Then for $1 \leqq i \leqq 8$, $\overline{M \cap Z_i}$ has at most two components. By considering $\psi^{-1}[\text{int } Z_i]$ as an arc in T, where int Z_i denotes the interior of the arc Z_i, let $l(M \cap Z_i)$ equal the sum of the diameters of the components of $\overline{\psi^{-1}[M \cap \text{int } Z_i]}$, where $l(M \cap Z_i) = 0$ if M does not intersect Z_i. Denote by d the metric on S determined as follows: if each of x and y belongs to S, then

$$d(x, y) = \inf \{\sum_{i=1}^{8} l(M \cap Z_i) | M \in C(S) \text{ and } M \text{ contains } x \text{ and } y\}.$$

We have that the distance between two points of S coincides with the length of a shortest arc in S which contains them, and the diameter of S is 2.

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We now define a relation G on $[T \times \{1, 2\}] \times [T \times \{1, 2\}]$ which will induce a mapping $g:S \to S$. The relation G is given as follows:

$$G((rA, i)) = \begin{cases} (f(rA), i) & \text{if } 0 \leqq r \leqq 1/4, \\ & 3/8 \leqq r \leqq 1/2, \text{ or} \\ & 5/8 \leqq r \leqq 3/4, \\ (f(rA), j) & \text{where } j \in \{1, 2\}, j \neq i, \\ & \text{if } 1/4 \leqq r \leqq 3/8, \\ & 1/2 \leqq r \leqq 5/8, \text{ or} \\ & 3/4 \leqq r \leqq 1. \end{cases}$$

$$G((rB, i)) = \begin{cases} (f(rB), i) & \text{if } 0 \leqq r \leqq 1/3 \text{ or,} \\ & 1/2 \leqq r \leqq 2/3, \\ (f(rB), j) & \text{where } j \in \{1, 2\}, j \neq i, \\ & \text{if } 1/3 \leqq r \leqq 1/2 \text{ or} \\ & 2/3 \leqq r \leqq 1. \end{cases}$$

$$G((rC, i)) = \begin{cases} (f(rC), i) & \text{if } 0 \leqq r \leqq 1/2, \\ (f(rC), j) & \text{where } j \in \{1, 2\}, j \neq i, \\ & \text{if } 1/2 \leqq r \leqq 1. \end{cases}$$

Suppose that $s \in S$ such that $s \in \{A, B, C, J, \frac{1}{2}A\}$, then $\psi^{-1}(s)$ is a nondegenerate element of D, say $\psi^{-1}(s) = \{(x, 1), (x, 2)\}$, and $f(x) \in \{A, B, C\}$. Thus each of $((x, 1), (f(x), 1))$, $((x, 1), (f(x), 2))$, $((x, 2), (f(x), 1))$, and $((x, 2), (f(x), 2))$ belongs to G. But $f(x) \in \{A, B, C\}$ implies that

$$\psi((f(x), 1)) = \psi((f(x), 2));$$

consequently, $\psi G \psi^{-1}(s)$ denotes exactly one element of S. In case $s \in S \setminus \{A, B, C, J, \frac{1}{2}A\}$, say $s = (x, i)$, and each of $((x, i), (f(x), 1))$, and $((x, i), (f(x), 2))$ belongs to G, then $f(x) \in \{J, \frac{1}{2}A\}$, and hence

$$\psi((f(x), 1)) = \psi((f(x), 2)).$$

So again $\psi G \psi^{-1}(s)$ denotes exactly one element of S. Thus $g:S \to S$ defined by $g(s) = \psi G \psi^{-1}(s)$ for each $s \in S$ is a piecewise linear mapping from S onto S. Figure 3 provides schematic representations of the mappings $g|Z_i$ for each integer $1 \leqq i \leqq 8$. By superimposing the eight representations in Figure 3, we obtain the schematic of the mapping g given in Figure 4.

For each positive integer n, put $S_n = S$ and $g_n^{n+1} = g$. Let

$$X = \varprojlim \{S_n, g_n^{n+1}\}.$$

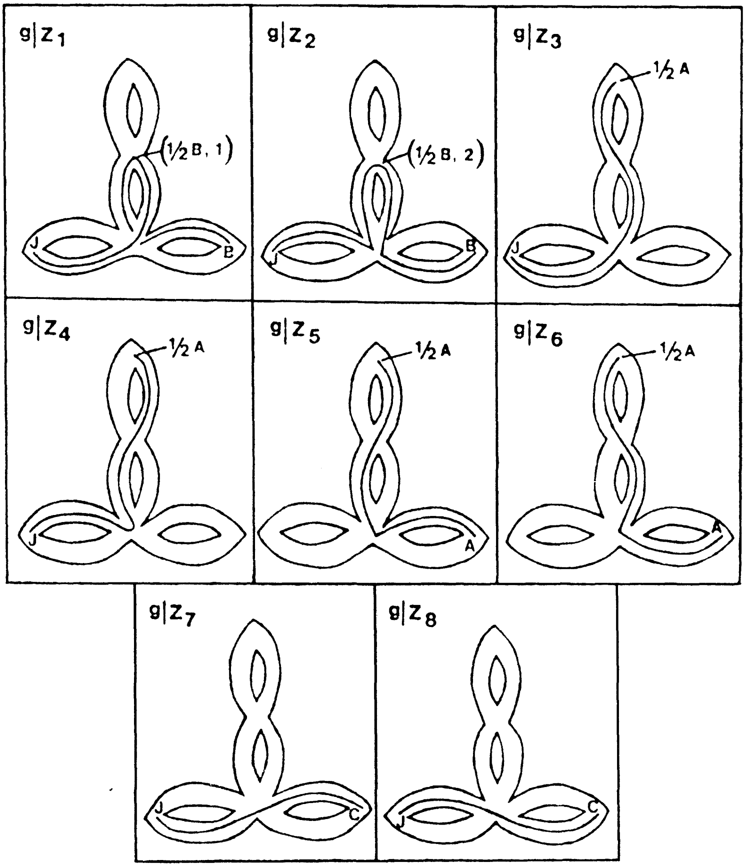

Figure 3

We denote by ρ the metric on X determined by d; the diameter of X does not exceed 2.

Throughout the remainder of this paper Y and X will denote the continua as defined in Examples 4.1 and 4.2, respectively.

In [**4**, Theorem 1] it is proved that every nondegenerate, proper subcontinuum of Y is chainable. The proof of the following theorem is similar.

THEOREM 4.3. *Every nondegenerate, proper subcontinuum of X is an arc.*

Proof. Suppose that α is a nondegenerate, proper subcontinuum of X. Let N denote a positive integer such that if m is an integer and $m \geqq N$,

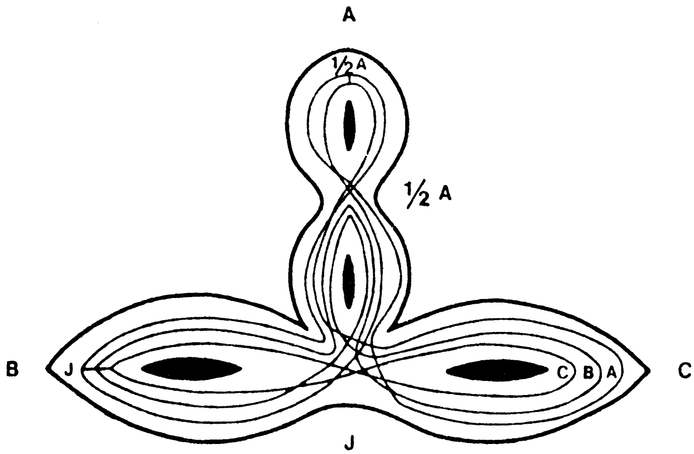

Figure 4

then the projection $\pi_m[\alpha]$ is a proper subcontinuum of S_m. Suppose, by way of contradiction, that there exist infinitely many integers i for which $J \in \pi_i[\alpha]$. Select positive integers j and k such that $j \geqq N + 3$, $k \geqq j + 2$, $J \in \pi_j[\alpha]$, and $J \in \pi_k[\alpha]$. If $n \geqq 2$, then $g^n(J) = C$. Thus

$$g_j^k[\pi_k[\alpha]] = g^{k-j}[\pi_k[\alpha]]$$

contains $\{J, C\}$ as a subset, is a subcontinuum of S_j, and contains at least one of Z_7 or Z_8 as a subset. Suppose that $Z_7 \subset g_j^k[\pi_k[\alpha]]$, then

$$g_{j-1}^k[\pi_k[\alpha]] \supset Z_1 \cup Z_8, \quad g_{j-2}^k[\pi_k[\alpha]] \supset S \backslash (Z_5 \cup Z_6),$$

and hence

$$g_{j-3}^k[\pi_k[\alpha]] = S.$$

But $j - 3 \geqq N$, and hence $g_{j-3}^k[\pi_k[\alpha]] = \pi_{j-3}[\alpha]$ is a proper subcontinuum of S_{j-3}, a contradiction. Similarly, if $Z_8 \subset g_j^k[\pi_k[\alpha]]$, then

$$\pi_{j-3}[\alpha] = S_{j-3}.$$

Therefore, there exists a positive integer N_1 such that if m is an integer and $m \geqq N_1$, then $\pi_m[\alpha]$ does not contain J. We now consider two cases.

Case 1. Suppose that there exist infinitely many integers i for which $C \in \pi_i[\alpha]$. Since C is a fixed point of the mapping g, it follows that $C \in \pi_j[\alpha]$ for each positive integer j. Thus $\pi_{N_1}[\alpha]$ is an arc containing C and contained in $(Z_7 \cup Z_8)\backslash\{J\}$, and moreover, if $m > N_1$, then $g|\pi_m[\alpha]$ is a linear mapping of arcs in $(Z_7 \cup Z_8)\backslash\{J\}$. Hence α is homeomorphic to

$$\varprojlim \{\pi_n[\alpha], g_n^{n+1}|\pi_{n+1}[\alpha], n \geqq N_1\};$$

so α is an arc.

Case 2. Suppose that there exists an integer $N_2 \geqq N_1$ such that if m is an integer and $m \geqq N_2$, then $\pi_m[\alpha]$ does not contain C. Now

$$\begin{aligned}
&g^{-1}(C) = \{A, B, C\}, \quad (g^{-1})^2(C) = \{A, B, C, J, \tfrac{1}{2}A\}, \quad \text{and} \\
&(g^{-1})^3(C) = \{A, B, C, J, \tfrac{1}{2}A\} \\
&\cup \{ (rA, i)|r = 1/4, 3/8, 5/8, \text{ or } 3/4, \text{ and } i = 1, 2\} \\
&\cup \{ rB, i)|r = 1/3, 1/2, \text{ or } 2/3, \text{ and } i = 1, 2\} \\
&\cup \{ (\tfrac{1}{2}C, i)|i = 1, 2\}.
\end{aligned}$$

Thus if k is an integer greater than $N_2 + 3$, then

$$\pi_k[\alpha] \subset S\backslash(g^{-1})^3(C).$$

From the definition of g, we have that the mapping g restricted to a component of $S\backslash(g^{-1})^3(C)$ is a linear mapping. Whence $g|\pi_k[\alpha]$ is a homeomorphism of an arc in $S\backslash(g^{-1})^3(C)$ to an arc in $S\backslash(g^{-1})^3(C)$. Therefore, as in Case 1, we have that α is homeomorphic to an inverse limit on arcs having linear bonding maps, and so α is an arc. This concludes the proof of Theorem 4.3.

The next portion of this section is concerned with proving that X has the cone = hyperspace property. Ultimately, this will be accomplished by defining a selection ν for $C(X)\backslash\{X\}$ and a Whitney map μ for $C(X)$ which satisfy the hypothesis of Theorem 3.1. However, some additional notation and terminology must first be introduced.

Let η denote the Hausdorff metric on $C(S)$, where S is the decomposition space with metric d defined in Example 4.2. We define a function $\lambda:C(S) \to [0, 6]$ as follows: for each element $M \in C(S)$, let

$$\lambda(M) = \sum_{i=1}^{8} l(M \cap Z_i),$$

where $l(M \cap Z_i)$ is also defined in Example 4.2.

LEMMA 4.4. *The function* $\lambda:C(S) \to [0, 6]$ *is a Whitney map for* $C(S)$.

Proof. For each $x \in S$,

$$\lambda(\{x\}) = \sum_{i=1}^{8} l(\{x\} \cap Z_i), \quad \text{and}$$

$$l(\{x\} \cap Z_i) = 0 \quad \text{for each } 1 \leqq i \leqq 8.$$

Thus $\lambda(\{x\}) = 0$. If each of M and N belongs to $C(S)$ and N is a proper subset of M, then $N \cap Z_k$ is a proper subset of $M \cap Z_k$ for some $1 \leqq k \leqq 8$, and hence $l(N \cap Z_k) < l(M \cap Z_k)$. If $i \neq k$ and $1 \leqq i \leqq 8$, then $l(N \cap Z_i) \leqq l(M \cap Z_i)$. Thus $\lambda(N) < \lambda(M)$. Clearly

$$\lambda(S) = \sum_{i=1}^{8} l(Z_i) = 6.$$

To see that λ is continuous, let $M \in C(S)$ and $\epsilon > 0$. Let $0 < \delta < \epsilon/16$. Suppose that $V \in C(S)$ with $\eta(V, M) < \delta$, then

$$N_d(\delta, M) \supset V \quad \text{and} \quad N_d(\delta, V) \supset M.$$

If $1 \leqq i \leqq 8$, then

$$l(V \cap Z_i) < l(M \cap Z_i) + 2\delta < l(M \cap Z_i) + \epsilon/8.$$

Thus

$$\lambda(V) = \sum_{i=1}^{8} l(V \cap Z_i) < \sum_{i=1}^{8} [l(M \cap Z_i) + \epsilon/8] = \lambda(M) + \epsilon.$$

Similarly, $\lambda(M) < \lambda(V) + \epsilon$, and hence $|\lambda(M) - \lambda(V)| < \epsilon$. This completes the proof that λ is a Whitney map for $C(S)$.

We denote by H the metric on

$$C_\infty(X) = \lim_{\leftarrow} \{C(S_n), \hat{g}_n^{n+1}\}$$

determined by η; that is, if each of α and β is an element of $C_\infty(X)$, then

$$H(\alpha, \beta) = \sum_{i=1}^{\infty} 2^{-i}\eta(\pi_i(\alpha), \pi_i(\beta)).$$

Considering the nature of the homeomorphism from $C_\infty(Z)$ to $C(Z)$ given in the proof of [**12**, 1.169], where Z denotes any continuum which is the inverse limit of an inverse limit sequence, we shall consider $C_\infty(X)$ as being identical with $C(X)$. Thus for $\alpha \in C(X)$, we denote $\pi_n(\alpha)$ by α_n, for each positive integer n, and we write $\alpha = (\alpha_1, \alpha_2, \dots)$ or

$$\alpha = \lim_{\leftarrow} \{\alpha_n, g_n^{n+1}|\alpha_{n+1}\}$$

when considering α as a point of $C(X)$ or a subcontinuum of X, respectively.

We are now ready to define the function $\nu: C(X)\backslash\{X\} \to X$, and prove that ν is a selection for $C(X)\backslash\{X\}$. Recall from the proof of Theorem 4.3 that if α is a nondegenerate, proper subcontinuum of X, then there exists a positive integer, denoted here by N, such that if $m \geqq N$, then $\alpha_m \subset S\backslash\{A, B, J, \frac{1}{2}A\}$ and $g_m^{m+1}|\alpha_{m+1}$ is linear, hence midpoint preserving. Thus there exists exactly one point of α, which we denote by $\nu(\alpha)$, such that for each $m \geqq N$, $\pi_m(\nu(\alpha))$ is the midpoint of the arc α_m. If $\alpha \in F_1(X)$, say α

$= \{x\}$ for some $x \in X$, then we put $\nu(\alpha) = x$. Thus ν is a function from $C(X)\backslash\{X\}$ onto X.

THEOREM 4.5. *The function* $\nu:C(X)\backslash\{X\} \to X$ *is a selection for* $C(X)\backslash\{X\}$.

Proof. Clearly $\nu(\alpha) \in \alpha$ for each $\alpha \in C(X)\backslash\{X\}$. Hence we need only to show that ν is continuous. Suppose that $\alpha \in C(X)\backslash\{X\}$ and $\epsilon > 0$. Since the projection mappings $\pi_n:X \to S_n$ are $2^{-(n-1)}$-mappings, there exists a positive integer N' and a positive sequence $\delta_{N'}, \delta_{N'+1}, \ldots$ such that if $m \geqq N'$ and each of $x = (x_1, x_2, \ldots)$ and $y = (y_1, y_2, \ldots)$ belongs to X with $d(x_m, y_m) < \delta_m$, then $\rho(x, y) < \epsilon$. Let N be a positive integer such that $N \geqq N'$ and if $m \geqq N$, then

$$\alpha_m \subset S\backslash\{A, B, J, \tfrac{1}{2}A\}.$$

Put $0 < \delta < \delta_N$ such that $2\delta < \lambda(\alpha_N)$ and if $K \in C(S)$ with $\eta(\mathrm{K}, \alpha_N) < \delta$, then K is a subset of the component of $S\backslash\{A, B, J, \frac{1}{2}A\}$ containing α_N. Thus for $K \in C(S)$ with $\eta(K, \alpha_N) < \delta$, it follows that the midpoint of the arc K is within a distance δ of the midpoint of the arc α_N. Since the projection π_N is a mapping, there exists $\delta_0 > 0$ such that if $\beta \in C(X)\backslash\{X\}$ and $H(\alpha, \beta) < \delta_0$, then $\eta(\alpha_N, \beta_N) < \delta$. Denote $\nu(\alpha)$ by $(x_1, x_2, \ldots)$ and $\nu(\beta)$ by $(y_1, y_2, \ldots)$. Since each of α_N and β_N is a subset of $S\backslash\{A, B, J, \frac{1}{2}A\}$ then x_N and y_N are the midpoints of α_N and β_N, respectively. Thus $d(x_N, y_N) < \delta < \delta_N$, and hence

$$\rho(\nu(\alpha), \nu(\beta)) < \epsilon.$$

This concludes the proof that $\nu:C(X)\backslash\{X\} \to X$ is a selection for $C(X)\backslash\{X\}$.

We now define a function $\Lambda:C(X)\backslash\{X\} \to [0, +\infty)$ which we shall use to define a Whitney map μ on $C(X)$. Let

$$\Lambda:C(X)\backslash\{X\} \to [0, +\infty)$$

be given by

$$\Lambda(\alpha) = \sum_{i=1}^{\infty} \lambda(\alpha_i) \quad \text{for each } \alpha \text{ in } C(X)\backslash\{X\}.$$

LEMMA 4.6. *The function* $\Lambda:C(X)\backslash\{X\} \to [0, +\infty)$ *is continuous.*

Proof. Suppose that $\alpha \in C(X)\backslash\{X\}$. We wish to show that $\sum_{i=1}^{\infty} \lambda(\alpha_i)$ converges. If $\alpha \in F_1(X)$, then α_i is degenerate and so $\lambda(\alpha_i) = 0$. Thus

$$\sum_{i=1}^{\infty} \lambda(\alpha_i) = 0.$$

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Suppose that α is nondegenerate. From the definition of λ, we have that $0 \leqq \lambda(\alpha_i) \leqq 6$ for each positive integer i. There exists a positive integer N such that if $m \geqq N$, then $\alpha_m \subset S\backslash\{A, B, J, \frac{1}{2}A\}$ and $g_m^{m+1}|\alpha_{m+1}$ is a linear mapping; moreover, from the definition of the mapping g, we have that

$$\lambda(\alpha_{m+1}) \leqq [\lambda(\alpha_m)]/2 \quad \text{for each } m \geqq N.$$

Therefore $\sum_{i=1}^{\infty} \lambda(\alpha_i)$ converges.

To establish that Λ is continuous at α, let $\epsilon > 0$. Since each of λ and $\hat{g}$ is uniformly continuous, then there exists a positive sequence $\delta_1, \delta_2, \ldots$ satisfying the following conditions:

1) if i is a positive integer and each of M and V belongs to $C(S)$ with $\eta(M, V) < \delta_i$, then

$$|\lambda(M) - \lambda(V)| < \epsilon/2^{i+1}.$$

2) if i is an integer, $i \geqq 2$, and each of M and V belongs to $C(S)$ with $\eta(M, V) < \delta_i$, then

$$\eta(\hat{g}^n(M), \hat{g}^n(V)) < \delta_{i-n} \quad \text{for } 1 \leqq n \leqq i - 1.$$

Let r be a positive integer such that if $m \geqq r$, then

$$\alpha_m \subset S\backslash\{A, B, J, \tfrac{1}{2}A\} \quad \text{and} \quad \lambda(\alpha_m) < \epsilon/2.$$

Put $0 < \delta < \delta_r$ such that if $K \in C(S)$ and $\eta(K, \alpha_r) < \delta$, then $K \subset S\backslash\{A, B, J, \frac{1}{2}A\}$.

Since π_r is continuous, there exists a positive number δ_0 such that if $\beta \in C(X)\backslash\{X\}$ and $H(\alpha, \beta) < \delta_0$, then $\eta(\alpha_r, \beta_r) < \delta$.

Now

$$\begin{aligned}|\Lambda(\alpha) - \Lambda(\beta)| &= \left|\sum_{i=1}^{\infty} \lambda(\alpha_i) - \sum_{i=1}^{\infty} \lambda(\beta_i)\right| \\ &\leqq \left(\sum_{i=1}^{r-1} |\lambda(\alpha_i) - \lambda(\beta_i)|\right) + |\lambda(\alpha_r) - \lambda(\beta_r)| \\ &\quad + \left(\sum_{i=r+1}^{\infty} |\lambda(\alpha_i) - \lambda(\beta_i)|\right).\end{aligned}$$

Since $\eta(\alpha_r, \beta_r) < \delta < \delta_r$, then from (1) we have that

$$|\lambda(\alpha_r) - \lambda(\beta_r)| < \epsilon/2^{r+1}.$$

If $1 \leqq n \leqq r - 1$, then from (2),

$$\eta(\hat{g}^n(\alpha_r), \hat{g}^n(\beta_r)) < \delta_{r-n}.$$

But $\hat{g}^n(\alpha_r) = \alpha_{r-n}$ and $g^n(\beta_r) = \beta_{r-n}$. Thus

$$\eta(\alpha_{r-n}, \beta_{r-n}) < \delta_{r-n},$$

and hence from (1),

$$|\lambda(\alpha_{r-n}) - \lambda(\beta_{r-n})| < \epsilon/2^{r-n+1}.$$

So we have that

$$|\Lambda(\alpha) - \Lambda(\beta)| < \sum_{i=1}^{r} \epsilon/2^{i+1} + \sum_{i=r+1}^{\infty} |\lambda(\alpha_i) - \lambda(\beta_i)|$$
$$= \epsilon/2 - \epsilon/2^{r+1} + \sum_{i=r+1}^{\infty} |\lambda(\alpha_i) - \lambda(\beta_i)|.$$

By the nature of the bonding map g and the choice of the integer r, for each integer $j \geqq 1$, we have that

$$0 \leqq \lambda(\alpha_{r+j}) \leqq \lambda(\alpha_r)/2^j < \epsilon/2^{j+1}.$$

Since $\eta(\alpha_r, \beta_r) < \delta < \delta_r$, then

$$0 \leqq \lambda(\beta_{r+j}) \leqq \lambda(\beta_r)/2^j < (\epsilon/2 + \epsilon/2^{r+1})/2^j.$$

Hence

$$|\lambda(\alpha_{r+j}) - \lambda(\beta_{r+j})| < (\epsilon/2 + \epsilon/2^{r+1})/2^j.$$

Consequently,

$$\sum_{i=r+1}^{\infty} |\lambda(\alpha_i) - \lambda(\beta_i)| < \sum_{i=1}^{\infty} (\epsilon/2 + \epsilon/2^{r+1})/2^i = \epsilon/2 + \epsilon/2^{r+1}.$$

Thus

$$|\Lambda(\alpha) - \Lambda(\beta)| < (\epsilon/2 - \epsilon/2^{r+1}) + (\epsilon/2 + \epsilon/2^{r+1}) = \epsilon,$$

and hence

$$\Lambda: C(X)\backslash\{X\} \to [0, +\infty)$$

is continuous. This concludes the proof of Lemma 4.6.

Let $\Gamma:[0, +\infty) \to [0, 1)$ be a homeomorphism, and define μ by $\Gamma\Lambda \cup \{(X, 1)\}$.

THEOREM 4.7. *The function* $\mu: C(X) \to [0, 1]$ *is a Whitney map for* $C(X)$.

Proof. The proof follows immediately from Lemmas 4.4 and 4.6, and the definition of μ.

Having now defined a selection ν for $C(X)\backslash\{X\}$ and a Whitney map μ for $C(X)$ with $\mu(X) = 1$, we need only to prove that $\nu|\mu^{-1}(t)$ is a homeomorphism from $\mu^{-1}(t)$ onto X for $0 \leqq t < 1$ in order to establish that X has the cone = hyperspace property.

THEOREM 4.8. *The selection $\nu|\mu^{-1}(t)$ is homeomorphism from $\mu^{-1}(t)$ onto X for $0 \leqq t < 1$.*

Proof. Clearly, if $t = 0$, then $\nu|\mu^{-1}(t)$ is homeomorphism from $\mu^{-1}(0) = F_1(X)$ onto X. Suppose then that $0 < t < 1$. We first show that $\nu|\mu^{-1}(t)$ is a mapping from $\mu^{-1}(t)$ onto X, that is if $x \in X$, then there exists $\alpha \in \mu^{-1}(t)$ such that $\nu(\alpha) = x$. To establish the existence of such an α, we first prove the following:

1) if $x \in X$ and $a > 0$, then there exists $\gamma \in C(X)$ such that $\nu(\gamma) = x$ and $\Lambda(\lambda) > a$.

We denote $x \in X$ by $(x_1, x_2, \dots)$ and let k be an integer such that $6k > a$. We consider two cases.

Case 1. Suppose that $x_j \notin Z_7 \cup Z_8$ for infinitely many integers j. Then there exists a positive integer N such that if $m \geqq N$, then

$$x_m \in S\backslash\{A, B, C, J, \tfrac{1}{2}A\}.$$

We choose integers r and s so that

$$r \geqq \max\{k + 3, N\}, \quad s \geqq r + 2, \quad \text{and} \quad x_r \notin Z_7 \cup Z_8.$$

Thus $x_s \in \operatorname{int} Z_q$ for some integer $1 \leqq q \leqq 8$. Denote by γ_s the maximal arc in Z_q having midpoint x_s. Then $\gamma_s \cap \{A, B, C, J, \frac{1}{2}A\}$ is nonempty, and hence $g^2[\gamma_s]$ contains C. Since C is a fixed point of the mapping g, then for $n \geqq 2$, $g^n[\gamma_s]$ contains C. Consequently, $g^{s-r}[\gamma_s]$ is a subcontinuum of S containing both C and a point not in $Z_7 \cup Z_8$, namely x_r, and hence

$$g^{s-r+3}[\gamma_s] = S.$$

For each positive integer $j < s$, put $\gamma_j = g^{s-j}[\gamma_s]$. Now again we make use of the fact that the mapping g restricted to a component of $g^{-1}[S\backslash\{A, B, J, \frac{1}{2}A\}]$ is linear; for each positive integer n, let γ_{s+n} be the closure of the component of $g^{-1}[\operatorname{int} \gamma_{s+n-1}]$ which contains x_{s+n}. Then x_{s+n} is the midpoint of the arc γ_{s+n}, and for $n \geqq 2$, γ_{s+n} does not intersect $\{A, B, J, \frac{1}{2}A\}$. Thus if $\gamma = (\gamma_1, \gamma_2, \dots)$, then $\gamma \in C(X)\backslash\{X\}$ and $\nu(\gamma) = x$.

Since $g^{s-r+3}[\gamma_s] = S$, then

$$\gamma_{r-3} = g^s_{r-3}[\gamma_s] = S.$$

But $r - 3 \geqq k$ and $\lambda(S) = 6$, so

$$\Lambda(\gamma) = \sum_{i=1}^{\infty} \lambda(\gamma_i) = \sum_{i=1}^{k} \lambda(\gamma_i) + \sum_{i=k+1}^{\infty} \lambda(\gamma_i)$$

$$= 6k + \sum_{i=k+1}^{\infty} \lambda(\gamma_i) > a.$$

Case 2. Suppose that there exists a positive integer N' such that if m is an integer and $m \geqq N'$, then $x_m \in Z_7 \cup Z_8$. This supposition and the definition of the mapping g imply that there exists an integer $N \geqq N'$ such that $d(x_j, C) < 1/2$ for each integer $j \geqq N$.

We put $r = \max \{k + 4, N\}$, and let γ_r be an arc in $Z_7 \cup Z_8 \backslash \{J\}$ such that x_r is the midpoint of γ_r, $C \in \gamma_r$, and $\gamma_r \cap \{ (\frac{1}{2}C, 1), (\frac{1}{2}C, 2) \}$ is nonempty. Then γ_r does not intersect $\{A, B, J, \frac{1}{2}A\}$ and $g[\gamma_r]$ must contain one of Z_7 or Z_8 as a subset. Hence $g^4[\gamma_r] = S$. For each positive integer j such that $j < r$, we put $\gamma_j = g^{r-j}[\gamma_r]$. From the definition of the mapping g, we have that the component of $g^{-1}[(Z_7 \cup Z_8)\backslash\{J\}]$ which contains C is mapped linearly by g onto $(Z_7 \cup Z_8)\backslash\{J\}$. Thus for each positive integer n, we denote by γ_{r+n} the component of $g^{-1}[\gamma_{r+n-1}]$ which contains x_{r+n}; clearly x_{r+n} is the midpoint of γ_{r+n}. Hence if $\gamma = (\gamma_1, \gamma_2, \dots)$, then $\gamma \in C(X)\backslash\{X\}$ and $\nu(\gamma) = x$.

Now

$$\Lambda(\gamma) = \sum_{i=1}^{\infty} \lambda(\gamma_i) = \sum_{i=1}^{k} \lambda(\gamma_i) + \sum_{i=k+1}^{\infty} \lambda(\gamma_i).$$

Since $\gamma_{r-4} = g^r_{r-4}[\gamma_r] = g^4[\gamma_r] = S$, $r - 4 \geqq k$, and $\lambda(S) = 6$, it follows that

$$\sum_{i=1}^{k} \lambda(\gamma_i) + \sum_{i=k+1}^{\infty} \lambda(\gamma_i) = 6k + \sum_{i=k+1}^{\infty} \lambda(\gamma_i) > a.$$

This establishes statement (1).

Thus for $0 < t < 1$ and $x \in X$, we have that $\Gamma^{-1}(t) > 0$, and hence there exists $\gamma = (\gamma_1, \gamma_2, \dots)$, belonging to $C(X)\backslash\{X\}$ such that

$$\nu(\gamma) = x \quad \text{and} \quad \Lambda(\gamma) > \Gamma^{-1}(t).$$

But $\Lambda(\gamma) > \Gamma^{-1}(t)$ implies that $\mu(\gamma) > t$; thus γ is an arc in X such that $\nu(\gamma) = x$ and γ is "above" the Whitney level $\mu^{-1}(t)$.

Let N be a positive integer such that if m is an integer and $m \geqq N$, then $\gamma_m \subset S \backslash \{A, B, J, \frac{1}{2}A\}$. Hence x_m is the midpoint of γ_m for each $m \geqq N$. Consider $C(\gamma_i)$ as a subset of $C(S_i)$ for $i = 1, 2, \dots$. For each integer $m \geqq N$, let

$$U_m = \{V \in C(\gamma_m) | \quad V = \{x_m\} \text{ or } x_m \text{ is the midpoint of } V.$$

Thus U_m is an order arc in $C(\gamma_m)$ from $\{x_m\}$ to γ_m; we refer the reader to [**12**, Chapter I, Section A] for a discussion of order arcs in the hyperspaces of sets. Since $g|\gamma_{m+1}$ is linear, thus midpoint preserving, then $\hat{g}|U_{m+1}$ is a homeomorphism from the arc U_{m+1} onto the arc U_m. If j is an integer such that $1 \leqq j < N$, let

$$U_j = \hat{g}^{N-j}[U_N];$$

thus U_j is a subcontinuum of $C(S_j)$. Put

$$U = \lim_{\leftarrow} \{U_n, \hat{g}_n^{n+1} | U_{n+1}\}.$$

Then U is a subcontinuum of $C(X)$ which is homeomorphic to

$$\lim_{\leftarrow} \{U_n, \hat{g}_n^{n+1} | U_{n+1}, n \geqq N\},$$

so U is an order arc in $C(X)$ from $\{x\}$ to γ, and $U \subset \nu^{-1}[x]$. Now $\mu[U] = [0, \mu(\gamma)]$ and $\mu(\gamma) > t$, thus there exists $\alpha \in U$ such that $\mu(\alpha) = t$. Therefore $\nu|\mu^{-1}(t)$ maps $\mu^{-1}(t)$ onto X.

We wish to show that $\nu|\mu^{-1}(t)$ is one-to-one. Suppose that $\nu(\alpha) = \nu(\beta)$ for $\alpha, \beta \in \mu^{-1}(t)$. Denote $\nu(\alpha)$ by $(x_1, x_2, \dots)$. Then there exists a positive integer K such that if m is an integer and $m \geqq K$, then x_m is the midpoint of each α_m and β_m. Thus for $m \geqq K$, we have that $\alpha_m \subset \beta_m$ or $\beta_m \subset \alpha_m$, and hence $\alpha \subset \beta$ or $\beta \subset \alpha$. Since $\mu(\alpha) = t = \mu(\beta)$, this implies that $\alpha = \beta$. This concludes the proof of Theorem 4.8, and thus establishes that X has the cone = hyperspace property.

The final portion of this section is concerned with proving that the continuum X is neither chainable nor circle-like.

In [**9**] T. A. Moebes proved that Y is not weakly chainable, that is, Y is not the continuous image of a chainable continuum. Next, we define a mapping Φ from X onto Y thus establishing that X is not chainable, in fact not weakly chainable. We subsequently use the mapping Φ in proving that X is not circle-like.

Let $p:T \times \{1, 2\} \to T$ denote the projection mapping. For each element $s \in S$, put $k(s) = p[\psi^{-1}(s)]$, where $\psi:T \times \{1, 2\} \to S$ is the quotient map as defined in Example 4.2. Thus $k:S \to T$ collapses S onto T in the natural manner. From the definition of the mapping g, we have that $kg = fk$, and thus inductively, $kg^n = f^n k$ for each positive integer n. If $x = (x_1, x_2, \dots)$ belongs to X, let

$$\Phi(x) = (k(x_1), k(x_2), \dots).$$

Clearly Φ is a well-defined function on X. Furthermore, if n is a positive integer, then

$$f_n^{n+1}(k(x_{n+1})) = k(g_n^{n+1}(x_{n+1})) = k(x_n),$$

and thus $\Phi(x) \in Y$. It follows immediately that Φ is continuous and maps X onto Y.

THEOREM 4.9. *The continuum X is neither chainable nor circle-like.*

Proof. As previously mentioned, since Y is not weakly chainable and Φ is a mapping from X onto Y, then X is not chainable, in fact X is not weakly chainable.

Suppose, by way of contradiction, that X is circle-like. Then X is homeomorphic to the inverse limit of an inverse limit sequence in which each factor space is the unit circle S^1. From [**8**, Theorem 3], if $\epsilon_1 > 0$, then there exists a positive integer n and mappings $\gamma:S \to S^1$ and $\beta:S^1 \to S$ such that

$$d(g^n(z), \beta\gamma(z)) < \epsilon_1 \quad \text{for each } z \in S.$$

Let $\tau:T \to Z_1 \cup Z_3 \cup Z_5 \cup Z_7$ denote the embedding such that $k\tau$ is the identity mapping on T. Then for each $x \in T$, we have that

$$f^n(x) = kg^n\tau(x);$$

also, if each of y and z belongs to S, then

$$d_1(k(y), k(z)) \leqq d(y, z).$$

Hence

$$d_1(f^n(x), k\beta\gamma\tau(x)) = d_1(kg^n\tau(x), k\beta\gamma\tau(x))$$

and

$$d_1(k(g^n\tau(x)), k(\beta\gamma\tau(x))) \leqq d(g^n\tau(x), \beta\gamma\tau(x)).$$

But

$$d(g^n(z), \beta\gamma(z)) < \epsilon_1 \quad \text{for each } z \in S,$$

thus

$$d_1(f^n(x), k\beta\gamma\tau(x)) < \epsilon_1 \quad \text{for each } x \in T.$$

So for each $\epsilon_1 > 0$, there exists a positive integer n and mappings $\gamma\tau:T \to S^1$ and $k\beta:S^1 \to T$ such that

$$d_1(f^n(x), k\beta\gamma\tau(x)) < \epsilon_1 \quad \text{for each } x \in T.$$

Let $\epsilon > 0$. There exists a positive integer N and a positive sequence $\delta_N, \delta_{N+1}, \delta_{N+2}, \dots$ such that if $m \geqq N$ and each of $x = (x_1, x_2, x_3, \dots)$ and y

$= (y_1, y_2, y_3, \ldots)$ belongs to Y with $d_1(x_m, y_m) < \delta_m$, then $\rho_1(x, y) < \epsilon$ and $\pi_m: Y \to T_m$ is an ϵ-mapping. As established above, there exists a positive integer n and mappings γ, τ, k, and β such that

$$d_1(f^n(x), k\beta\gamma\tau(x)) < \delta_N/2 \quad \text{for each } x \in T.$$

Thus the mapping $\gamma\tau\pi_{N+n}$ is an ϵ-mapping from Y to S^1. Since Y is not chainable, this implies that Y is circle-like. However, in **[2]** C. E. Burgess proved that a continuum which is both tree-like and circle-like is chainable. This is a contradiction, since Y is tree-like but not chainable. Therefore X is not circle-like.

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