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Stability of Biharmonic Legendrian Submanifolds in Sasakian Space Forms

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Abstract. Biharmonic maps are defined as critical points of the bienergy. Every harmonic map is a stable biharmonic map. In this article, the stability of nonharmonic biharmonic Legendrian submanifolds in Sasakian space forms is discussed.

1 Introduction

The study of Legendrian submanifolds in contact manifolds from the Riemannian geometric point of view was initiated in the 1970's. In particular, the class of minimal Legendrian submanifolds is one of the most interesting objects of study from both the geometric and the physical points of view. It is important to introduce classes which include such submanifolds.

A natural extension of the class of minimal submanifolds is the class of those with parallel mean curvature vector field. During the last three decades, many geometers have obtained interesting results on nonminimal submanifolds with parallel mean curvature vector field.

On the other hand, it is known that there exist no Legendrian submanifolds with parallel mean curvature vector field in Sasakian manifolds, apart from the minimal ones [13]. Thus, when the ambient space is a Sasakian manifold, we need to consider some other extensions of minimal Legendrian submanifolds. In this paper, we consider an extension from a variational point of view.

Eells and Sampson introduced the notion of *biharmonic maps*, which are critical points of bienergy functionals [8]. Harmonic maps are biharmonic maps; however the converse is not true in general. Recently, nonharmonic biharmonic submanifolds have been investigated intensively. In particular, nonharmonic biharmonic Legendrian submanifolds in Sasakian space forms of low dimension have been classified [9, 12]. The purpose of this paper is to establish the second variation formula for nonharmonic biharmonic Legendrian submanifolds in Sasakian space forms of general dimension and then to investigate their stability.

2 Preliminaries

A $(2n+1)$-dimensional differentiable manifold N^{2n+1} is called a *contact manifold* if there exists a globally defined 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. On a contact

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manifold there exists a unique global vector field ξ satisfying

$$d\eta(\xi, X) = 0, \quad \eta(\xi) = 1,$$

for all $X \in TN^{2n+1}$. The vector field ξ is called a *Reeb vector field*.

Moreover, it is well known that there exist a tensor field ϕ of type $(1,1)$ and a Riemannian metric g which satisfy

$$\phi^2 = -I + \eta \otimes \xi, \quad g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y),$$
$$g(\xi, X) = \eta(X), \quad d\eta(X,Y) = g(X, \phi Y),$$

for all $X, Y \in TN^{2n+1}$ (see [2]).

The structure (ϕ, ξ, η, g) is called a *contact metric structure*, and the manifold N^{2n+1} with a contact metric structure is said to be a *contact metric manifold.* A contact metric manifold is said to be a *Sasakian manifold* if it satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$ on N^{2n+1}, where $[\phi, \phi]$ is the Nijenhuis torsion of ϕ. On Sasakian manifolds, we have

$$(2.1) \qquad (\bar{\nabla}_X \phi)Y = g(X,Y)\xi - \eta(Y)X, \quad \text{and} \quad \bar{\nabla}_X \xi = -\phi X,$$

for any vector fields X and Y, where $\bar{\nabla}$ is the Levi–Civita connection of N^{2n+1}. In some respects, Sasakian manifolds may be viewed as odd-dimensional analogues of Kähler manifolds.

A tangent plane in T_pN^{2n+1} which is invariant under ϕ is called a *ϕ-section* (see [2]). The sectional curvature of ϕ-section is called a ϕ-sectional curvature. If the ϕ-sectional curvature is constant on N^{2n+1}, then N^{2n+1} is said to be of *constant ϕ-sectional curvature*. Complete and connected Sasakian manifolds of constant ϕ-sectional curvature are called *Sasakian space forms*. Denote Sasakian space forms of constant ϕ-sectional curvature ϵ by $N^{2n+1}(\epsilon)$. The curvature tensor $\bar{R}$ of $N^{2n+1}(\epsilon)$ is given by

$$(2.2) \qquad \bar{R}(X,Y)Z = \frac{\epsilon+3}{4}\{g(Y,Z)X - g(Z,X)Y\} + \frac{\epsilon-1}{4}\Big\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X + g(X,Z)\eta(Y)\xi - g(Y,Z)\eta(X)\xi + g(Z,\phi Y)\phi X - g(Z,\phi X)\phi Y + 2g(X,\phi Y)\phi Z\Big\}.$$

Let M^m be a submanifold in a contact manifold N^{2n+1}. If η restricted to M^m vanishes, then M^m is called an *integral submanifold*; in particular if $m = n$, it is called a *Legendrian submanifold.*

Let $f\colon M^m \to N^{2n+1}(\epsilon)$ be an isometric immesion. Denote the Levi–Civita connection of $N^{2n+1}(\epsilon)$ (resp. M^m) by $\bar{\nabla}$ (resp. ∇). Let ∇^f be the induced connection by f on the bundle f^*TN^{2n+1}, which is the pull-back of $\bar{\nabla}$.

The formulas of Gauss and Weingarten are given respectively by

$$(2.3) \qquad \nabla^f_X df(Y) = df(\nabla_X Y) + h(X,Y),$$

$$(2.4) \qquad \nabla^f_X V = -df(A_V X) + D_X V,$$

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where $X, Y \in TM^m$, $V \in T^{\perp}M^m$, h, A and D are the second fundamental form, the shape operator and the normal connection, respectively. We identify $df(X)$ and X for any vector fields X on M^m. The following relation holds:

$$\langle A_V X, Y\rangle = \langle h(X, Y), V\rangle, \tag{2.5}$$

where $\langle\ ,\ \rangle := g(\ ,\ )$.

The mean curvature vector H is given by $H = \frac{1}{m}$ trace h. The length of H is called the *mean curvature*. If the mean curvature vector vanishes on M^m everywhere, then M^m is called a *minimal submanifold*.

In this paper, submanifolds and immersions mean isometrically immersed manifolds and isometric immersions, respectively.

3 Biharmonic Legendrian Submanifolds

Let M^m and N^n be Riemannian manifolds and $f\colon M^m \to N^n$ a smooth map. The *tension field* $\tau(f)$ of f is a section of the vector bundle f^*TN defined by

$$\tau(f) := \operatorname{tr}(\nabla^f df) = \sum_{i=1}^{m}\{\nabla^f_{e_i} df(e_i) - df(\nabla_{e_i} e_i)\},$$

where $\{e_i\}$ denotes a local orthonormal frame field of M.

A smooth map f is said to be a *harmonic map* if its tension field vanishes. It is well known that f is harmonic if and only if f is a critical point of the *energy*

$$E(f) = \int_\Omega |df|^2\, dv_g$$

over every compact domain Ω of M, where $|\cdot|$ denotes the Hilbert–Schmidt norm.

J. Eells and J. H. Sampson [8] suggested studying *k-harmonic maps* which are critical points of k-energy E_k:

$$E_k(f) = \int_\Omega |(d + d^*)^k f|^2\, dv_g,$$

where d^* is the codifferential operator.

Clearly, a 1-harmonic map is a harmonic map. In case of $k = 2$, we have

$$E_2(f) = \int_\Omega |\tau(f)|^2\, dv_g.$$

The functional E_2 is frequently called the *bienergy*. The Euler–Lagrange equation of the functional E_2 was computed by Jiang [10] as follows:

$$\mathcal{J}_f(\tau(f)) = 0, \tag{3.1}$$

where the operator $\mathcal{J}_f$ is the *Jacobi operator* defined by

$$\mathcal{J}_f(V) := \bar{\Delta}_f V - \mathcal{R}_f(V), \qquad V \in \Gamma(f^*TN), \tag{3.2}$$

$$\bar{\Delta}_f := -\sum_{i=1}^{m}(\nabla^f_{e_i}\nabla^f_{e_i} - \nabla^f_{\nabla_{e_i}e_i}), \quad \mathcal{R}_f(V) := \sum_{i=1}^{m} R^N(V, df(e_i))df(e_i), \tag{3.3}$$

where R^N is the curvature tensor of N.

In particular, if N is the Euclidean n-space $\mathbf{E}^n$ and $f = (x_1, \dots, x_n)$ is an immersion from M into $\mathbf{E}^n$, then

$$\mathcal{J}_f(\tau(f)) = (-\Delta_M\Delta_M x_1, \dots, -\Delta_M\Delta_M x_n),$$

where Δ_M is the Laplace operator acting on $C^\infty(M)$. Thus the 2-harmonicity for an immersion into Euclidean space is equivalent to the biharmonicity in the sense of Chen [6]. For this reason, 2-harmonic maps are frequently called *biharmonic maps*. Every harmonic map is a stable biharmonic map [10]. Nonharmonic biharmonic maps are said to be proper.

Now let $f\colon M^2 \to N^5(\epsilon)$ be a Legendrian immersion into Sasakian space forms. Then from (3.1), (3.2), (3.3), and (2.2) we see that f is biharmonic if and only if

$$\bar{\Delta}_f H = \left(\frac{5\epsilon + 3}{4}\right) H.$$

In [12] the author determined the intrinsic and the extrinsic structures of Legendrian surfaces satisfying $\bar{\Delta}_f H = \lambda H$ for a constant λ.

Theorem 1 ([12]) *Let $f\colon M^2 \to N^5(\epsilon)$ be a nonminimal Legendrian immersion satisfying $\bar{\Delta}_f H = \lambda H$ for a constant λ. Then there exists a suitable local coordinate system $\{u, v\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:*

(i) $g = du^2 + dv^2$,

(ii)
$$h(\partial_u, \partial_u) = \sqrt{\lambda - 1}\cos\theta\phi\partial_u,$$
$$h(\partial_v, \partial_v) = \sqrt{\lambda - 1}\sin\theta\phi\partial_u,$$
$$h(\partial_u, \partial_v) = \sqrt{\lambda - 1}\sin\theta\phi\partial_v,$$

where $\partial_u = \frac{\partial}{\partial u}$, $\partial_v = \frac{\partial}{\partial v}$, and θ is a constant which satisfies

$$\sin\theta(\cos\theta - \sin\theta) = \frac{\epsilon + 3}{4(1 - \lambda)}. \tag{3.4}$$

Conversely, suppose that θ, $\lambda(\ \ > 1)$ and ϵ are constants satisfying (3.4). *Let $g = du^2 + dv^2$ be the metric tensor on a simply-connected domain $V \subset \mathbf{R}^2$. Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendrian immersion f of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by* (ii). *Moreover such an immersion satisfies $\bar{\Delta}_f H = \lambda H$.*

In case $\lambda = \frac{5\epsilon+3}{4}$ in Theorem 1, we get the following.

Corollary 2 ([12]) *Let M^2 be a proper biharmonic Legendrian surfaces in $N^5(\epsilon)$. Then $\epsilon \geq \frac{-11+32\sqrt{2}}{41}$ and at each point $p \in M^2$ there exists a suitable local coordinate system $\{u, v\}$ on a neighborhood of p such that the metric tensor g and the second fundamental form h take the following forms:*

(i) $g = du^2 + dv^2$,

(ii)

$$h(\partial_u, \partial_u) = \frac{\epsilon - 1}{\alpha}\phi\partial_u, \quad h(\partial_v, \partial_v) = \Big(\alpha - \frac{\epsilon - 1}{\alpha}\Big)\phi\partial_u,$$

$$h(\partial_u, \partial_v) = \Big(\alpha - \frac{\epsilon - 1}{\alpha}\Big)\phi\partial_v,$$

where

$$\alpha = \sqrt{\frac{13\epsilon - 9 \pm \sqrt{41\epsilon^2 + 22\epsilon - 47}}{8}} \quad (\neq 0).$$

Conversely, suppose that ϵ is a constant satisfying $\epsilon \geq (-11 + 32\sqrt{2})/41$ and let g be the metric tensor on a simply-connected domain $V \subset \mathbf{R}^2$ defined by (i). *Then, up to rigid motions of $N^5(\epsilon)$, there exists a unique Legendrian immersion of (V, g) into $N^5(\epsilon)$ whose second fundamental form is given by* (ii). *Moreover such an immersion is proper biharmonic.*

We consider the complex Euclidean $(n + 1)$-space $\mathbf{C}^{n+1}$ and identify

$$z = (x_1 + iy_1, \dots, x_{n+1} + iy_{n+1}) \in \mathbf{C}^{n+1}$$

with $(x_1, \dots, x_{n+1}, y_1, \dots, y_{n+1}) \in \mathbf{E}^{2n+2}$. Let J be its usual almost complex structure. It is well known [2] that a Sasakian space form $N^{2n+1}(1)$ is isomorphic to $S^{2n+1}(1)$ endowed with the Sasakian structure induced by J of $\mathbf{C}^{n+1}$.

We can explicitly represent proper biharmonic Legendrian immersions into $S^5(1)$ in $\mathbf{C}^3$ as follows:

Corollary 3 ([12]) *Let $f: M^2 \to S^5(1) \subset \mathbf{C}^3$ be a proper biharmonic Legendrian immersion. Then the position vector $f = f(u, v)$ of M^2 in $\mathbf{C}^3$ is given by*

$$f(u, v) = \frac{1}{\sqrt{2}}(e^{iu}, ie^{-iu} \sin\sqrt{2}v, ie^{-iu} \cos\sqrt{2}v). \tag{3.5}$$

Remark. (i) We see that (3.5) is doubly periodic. More precisely, it is periodic with period 2π with respect to u and $\sqrt{2}\pi$ with respect to v. Thus, it is a proper biharmonic Legendrian embedding from the flat torus $T^2 = R^2/\Lambda$ with Λ generated by $\{(2\pi, 0), (0, \sqrt{2}\pi)\}$.

(ii) Let $f: M \to E^n$ be an immersion. If the position vector f can be written as

$$f = f_1 + f_2, \quad \Delta_M f_1 = \lambda_1 f_1, \quad \Delta_M f_2 = \lambda_2 f_2,$$

for two different constants λ_1 and λ_2, then f is said to be of 2-*type*. Compact 2-type submanifolds are characterized by the minimal polynomial criterion which establishes an existence of a polynomial of degree 2 such that $P(\bar{\Delta}_f)(\tilde{H}) = 0$ [5]. Here $\tilde{H}$ is the mean curvature vector field of f. Since (3.5) satisfies $(\bar{\Delta}_f^2 - 4\bar{\Delta}_f + 3I)(\tilde{H}) = 0$, where I is the identity transformation of E^n, we obtain that f is biharmonic by [4, Proposition 4.1] and 2-type by [5]. In fact, we put

$$f_1(u,v) := \frac{1}{\sqrt{2}}(e^{iu}, 0, 0) \quad \text{and} \quad f_2(u,v) := \frac{1}{\sqrt{2}}(0, ie^{-iu}\sin\sqrt{2}v, ie^{-iu}\cos\sqrt{2}v).$$

Then we have $f = f_1 + f_2$, $\Delta_M f_1 = f_1$ and $\Delta_M f_2 = 3f_2$. Note that (3.5) is not of the type given by [4, Theorem 3.9 or 3.13].

(iii) We put $g_1(u) = (\cos u, \sin u)$ and $g_2(v) = \frac{1}{\sqrt{2}}(1, \sin\sqrt{2}v, \cos\sqrt{2}v) \in S^2(1)$. Then $f(u,v)$ can be written as $f(u,v) = g_1 \otimes g_2$ [7]. Note that g_2 is a proper biharmonic curve in $S^2(1)$ [4].

4 Stability of Biharmonic Legendrian Submanifolds

In [10] Jiang obtained the second variation formula for the bienergy E_2. But it is difficult to compute the formula when the ambient space is not locally symmetric. We remark that Sasakian space forms are not locally symmetric in general. In this section, we shall compute the second variation formula for a biharmonic Legendrian immersion into Sasakian space forms in a similar way as in [11].

Let $f\colon M^n \to N^{2n+1}(\epsilon)$ be a biharmonic Legendrian immersion from a compact n-dimensional manifold into a $(2n+1)$-dimensional Sasakian space form. Let $F\colon \mathbf{R} \times M^n \to N^{2n+1}(\epsilon)$ be a smooth variation of f such that $F(0,p) = f(p)$ for any $p \in M$. Let $\left(\frac{\partial}{\partial t}\right)_{(t,p)}$ and $X_{(t,p)}$ be the vector fields which are the extension of $\frac{\partial}{\partial t}$ on $\mathbf{R}$ and X on M^n to $\mathbf{R} \times M^n$, respectively. We put $f_t(p) = F(t,p)$. The corresponding variational vector field V is given by

$$V(p) = \frac{d}{dt}\bigg|_{t=0} f_t(p) = dF\left(\frac{\partial}{\partial t}\right)_{(0,p)}.$$

We recall the following from [11].

$$\text{(4.1)} \qquad \frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} E_2(f_t) = \int_{M^n} \langle I(V), V\rangle \, dv_g,$$

where

$$\text{(4.2)} \qquad I(V) = \tilde{\nabla}_{\frac{\partial}{\partial t}}\{-\bar{\Delta}_{f_t}\tau_t - \mathrm{trace} R^N(df_t\,\cdot, \tau_t)df_t\,\cdot\}\big|_{t=0},$$

$\tilde{\nabla} = \nabla^F$ and $\tau_t = \tau(f_t)$.

If (4.1) is non-negative for any vector field V, then f or M^n is said to be stable. Otherwise it is said to be *unstable*.

We shall calculate (4.2) more precisely.

$$
\begin{aligned}
(4.3)\quad & -\tilde{\nabla}_{\frac{\partial}{\partial t}}\bar{\Delta}_{f_t}\tau_t = \sum\Big(\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{e_i}\tilde{\nabla}_{e_i}\tau_t - \tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{\nabla_{e_i}e_i}\tau_t\Big)\\
&= \sum\Big\{R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\Big)(\tilde{\nabla}_{e_i}\tau_t) + \tilde{\nabla}_{e_i}\tilde{\nabla}_{\frac{\partial}{\partial t}}\tilde{\nabla}_{e_i}\tau_t + \tilde{\nabla}_{[\frac{\partial}{\partial t}, e_i]}\tilde{\nabla}_{e_i}\tau_t\Big\}\\
&\quad - \sum\Big\{R^N\Big(dF\Big(\frac{\partial}{\partial t}\Big), dF(\nabla_{e_i}e_i)\Big)\tau_t + \tilde{\nabla}_{\nabla_{e_i}e_i}\tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t + \tilde{\nabla}_{[\frac{\partial}{\partial t}, \nabla_{e_i}e_i]}\tau_t\Big\}.
\end{aligned}
$$

As in [11], we have

$$
(4.4)\qquad \tilde{\nabla}_{\frac{\partial}{\partial t}}\tau_t\big|_{t=0} = -\bar{\Delta}_f V - \operatorname{trace} R^N(df\,\cdot, V)df\,\cdot = -\mathcal{J}_f(V).
$$

Let $\{e_i\}$ be a geodesic frame field around an arbitrary point $p \in M^n$. Then from (4.3) and (4.4), when $t = 0$, at p we get the following.

Lemma 4

$$
(4.5)\qquad -\tilde{\nabla}_{\frac{\partial}{\partial t}}\bar{\Delta}_{f_t}\tau_t\big|_{t=0} = \sum\big\{R^N(V, e_i)(\bar{\nabla}_{e_i}\tau) + \bar{\nabla}_{e_i}(R^N(V, e_i)\tau)\big\} + \bar{\Delta}_f\mathcal{J}_f V,
$$

where $\bar{\nabla} = \nabla^f$, $\tau = \tau_0$.

We need the following lemma in order to compute (4.5) more precisely.

Lemma 5

$$
\begin{aligned}
(4.6)\quad R^N(V, e_i)(\bar{\nabla}_{e_i}\tau) &= \frac{\epsilon + 3}{4}\Big(\langle e_i, \bar{\nabla}_{e_i}\tau\rangle V - \langle\bar{\nabla}_{e_i}\tau, V\rangle e_i\Big)\\
&+ \frac{\epsilon - 1}{4}\Big\{\eta(V)\eta(\bar{\nabla}_{e_i}\tau)e_i - \langle e_i, \bar{\nabla}_{e_i}\tau\rangle\eta(V)\xi + \langle\bar{\nabla}_{e_i}\tau, \phi e_i\rangle\phi V\\
&\qquad - \langle\bar{\nabla}_{e_i}\tau, \phi V\rangle\phi e_i + 2\langle V, \phi e_i\rangle\phi(\bar{\nabla}_{e_i}\tau)\Big\},
\end{aligned}
$$

$$
\begin{aligned}
(4.7)\quad \bar{\nabla}_{e_i}(R^N(V, e_i)\tau) &= -\frac{\epsilon + 3}{4}\bar{\nabla}_{e_i}(\langle\tau, V\rangle e_i)\\
&+ \frac{\epsilon - 1}{4}\Big\{\bar{\nabla}_{e_i}\Big(\langle\tau, \phi e_i\rangle\phi V - \langle\tau, \phi V\rangle\phi e_i + 2\langle V, \phi e_i\rangle\phi\tau\Big)\Big\}.
\end{aligned}
$$

Proof By using the fact that τ is normal to M^n and ξ, we can easily obtain (4.6) and (4.7) from (2.2). ■

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We continue to calculate (4.2). Using (2.1) and (2.2), we have

$$
\begin{aligned}
(4.8)\quad & -\tilde\nabla_{\frac{\partial}{\partial t}} \operatorname{trace} R^N(dF\cdot,\tau_t)dF\cdot \\
&= -\frac{\epsilon+3}{4}\sum \tilde\nabla_{\frac{\partial}{\partial t}}\Big\{ \langle \tau_t, dF(e_i)\rangle dF(e_i) - \langle dF(e_i), dF(e_i)\rangle \tau_t \Big\} \\
&\quad - \frac{\epsilon-1}{4}\sum \tilde\nabla_{\frac{\partial}{\partial t}}\Big\{ \eta(dF(e_i))\eta(dF(e_i))\tau_t - \eta(\tau_t)\eta(dF(e_i))dF(e_i) \\
&\quad + \langle dF(e_i), dF(e_i)\rangle \eta(\tau_t)\xi - \langle \tau_t, dF(x_i)\rangle \eta(dF(e_i))\xi \\
&\quad + 3\langle dF(e_i), \phi\tau_t\rangle \phi(dF(e_i)) - \langle dF(e_i), \phi(dF(e_i))\rangle \phi\tau_t \Big\}, \\
&= -\frac{\epsilon+3}{4}\sum \Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t, dF(e_i)\rangle dF(e_i) + \langle \tau_t, \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i)\rangle dF(e_i) \\
&\quad + \langle \tau_t, dF(e_i)\rangle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i) - 2\langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), dF(e_i)\rangle \tau_t - \langle dF(e_i), dF(e_i)\rangle \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t \Big\} \\
&\quad - \frac{\epsilon-1}{4}\sum \bigg[2\langle dF(e_i),\xi\rangle \Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), \xi\rangle - \Big\langle dF(e_i), \phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle \Big\}\tau_t \\
&\quad + \eta(dF(e_i))^2 \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t - \Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t, \xi\rangle - \Big\langle \tau_t, \phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle \Big\} \eta(dF(e_i))dF(e_i) \\
&\quad - \eta(\tau_t)\Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), \xi\rangle - \Big\langle dF(e_i), \phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle \Big\} dF(e_i) \\
&\quad - \eta(\tau_t)\eta(dF(e_i))\tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i) + 2\langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), dF(e_i)\rangle \eta(\tau_t)\xi \\
&\quad + \langle dF(e_i), dF(e_i)\rangle \Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t, \xi\rangle - \Big\langle \tau_t, \phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle \Big\}\xi \\
&\quad - \langle dF(e_i), dF(e_i)\rangle \eta(\tau_t)\phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big) - \langle \tilde\nabla_{\frac{\partial}{\partial t}}\tau_t, dF(e_i)\rangle \eta(dF(e_i))\xi \\
&\quad - \langle \tau_t, \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i)\rangle \eta(dF(e_i))\xi \\
&\quad - \langle \tau_t, dF(e_i)\rangle \Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), \xi\rangle - \Big\langle dF(e_i), \phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle \Big\}\xi \\
&\quad + \langle \tau_t, dF(e_i)\rangle \eta(dF(e_i))\phi\Big(dF\Big(\frac{\partial}{\partial t}\Big)\Big) + 3\Big\{ \langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), \phi\tau_t\rangle \phi(dF(e_i)) \\
&\quad + \langle dF(e_i), \Big\langle dF\Big(\frac{\partial}{\partial t}\Big), \tau_t\Big\rangle \xi - \eta(\tau_t) dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde\nabla_{\frac{\partial}{\partial t}}\tau_t)\rangle \phi(dF(e_i)) \\
&\quad + \langle dF(e_i), \phi\tau_t\rangle \Big(\Big\langle dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\Big\rangle \xi - \eta(dF(e_i)) dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i))\Big) \Big\} \\
&\quad - \langle \tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i), \phi(dF(e_i))\rangle \phi\tau_t \\
&\quad - \Big\langle dF(e_i), \Big\langle dF\Big(\frac{\partial}{\partial t}\Big), dF(e_i)\Big\rangle \xi - \eta(dF(e_i)) dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde\nabla_{\frac{\partial}{\partial t}} dF(e_i))\Big\rangle \phi\tau_t \\
&\quad - \langle dF(e_i), \phi(dF(e_i))\rangle \Big\{ \Big\langle dF\Big(\frac{\partial}{\partial t}\Big), \tau_t\Big\rangle \xi - \eta(\tau_t) dF\Big(\frac{\partial}{\partial t}\Big) + \phi(\tilde\nabla_{\frac{\partial}{\partial t}}\tau_t)\Big\} \bigg].
\end{aligned}
$$

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We need the following lemma.

Lemma 6 $\tilde{\nabla}_{\frac{\partial}{\partial t}} dF(e_i)\big|_{t=0} = \tilde{\nabla}_{e_i} dF(\frac{\partial}{\partial t})\big|_{t=0} = \bar{\nabla}_{e_i} V.$

From (4.8) and Lemma 6 we deduce the following.

Lemma 7

$$
\begin{aligned}
-\tilde{\nabla}_{\frac{\partial}{\partial t}} & \operatorname{trace} R^N(dF\cdot, \tau_t)dF\cdot|_{t=0} \\
&= -\frac{\epsilon+3}{4}\Big\{ -(\mathcal{J}_f V)^\top + \sum\Big(\langle \tau, \bar{\nabla}_{e_i} V\rangle e_i - 2\langle \bar{\nabla}_{e_i} V, e_i\rangle \tau\Big) + n\mathcal{J}_f V\Big\} \\
&\quad - \frac{\epsilon-1}{4}\Big\{ -n\big(\langle \mathcal{J}_f V, \xi\rangle + \langle \tau, \phi V\rangle\big)\xi + 3(\mathcal{J}_f V)^\perp \\
&\qquad + 3\sum\Big(\langle \bar{\nabla}_{e_i} V, \phi\tau\rangle \phi e_i + \langle e_i, \phi\tau\rangle\big(\langle V, e_i\rangle \xi + \phi(\bar{\nabla}_{e_i} V)\big)\Big)\Big\},
\end{aligned}
$$

where $(\mathcal{J}_f V)^\top$ *(resp.* $(\mathcal{J}_f V)^\perp$*) denotes the tangent (resp. normal) part of* $\mathcal{J}_f V$*.*

Consequently, we obtain the second variation formula as follows.

Theorem 8 *Let f be a biharmonic Legendrian immersion from a compact n-dimensional manifold M^n into a Sasakian space form $N^{2n+1}(\epsilon)$. Let $\{f_t\}$ be a smooth variation of f such that $f_0 = f$ and V are the corresponding variational vector field. Then we have*

$$
\frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} E_2(f_t) = \int_{M^n} \langle I(V), V\rangle \, dv_g, \tag{4.9}
$$

where

$$
\begin{aligned}
I(V) = &-\frac{\epsilon+3}{4}\Big\{ |\tau|^2 V + 2\operatorname{trace}\langle \nabla^f_{\cdot}\tau, V\rangle\cdot + 2\operatorname{trace}\langle \tau, \nabla^f_{\cdot} V\rangle\cdot + \langle \tau, V\rangle \tau \\
&\qquad - 2\operatorname{trace}\langle \nabla^f_{\cdot} V, \cdot\rangle \tau - (\mathcal{J}_f V)^\top + n\mathcal{J}_f V\Big\} \\
&+ \frac{\epsilon-1}{4}\Big\{ \eta(V)\operatorname{trace}(\eta(\nabla^f_{\cdot}\tau)\cdot) + |\tau|^2\eta(V)\xi + 2\operatorname{trace}\langle \nabla^f_{\cdot}\tau, \phi\cdot\rangle \phi V \\
&\qquad - 2\operatorname{trace}\langle \nabla^f_{\cdot}\tau, \phi V\rangle \phi\cdot - 4\phi(\nabla^f_{(\phi V)^\top} V) - 2\langle V, \phi\tau\rangle \xi + \eta(V)\phi\tau \\
&\qquad - 4\phi(\nabla^f_{\phi\tau} V) + 2\operatorname{trace}\langle \tau, \phi(\nabla^f_{\cdot} V)\rangle \phi\cdot \\
&\qquad - 3\langle \tau, \phi V\rangle \phi\tau + 2\operatorname{trace}\langle \nabla^f_{\cdot} V, \phi\cdot\rangle \phi\tau + 2n\eta(V)\phi\tau \\
&\qquad + 2\eta(V)(\phi V)^\top + n\eta(\mathcal{J}_f V)\xi - 3(\mathcal{J}_f V)^\perp\Big\} + \bar{\Delta}_f \mathcal{J}_f V.
\end{aligned}
$$

Proof When we compute (4.7) at p, we use the following:

$$
\bar{\nabla}_{e_i}(\phi V) = \langle e_i, V\rangle \xi - \eta(V)e_i + \phi(\bar{\nabla}_{e_i} V), \quad \bar{\nabla}_{e_i}(\phi e_i) = \langle e_i, e_i\rangle \xi + \phi(h(e_i, e_i)).
$$

Combining Lemmas 4, 5 and 7 we get (4.9). ■

Observe that if $\epsilon = 1$, formula (4.9) agrees with [11, (2.2)].

We put $F(X) := \langle h(X,X), \phi X\rangle$ for a vector field X of M^n. Then $F(\phi\tau)$ is globally defined on M^n. In terms of $\|\tau\|$ and $F(\phi\tau)$, we give the sufficient conditions for proper biharmonic Legendrian submanifolds to be unstable.

Theorem 9 *Let M^n be a compact proper biharmonic Legendrian submanifold in a Sasakian space form $N^{2n+1}(\epsilon)$. If*

$$\int_{M^n}\Big\{(\epsilon+3)||\tau||^4 - 3(\epsilon-1)F(\phi\tau)\Big\}\,dv_g > 0,$$

then M^n is unstable.

Proof We take τ as the variational vector field V. By Theorem 8, (2.3), (2.4), and (2.5) we have

$$\langle I(\tau),\tau\rangle = -(\epsilon+3)\|\tau\|^4 - 3(\epsilon-1)\langle h(\phi\tau,\phi\tau),\tau\rangle.$$

This completes the proof. ■

Inoguchi [9] determined proper biharmonic Legendrian curves of 3-dimensionl Sasakian space forms.

Theorem 10 **([9])** *Let $\gamma\colon I \to N^3(\epsilon)$ be a proper biharmonic Legendrian curve. Then $\epsilon > 1$ and γ is a Legendrian helix of curvature $\sqrt{\epsilon-1}$.*

It follows from Theorem 10 and Corollary 2(ii) that in case $n = 1$ or 2, then $\epsilon > -3$ and $F(\phi\tau) = -n\|\tau\|^2(\epsilon-1)$. Therefore applying Theorem 12 we state the following.

Corollary 11 *Let M^n be a compact proper biharmonic Legendrian submanifold in Sasakian space form $N^{2n+1}(\epsilon)$. If $n \leq 2$, then M^n is unstable.*

There is a special vector field along submanifolds in contact manifolds, *i.e.,* Reeb vector field ξ. Thus, it is natural and interesting to consider variations

$$V \in \mathrm{Span}\{\xi\} := \{a\xi \mid a \in C^\infty(M)\}.$$

We call such variations *R-variations*. If the second variation (4.1) under any *R*-variation is non-negative, f or M^n is said to be *R-stable*. Otherwise it is said to be *R-unstable*.

Proposition 12 *Let M^n be a compact proper biharmonic Legendrian submanifold in Sasakian space form $N^{2n+1}(\epsilon)$. Then we have*

$$\int_{M^n}\langle I(a\xi), a\xi\rangle\,dv = \int_{M^n}\Big\{(\Delta_{M^n}a)^2 + \frac{19-3\epsilon}{4}(\Delta_{M^n}a)a\Big\}\,dv_g,$$

where $a \in C^\infty(M^n)$.

Proof Let f be a proper biharmonic Legendrian immersion from M^n into $N^{2n+1}(\epsilon)$. We take $a\xi$ as the variational vector field, where $a \in C^\infty(M^n)$. We can easily see the following:

$$\bar{\Delta}_f(a\xi) = (\Delta_{M^n} a + na)\xi + 2\phi \operatorname{grad} a + a\phi\tau, \tag{4.10}$$

$$\mathcal{R}_f(a\xi) = an\xi. \tag{4.11}$$

By using Theorem 8, (4.10), (4.11), and Stokes' theorem, we obtain

$$\begin{aligned}
&\int_{M^n} \langle I(a\xi), a\xi\rangle \, dv \\
&\quad = \int_{M^n} \Big\{ -a^2|\tau|^2 + \langle \bar{\Delta}_f(\mathcal{J}_f(a\xi)), a\xi\rangle + \frac{3-3\epsilon-4n}{4}\langle \mathcal{J}_f(a\xi), a\xi\rangle \Big\} \, dv_g \\
&\quad = \int_{M^n} \Big\{ -a^2|\tau|^2 + \langle \mathcal{J}_f(a\xi), \bar{\Delta}_f(a\xi)\rangle + \frac{3-3\epsilon-4n}{4}(\Delta_{M^n} a)a \Big\} \, dv_g \\
&\quad = \int_{M^n} \Big\{ (\Delta_{M^n} a)^2 + n(\Delta_{M^n} a)a + 4\|\operatorname{grad} a\|^2 + \frac{3-3\epsilon-4n}{4}(\Delta_{M^n} a)a \Big\} \, dv_g \\
&\quad = \int_{M^n} \Big\{ (\Delta_{M^n} a)^2 + \frac{19-3\epsilon}{4}(\Delta_{M^n} a)a \Big\} \, dv_g.
\end{aligned}$$

This completes the proof. ∎

Theorem 13 *Let M^n be a compact proper biharmonic Legendrian submanifold in Sasakian space form $N^{2n+1}(\epsilon)$. Then M^n is R-stable if and only if $\lambda_1 \geq \frac{3\epsilon-19}{4}$, where λ_1 is the first non-zero eigenvalue of the Laplacian acting on $C^\infty(M^n)$.*

Proof For each $a \in C^\infty(M^n)$, we have the spectral decomposition (in L^2-sense):

$$a = \sum_{t\geq 0} a_t,$$

where $\Delta_{M^n} a_t = \lambda_t a_t$ and $0 = \lambda_0 < \lambda_1 < \lambda_2 < \cdots \uparrow \infty$. Since $\int_{M^n} a_i a_j \, dv = 0$ for $i \neq j$, from Proposition 12 we get

$$\int_{M^n} \langle I(a\xi), a\xi\rangle \, dv = \sum_{t\geq 1}(\lambda_t^2 + \frac{19-3\epsilon}{4}\lambda_t) \int_M a_t^2 \, dv.$$

If $\lambda_1 \geq \frac{3\epsilon-19}{4}$, we have $\int_{M^n}\langle I(a\xi), a\xi\rangle \, dv \geq 0$ for any function a and hence R-stable. Conversely, suppose that M^n is R-stable. If $\lambda_1 < \frac{3\epsilon-19}{4}$, then $\int_{M^n}\langle I(a_1\xi), a_1\xi\rangle \, dv < 0$ for an eigenfunction a_1 of λ_1. This is a contradiction. Therefore M^n must satisfy $\lambda_1 \geq \frac{3\epsilon-19}{4}$. ∎

Corollary 14 *Compact proper biharmonic Legendrian submanifolds of Sasakian space forms $N^{2n+1}(\epsilon)$ with $\epsilon \leq \frac{19}{3}$ are R-stable.*

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Remark. A compact proper biharmonic Legendrian surface in S^5 (see (3.5)) is unstable but R-stable.

Theorem 13 indicates that the spectral geometry of compact proper biharmonic Legendrian submanifolds in Sasakian space forms is important.

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References

[1] P. Baird and D. Kamissoko, *On constructing biharmonic maps and metrics.* Ann. Global Anal. Geom. **23**(2003), no. 1, 65–75.
[2] D. E. Blair, *Riemannian Geometry of Contact and Symplectic Manifolds.* Progress in Mathematics 203, Birkhäuser Boston, Boston, MA, 2002.
[3] R. Caddeo, S. Montaldo, and C. Oniciuc, *Biharmonic submanifolds of* S^3. Internat. J. Math. **12**(2001), no. 8, 867–876.
[4] ———, *Biharmonic submanifolds in spheres.* Israel J. Math. **130**(2002), 109-123.
[5] B.-Y. Chen, *Total Mean Curvature and Submanifold of Finite Type.* Series in Pure Mathematics 1, World Scientific Publishing, Singapore, 1984.
[6] ———, *A report on submanifolds of finite type.* Soochow J. Math. **22**(1996), no. 2, 117–337.
[7] F. Decruyenaere, F. Dillen, L. Verstraelen, and L. Vrancken, *The semiring of immersions of manifolds.* Beiträge Algebra Geom. **34**(1993), no. 2, 209-215.
[8] J. Eells and J. H. Sampson, *Harmonic mappings of Riemannian manifolds.* Amer. J. Math. **86**(1964), 109–160.
[9] J. Inoguchi, *Submanifolds with harmonic mean curvature vector field in contact* 3*-manifolds.* Colloq. Math. **100**(2004), no. 2, 163–179.
[10] G. Y. Jiang, 2*-harmonic maps and their first and second variational formulas.* Chinese Ann. Math. Ser. A **7**(1986), 389–402 (Chinese).
[11] C. Oniciuc, *On the second variation formula for biharmonic maps to a sphere.* Publ. Math. Debrecen. **61**(2002), no. 3-4, 613–622.
[12] T. Sasahara, *Legendre surfaces in Sasakian space forms whose mean curvature vectors are eigenvectors.* Publ. Math. Debrecen. **67**(2005), no. 3-4, 285–303.
[13] K. Yano and M. Kon, *Structures on manifolds.* Series in Pure Mathematics, 3, World Scientific Publishing, Singapore, 1984.

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