Geometrical Note, II.

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Figure 15.

In continuation of the results given in Vol. XI. of the *Proceedings* I note that the equations to AP, BQ, CR are respectively

$$\beta/nc = \gamma/mb : \gamma/na = \alpha/mc : \alpha/nb = \beta/ma :$$

and to AP′, BQ′, CR′ are

$$\beta/mc = \gamma/nb : \gamma/ma = \alpha/nc : \alpha/mb = \beta/na.$$

$$\text{Now}\quad \left.\begin{array}{lll} \text{BQ, CR} & \text{cut in} & \alpha_1; \\ \text{CR, AP} & & \beta_1; \\ \text{AP, BQ} & & \gamma_1 \end{array}\right\} \text{ and } \left.\begin{array}{lll} \text{BQ}', \text{CR}' & \text{cut in} & \alpha_2; \\ \text{CR}', \text{AP}' & & \beta_2; \\ \text{AP}', \text{BQ}' & & \gamma_2; \end{array}\right\}$$

where the points are given by the following

$$\left.\begin{array}{llll} (\alpha_1), & mnbc, & m^2ca, & n^2ab, \\ (\beta_1), & n^2bc, & mnca, & m^2ab, \\ (\gamma_1), & m^2bc, & n^2ca, & mnab, \end{array}\right\} \left.\begin{array}{llll} (\alpha_2), & mnbc, & n^2ca, & m^2ab, \\ (\beta_2), & m^2bc, & mnca, & n^2ab, \\ (\gamma_2), & n^2bc, & m^2ca, & mnab, \end{array}\right\}$$

where the common modulus is $2\Delta/(m^2+mn+n^2)abc$. Hence the triangles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, are concentroidal with ABC.

The straight lines $\alpha_1\alpha_2$, $\beta_1\beta_2$, $\gamma_1\gamma_2$ are parallel to the sides BC, CA, AB respectively, and since the equation to $\alpha_1\alpha_2$ is

$$-a\alpha(m^2+n^2)+mn(b\beta+c\gamma)=0,$$

it is seen that the above lines intersect in points α_3, β_3, γ_3 where α_3 is given by $a\alpha mn/(m^2-mn+n^2)=b\beta=c\gamma$, *i.e.*, the points are on the respective medians. From these coordinates we at once obtain that the modulus of similarity for the triangle $\alpha_3\beta_3\gamma_3$ is

$$(m-n)^2/(m^2+mn+n^2).$$

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The equations to the circles $\alpha_1\beta_1\gamma_1$; $\alpha_2, \beta_2, \gamma_2$; are

$$abc(m^2+mn+n^2)^2\Sigma a\beta\gamma = \Sigma a\alpha \,.\, \Sigma\{a\alpha\,.\,(m^2b^2+n^2c^2+2mnbc\cos A)\}$$
$$abc(m^2+mn+n^2)^2\Sigma a\beta\gamma = \Sigma a\alpha \,.\, \Sigma\{a\alpha\,.\,(n^2b^2+m^2c^2+2mnbc\cos A)\}$$

and their radical is $\Sigma[a\alpha(b^2-c^2)]=0$, *c.f.* (1).

The equation to the conic through the last-named six points is

$$mn(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)=(m^2-mn+n^2)(bc\beta\gamma+ca\gamma\alpha+ab\alpha\beta)\,;$$

this is an in-ellipse if $m^2-3mn+n^2=0$, *i.e.*, if

$$m/n=(3\pm\sqrt{5})/2.$$

It is similar and similarly situated to (4). The triangles $\alpha_1\beta_1\gamma_1$, $\alpha_2\beta_2\gamma_2$, are equal to one another and

$$=(m-n)^2\text{ABC}/(m^2+mn+n^2).$$

The perimeter of the hexagon $\alpha_1\beta_2\gamma_1\,\alpha_2\beta_1\gamma_2$

$$=(m-n)(m^2+n^2)/(m^2+mn+n^2)\times\text{the perimeter of ABC.}$$

The lines $\alpha_1\beta_2$, $\alpha_1\gamma_2$; are parallel respectively to AB, AC; with like results for the analogous lines.

The equation to $\alpha_1\alpha_3$ is

$$-mn(m+n)a\alpha+n^3b\beta+m^3c\gamma=0,$$

whence we see that if this line cuts CB in W, then

$$\text{CW}:\text{BW}=m^3:n^3.$$

If BQ′, CR; CR′, AP; AP′, BQ; intersect in α_4, β_4, γ_4, these points are given by

$$\left.\begin{array}{llll}(\alpha_4) & nbc, & mca, & mab,\\ (\beta_4) & mbc, & nca, & mab,\\ (\gamma_4) & mbc, & mca, & nab,\end{array}\right\}$$

and the modulus of similarity for $\alpha_4\beta_4\gamma_4$ is $(m-n)/(2m+n)$.

The equation to the circle $\alpha_4\beta_4\gamma_4$ is

$$(2m+n)^2abc\Sigma\alpha\beta\gamma = \Sigma a\alpha \,.\, \Sigma[ma\alpha\{-ma^2+(m+n)(b^2+c^2)\}].$$

Again, if CR′, BQ; AP′, CR; BQ′, AP; intersect in $\alpha_5, \beta_5, \gamma_5$, these points are given by

$$\left.\begin{array}{llll} (\alpha_5) & mbc, & nca, & nab, \\ (\beta_5) & nbc, & mca, & nab, \\ (\gamma_5) & nbc, & nca, & mab, \end{array}\right\}$$

and the modulus of similarity for $\alpha_5\beta_5\gamma_5$ is

$$(m-n)/(m+2n).$$

The equation to the circle $\alpha_5\beta_5\gamma_5$ is

$$(2n+m)^2abc\Sigma\alpha\beta\gamma = \Sigma a\alpha \,.\, \Sigma[na\alpha\{-ma^2+(m+n)(b^2+c^2)\}].$$

All the triangles are concentroidal with ABC.

If AP′$(\beta/mc = \gamma/nb)$ cuts QR$(-mna\alpha + n^2b\beta + m^2c\gamma = 0)$ in p', this point is given by

$$a\alpha/(m+n) = b\beta/m = c\gamma/n = \Delta,$$

i.e., p' is on the mid-parallel to BC, (say ZY), and is the mid point of QR. It is also readily seen that

$$Zp' = n\,.\,ZY.$$

Hence p', q', r' (analogous points to p') form the medial triangle of PQR.

In like manner if AP cuts Q′R′ in p, then pqr is the medial triangle of P′Q′R′, and $Zp = m\,.\,ZY$.

We see then that $pqrp'q'r'$ are the exact analogues on

YZ, ZX, XY of PQRP′Q′R′ on BC, CA, AB.

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I consider two envelopes, viz., of $\beta_1\gamma_4$, $\beta_4\gamma_3$.

The line $\beta_1\gamma_4$ is given by

$$ma\alpha(m+n) - b\beta(m^2+mn+n^2) + c\gamma mn = 0,$$

or, as it may be written,

$$m^2(b\beta + c\gamma) - m(a\alpha + b\beta + c\gamma) + b\beta = 0$$

The envelope, therefore, is

$$(a\alpha + b\beta + c\gamma)^2 = -4b\beta(b\beta + c\gamma),$$

an hyperbola of which the asymptotes are

$$\beta = 0, \ \beta b + c\gamma = 0$$

i.e., CA, and the parallel to BC through A.

The line $\beta_4\gamma_3$ is given by

$$n^2a\alpha + m(m-n)b\beta - mnc\gamma = 0,$$

or by $$n^2(a\alpha + 2b\beta + c\gamma) - n(3b\beta + c\gamma) + b\beta = 0.$$

The envelope of which is

$$(b\beta + c\gamma)^2 = 4ab\alpha\beta,$$

which is also an hyperbola, the asymptotes being given by

$$\alpha = 0, \ 3b\beta + c\gamma = 0.$$

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