ON SOME MATRIX THEOREMS OF FROBENIUS AND McCOY

J. K. GOLDHABER AND G. WHAPLES

1. Introduction. McCoy, following Frobenius, studied a problem which can be described as follows. Let k be an arbitrary field, k^c its algebraic closure, and $\mathfrak{A}$ any algebra of $n \times n$ matrices over k which contains the identity I. Define a *canonical ordering* to be a set of n mappings λ_i of $\mathfrak{A}$, or of a subset $\mathfrak{S}$ of $\mathfrak{A}$, into k^c such that the sequence $\lambda_1(A), \lambda_2(A), \ldots, \lambda_n(A)$, for each $A \in \mathfrak{S}$, consists of the characteristic values (roots of $\det(A - xI) = 0$) of A, each with the right multiplicity. Define a canonical ordering to be a *Frobenius ordering* if, for all non-commutative polynomials $f(x_1, x_2, \ldots, x_m)$ and all finite subsets $A_1, A_2, \ldots, A_m$ of $\mathfrak{A}$,

$$(1) \qquad \lambda_i f(A_1, A_2, \ldots, A_m) = f(\lambda_i(A_1), \ldots, \lambda_i(A_m)), \qquad i = 1, \ldots, n.$$

Say that $\mathfrak{A}$ has property F if it has a Frobenius ordering. (Previous authors defined F in an apparently weaker fashion, demanding that (1) holds only for elements of a fixed system of generators of $\mathfrak{A}$ rather than for all finite subsets; but a simple substitution argument shows that their definition is equivalent to ours. Also they assumed k to be algebraically closed.)

Frobenius [**3a**] proved that every commutative $\mathfrak{A}$ has property F; McCoy [**5**] proved that F is equivalent to the property

$$(M) \qquad \mathfrak{A}/\text{rad}\ \mathfrak{A} \text{ is commutative,}$$

where rad $\mathfrak{A}$ = radical of $\mathfrak{A}$ = maximal nilpotent left (or right, or two-sided) ideal in $\mathfrak{A}$; Goldhaber [4] proved F equivalent to

(P) For every $A, B \in \mathfrak{A}$ there is a canonical ordering, possibly defined only for A, B, and $A + B$, such that

$$(2) \qquad \lambda_i(A + B) = \lambda_i(A) + \lambda_i(B), \qquad i = 1, 2, \ldots, n.$$

There is also given in [**4**] a simple proof of the theorem of McCoy; however, the proof of a crucial lemma there (our Lemma 2) is not valid for all n unless k has characteristic 0.

In the present paper we give a simple proof of this lemma which avoids all trouble with the characteristic, prove McCoy's and Goldhaber's theorems without restriction on the field k, and show that if k is quasi-algebraically closed (i.e. is not the centre of any non-commutative division algebra) then P can be replaced by the weaker condition

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(P') The sum of every two nilpotent elements of $\mathfrak{A}$ is nilpotent.

To see that P implies P' recall that a matrix is nilpotent if and only if all its characteristic values are 0.

2. Equivalence of F, M, P, and P' for quasi-algebraically closed k. Throughout the paper $\mathfrak{A}$, k, k^c, λ_i retain the meaning given them in the introduction. All the algebras used are assumed to contain an identity element, and if they are matric algebras of any dimension they are assumed to contain the identity matrix of that dimension. (It is quite a simple matter to deduce theorems from our work about algebras which do not contain the identity matrix, but we omit this as not worth the effort.)

LEMMA 1. *If K is any field containing k, then* rad $(K \times_k \mathfrak{A})$ *contains* $K \times_k (\text{rad } \mathfrak{A})$.

For the necessary theory of the operation $K \times_k$ see [**1**], [**2**], or [**3**]. In [**1**] and [**3**] this operation is called "extending the ground field." Since rad $\mathfrak{A}$ is nilpotent there is an integer m such that every product of m elements of rad $\mathfrak{A}$ is 0. The product of any m elements of $K \times_k$ rad $\mathfrak{A}$ is a linear combination, over K, of products of m elements of rad $\mathfrak{A}$, and hence is also 0. Thus $K \times_k$ rad $\mathfrak{A}$ is a nilpotent ideal of $K \times_k \mathfrak{A}$, and Lemma 1 is proved.

LEMMA 2. *If $A \in \mathfrak{A}$ and $N \in$* rad $\mathfrak{A}$ *and x is an indeterminate, then*

(3) $$\det (A - xI) = \det (A + N - xI).$$

Let $k(x)$ be the field of rational functions of x over k. The matrix $(A - xI)$ has an inverse in $k(x) \times_k \mathfrak{A}$, and

(4) $$\det (A + N - xI) = \det (A - xI) \det (I + (A - xI)^{-1}N).$$

By Lemma 1, $(A - xI)^{-1} N$ is nilpotent, hence it is similar to a matrix with zeros on and above the main diagonal, hence the third determinant in (4) equals 1 and (3) follows.

THEOREM 1. *For every field k and every matric algebra $\mathfrak{A}$, F is equivalent to M.*

McCoy [**5**] and Goldhaber [**4**] give proofs of this theorem when k is algebraically closed.

Suppose that k is arbitrary and that $\mathfrak{A}$ satisfies M, i.e., $\mathfrak{A}/\text{rad } \mathfrak{A}$ is commutative. By Lemma 1, $(k^c \times_k \mathfrak{A})/\text{rad } (k^c \times_k \mathfrak{A})$ is a homomorphic image of $(k^c \times_k \mathfrak{A})/ (k^c \times_k \text{rad } \mathfrak{A})$, hence $k^c \times_k \mathfrak{A}$ has property M, hence it has property F. If we identify $k^c \times_k \mathfrak{A}$ with an algebra of matrices over k^c in the obvious way, then $\mathfrak{A}$ is contained in $k^c \times_k \mathfrak{A}$ and clearly $\mathfrak{A}$ also has property F.

LEMMA 4. *Let $\mathfrak{A}$ and $\mathfrak{A}^*$ be two matric algebras over k, let $A \rightarrow A^*$ be a homomorphism of $\mathfrak{A}$ into $\mathfrak{A}^*$ which maps the identity matrix of $\mathfrak{A}$ onto the identity matrix of $\mathfrak{A}^*$ and has precisely* rad $\mathfrak{A}$ *as its kernel. Then for every $A \in \mathfrak{A}$ the* *matrices A and A^* have the same set of characteristic values (though not in general with the same multiplicities).*

Let $f(x)$ and $f^*(x)$ be the minimum polynomials of A and of A^* respectively. Since $f^*(A)$ is nilpotent, $f(x)$ divides some power of $f^*(x)$. On the other hand, $f(A) = 0$ implies that $f(A^*) = 0$, and hence $f^*(x)$ divides $f(x)$. Our lemma now follows from the well-known fact that the minimum equation and the characteristic equation have the same set of roots.

THEOREM 2. *If k is quasi-algebraically closed, then F, M, P, and P' are equivalent.*

We already know that M implies F, F implies P, and P implies P', and thus it suffices to prove that P' implies M or, what is the same thing, that not M implies not P'.

Suppose then that $\mathfrak{A}/\text{rad}\ \mathfrak{A}$ is not commutative. Since it is a direct sum of simple algebras it must, in view of our assumption on k, contain a simple component which is a total matric algebra of dimension at least two (or order at least four) over k. Hence the algebra $\mathfrak{A}^*$ of Lemma 4 can be so chosen that it contains two elements A^* and B^* (images of elements A and B of $\mathfrak{A}$) which have in their upper left hand corners the elements

$$\begin{bmatrix} 0 & 0 & \\ 1 & 0 & \\ & & \\ \end{bmatrix}, \qquad \begin{bmatrix} 0 & 1 & \\ 0 & 0 & \\ & & \\ \end{bmatrix}$$

and which have zeros in all other positions.

A^* and B^* are clearly nilpotent; Lemma 4 tells us that their inverse images A and B are nilpotent (but not in rad $\mathfrak{A}$). Since $A^* + B^*$ is obviously not nilpotent, we see in just the same way that $A + B$ is not nilpotent. We have proved that not M implies not P'.

3. Equivalence of F, M, and P for arbitrary k. The next theorem requires the more elaborate methods of [**4**].

THEOREM 3. *If k is any field, $\mathfrak{A}$ any algebra over k, then F, M, and P are equivalent.*

In view of Theorem 2 and the well-known fact [**1**; **3**] that Galois fields are quasi-algebraically closed, we may, and shall, assume that *k has an infinite number of elements.*

Suppose that $\mathfrak{A}$ has property P. According to Theorem 6.1 of [**4**] (the proof of which does not require the algebraic closure of k, but only the existence of an infinite number of distinct elements in k), $\mathfrak{A}$ has a canonical ordering such that for any finite subset $A_1, A_2, \ldots, A_m$ of $\mathfrak{A}$, all $\alpha_r \in k$, and all $i = 1, 2, \ldots, n$,

$$\lambda_i\left(\sum_r \alpha_r A_r\right) = \sum_r \alpha_r \lambda_i(A_r). \tag{5}$$

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Let $A_1, A_2, \ldots, A_m$ be a linear k-basis for $\mathfrak{A}$, let $t_1, t_2, \ldots, t_m, x$ be commutative indeterminates over k, and consider the polynomial

$$\det\left(\sum_r t_r A_r - xI\right) - \prod_i \left(\sum_r t_r \lambda_i A_r - x\right). \tag{6}$$

From (5) it follows that for every specialization of the t_i and x into k, (6) is equal to zero. Consequently by [**6**, p. 70] we have

$$\det\left(\sum_r t_r A_r - xI\right) \equiv \prod_i \left(\sum_r t_r \lambda_i A_r - x\right), \tag{7}$$

each side of (7) being considered as an element of the ring $k\,[t_1, t_2, \ldots, t_m, x]$.

Now form the algebra $\mathfrak{A}^* = k^c \times_k \mathfrak{A}$ and, as before, consider its elements as $n \times n$ matrices with elements in k^c. The matrices $A_1, A_2, \ldots, A_m$ are a k^c-basis for $\mathfrak{A}^*$. (7) shows that if we use (5) to define a set of mappings $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathfrak{A}^*$ into k^c, allowing the α_r to be elements of k^c, the resulting set of mappings is a canonical ordering on $\mathfrak{A}^*$. It obviously satisfies P.

By Theorem 2, $\mathfrak{A}^*$ has property F, hence its subalgebra $\mathfrak{A}$ has property F.

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University of Connecticut
Indiana University

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