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Strichartz Inequalities for the Wave Equation with the Full Laplacian on the Heisenberg Group

Giulia Furioli, Camillo Melzi, and Alessandro Veneruso

Abstract. We prove dispersive and Strichartz inequalities for the solution of the wave equation related to the full Laplacian on the Heisenberg group, by means of Besov spaces defined by a Littlewood–Paley decomposition related to the spectral resolution of the full Laplacian. This requires a careful analysis due also to the non-homogeneous nature of the full Laplacian. This result has to be compared to a previous one by Bahouri, Gérard and Xu concerning the solution of the wave equation related to the Kohn Laplacian.

1 Introduction

The aim of this paper is to study Strichartz inequalities for the solution of the following Cauchy problem for the wave equation on the Heisenberg group $\mathbb{H}_n$ of topological dimension $2n+1$ and homogeneous dimension $N = 2n+2$:

$$(1) \qquad \begin{cases} \partial_t^2 u + \mathcal{L}u = f \in L^1((0,T), L^2(\mathbb{H}_n)) \\ u(0) = u_0 \in \dot{B}_2^{1,2}(\mathcal{L}) \\ \partial_t u(0) = u_1 \in L^2(\mathbb{H}_n) \end{cases}$$

where $\mathcal{L}$ is the full Laplacian on $\mathbb{H}_n$ (to be defined in Section 2) and the Besov spaces $\dot{B}_r^{\rho,q}(\mathcal{L})$ are defined by a Littlewood–Paley decomposition related to the spectral resolution of the full Laplacian (see Section 3). To our knowledge, the problem of establishing dispersive or Strichartz inequalities for solutions of partial differential equations in $\mathbb{H}_n$, or more generally on Lie groups, has been treated only in three recent papers. Bahouri, Gérard and Xu [BGX] studied the Cauchy problem analogous to (1) with the Kohn Laplacian Δ instead of the full Laplacian $\mathcal{L}$, using the Besov spaces $\dot{B}_r^{\rho,q}(\Delta)$, which contain $\dot{B}_r^{\rho,q}(\mathcal{L})$ for $\rho > 0$ (see Proposition 3.4). Furioli and Venedruso [FV] studied the corresponding Cauchy problem for the Schrödinger equation where they introduced the full Laplacian instead of the Kohn Laplacian, but still they used the Besov spaces $\dot{B}_r^{\rho,q}(\Delta)$. Del Hierro [D] generalised the results of [BGX] to H-type groups.

Strichartz and dispersive inequalities were first introduced in the Euclidean setting. They concern the so-called "dispersive equations"

$$\partial_t u = iP(D)u + f, \quad u(0) = u_0$$

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or, more generally, equations whose linear operator is related to oscillatory integrals as for the wave equation. For these equations, the question is whether a solution with L^2 initial data becomes eventually more "regular" due to some smoothing effect of the related linear operator combined with the integration in time appearing in the expression of the solution (as for example in formula (3)). This question has been addressed in $\mathbb{R}$ following the pioneering paper by Strichartz [S] on the Schrödinger equation, and has since been the subject of a huge literature concerning many different kinds of equations. In the beautiful introduction to their paper [KPV], Kenig, Ponce and Vega briefly touched on the history of the problem and showed that ellipticity is not essential for proving this kind of result; what is essential is the behavior of the real symbol of the operator $P(D)$: $P(D)f(x) = \int e^{ix\xi}P(\xi)\hat{f}(\xi)\,d\xi$. In the setting of the Heisenberg group, two operators have been considered: the Kohn Laplacian, which is not elliptic and the full Laplacian, which is elliptic. For both operators it is possible to prove this kind of result.

Let us begin by recalling the structure of the solution of the Cauchy problem (1). It is well known that the solution of (1) can be written as $u = v + w$, where v is the solution of (1) with $f = 0$ and w is the solution of (1) with $u_0 = u_1 = 0$. More precisely,

$$v(t) = \cos(t\sqrt{\mathcal{L}})u_0 + \frac{\sin(t\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}}u_1, \tag{2}$$

$$w(t) = \int_0^t \frac{\sin((t-\sigma)\sqrt{\mathcal{L}})}{\sqrt{\mathcal{L}}} f(\sigma)\,d\sigma. \tag{3}$$

We can now state the main results of this paper. As always when dealing with Strichartz inequalities, we prove first the following dispersive inequality on v.

Proposition 1.1 *Let $\rho \in [N - \frac{3}{2}, N - \frac{1}{2}]$ and $u_0 \in \dot{B}_1^{\rho,1}(\mathcal{L})$, $u_1 \in \dot{B}_1^{\rho-1,1}(\mathcal{L})$. Then there exists a constant $C > 0$, which does not depend on u_0, u_1, such that*

$$\|v(t)\|_{L^\infty(\mathbb{H}_n)} \leq C|t|^{-\frac{1}{2}}(\|u_0\|_{\dot{B}_1^{\rho,1}(\mathcal{L})} + \|u_1\|_{\dot{B}_1^{\rho-1,1}(\mathcal{L})}), \quad t \in \mathbb{R}^*.$$

We note the main difference between Proposition 1.1 and [BGX, Théorème 1.2]: in the hypotheses of the latter theorem, they obtain only the index $\rho = N - \frac{1}{2}$, which in that case is sharp because of the homogeneity property of the Kohn Laplacian Δ.

For every interval $I \subset \mathbb{R}$, we will denote by $L_I^p(X)$ the space $L^p(I, X)$. The Strichartz inequalities we have obtained are the following.

Theorem 1.2 *Let r_1, $r_2 \in [2, \infty]$. Let ρ_1, $\rho_2 \in \mathbb{R}$ and p_1, $p_2 \in [1, \infty]$ such that*

(i) $\frac{2}{p_i} = \frac{1}{2} - \frac{1}{r_i}$ *for* $i = 1, 2$;

(ii) $-\left(N - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{r_1}\right) + 1 \leq \rho_1 \leq -\left(N - \frac{3}{2}\right)\left(\frac{1}{2} - \frac{1}{r_1}\right) + 1$;

(iii) $-\left(N - \frac{1}{2}\right)\left(\frac{1}{2} - \frac{1}{r_2}\right) \leq \rho_2 \leq -\left(N - \frac{3}{2}\right)\left(\frac{1}{2} - \frac{1}{r_2}\right)$.

Let r_i', p_i' such that $\frac{1}{r_i'} + \frac{1}{r_i} = 1$ and $\frac{1}{p_i'} + \frac{1}{p_i} = 1$ for $i = 1, 2$. Then for every interval I which contains 0*, the following estimates hold:*

$$\|v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}^{\rho_1,2}_{r_1}(\mathcal{L}))} + \|\partial_t v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}^{\rho_1-1,2}_{r_1}(\mathcal{L}))} \leq C\,(\|u_0\|_{\dot{B}^{1,2}_2(\mathcal{L})} + \|u_1\|_{L^2(\mathbb{H}_n)}),$$

$$\|w\|_{L^{p_1}_{I}(\dot{B}^{\rho_1,2}_{r_1}(\mathcal{L}))} + \|\partial_t w\|_{L^{p_1}_{I}(\dot{B}^{\rho_1-1,2}_{r_1}(\mathcal{L}))} \leq C\,\|f\|_{L^{p_2'}_{I}(\dot{B}^{-\rho_2,2}_{r_2'}(\mathcal{L}))},$$

where the constant $C > 0$ depends neither on u_0, u_1, f nor on the interval I.

We can deduce from Theorem 1.2 the following result, which we compare to the analogous result by Bahouri, Gérard, and Xu.

Corollary 1.3 *Let u be the solution of the Cauchy problem* (1)*. If p and r satisfy $0 \leq \frac{2}{p} \leq \frac{1}{2} - \frac{1}{r}$ and $(N-1)(\frac{1}{2} - \frac{1}{r}) - 1 \leq \frac{1}{p} \leq N(\frac{1}{2} - \frac{1}{r}) - 1$, then there exists a constant $C > 0$, which does not depend on u_0, u_1, f, such that, for every interval I which contains* 0*, the following estimate holds:*

$$\|u\|_{L^p_I\left(L^r(\mathbb{H}_n)\right)} \leq C\left(\|u_0\|_{\dot{B}^{1,2}_2(\mathcal{L})} + \|u_1\|_{L^2(\mathbb{H}_n)} + \|f\|_{L^1_I\left(L^2(\mathbb{H}_n)\right)}\right).$$

In [BGX, Théorème 1.1], the solution of the wave equation with the Kohn Laplacian was proved to belong to $L^p_I\left(L^r(\mathbb{H}_n)\right)$ only for p and r satisfying $2N - 1 \leq p \leq \infty$ and $\frac{1}{p} = N(\frac{1}{2} - \frac{1}{r}) - 1$, which is a subset of the range of values of p and r we have found (since it is equivalent to $0 \leq \frac{2}{p} \leq \frac{1}{2} - \frac{1}{r}$ and $\frac{1}{p} = N(\frac{1}{2} - \frac{1}{r}) - 1$). The set of the admissible values $(\frac{1}{r}, \frac{1}{p})$ found in Corollary 1.3 is represented in Figure 1, where the result by Bahouri, Gérard, and Xu corresponds to the segment BC.

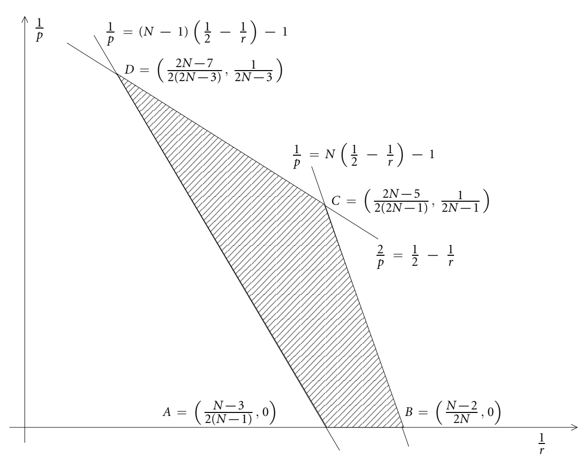

Figure 1

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Other results on the sharpness of the dispersive inequalities and remarks about the behaviour of the operator $e^{-it\sqrt{\mathcal{L}}}$ when analysed by the Besov spaces $\dot{B}_r^{\rho,q}(\Delta)$ can be found in Section 6.

2 Notation and Preliminaries

In this paper, $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}_+$ the set of positive real numbers. For $p \in [1, \infty]$ we denote by p' the conjugate index of p such that $\frac{1}{p} + \frac{1}{p'} = 1$. We will denote by C any positive constant, depending only on the group, which will not be necessarily the same at each occurrence.

In this section we recall some basic facts about harmonic analysis on the Heisenberg group. For the proofs and further information, see also [Ge1, Ge2, N, F, BJRW].

The Heisenberg group $\mathbb{H}_n$, $n \in \mathbb{Z}_+$, is the nilpotent Lie group whose underlying manifold is $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ with the following multiplication law:

$$(x, y, s)(x', y', s') = (x+x', y+y', s+s'+2(y\cdot x'-x\cdot y')), \quad x, x', y, y' \in \mathbb{R}^n,\ s, s' \in \mathbb{R}.$$

The Lie algebra of $\mathbb{H}_n$ is generated by the left-invariant vector fields $X_1, \dots, X_n$, $Y_1, \dots, Y_n$, S, where

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial s}, \qquad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial s}, \qquad S = \frac{\partial}{\partial s}.$$

We indicate an element $g = (x, y, s) \in \mathbb{H}_n$ as $g = (z, s)$, where $z = x + iy \in \mathbb{C}^n$. The family of dilations $\{\delta_r : r > 0\}$ given by $\delta_r(z, s) = (rz, r^2 s)$ makes $\mathbb{H}_n$ a stratified group of homogeneous dimension $N = 2n + 2$. The Kohn Laplacian

$$\Delta = -\sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

satisfies the homogeneity property $\Delta(f \circ \delta_r) = r^2(\Delta f \circ \delta_r)$, $r > 0$, while the full Laplacian $\mathcal{L} = \Delta - S^2$ is not invariant with respect to the dilation structure of $\mathbb{H}_n$.

The bi-invariant Haar measure dg on $\mathbb{H}_n$ coincides with the Lebesgue measure on $\mathbb{R}^{2n+1}$. The convolution of two functions f_1 and f_2 on G, defined by

$$f_1 * f_2(g) = \int_{\mathbb{H}_n} f_1(g{g'}^{-1}) f_2(g')\, dg', \quad g \in \mathbb{H}_n,$$

satisfies Young's inequality (where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$):

$$\|f_1 * f_2\|_{L^r(\mathbb{H}_n)} \le \|f_1\|_{L^p(\mathbb{H}_n)} \|f_2\|_{L^q(\mathbb{H}_n)}.$$

The convolution of $\varphi \in \mathcal{S}(\mathbb{H}_n)$ and $u \in \mathcal{S}'(\mathbb{H}_n)$, where $\mathcal{S}(\mathbb{H}_n)$ is the Schwartz space and $\mathcal{S}'(\mathbb{H}_n)$ is the space of tempered distributions, is defined as usual (see [V]). We say that a function f on $\mathbb{H}_n$ is radial if the value of $f(z, s)$ depends only on $|z|$ and s. We denote by $\mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$ and by $L^p_{\mathrm{rad}}(\mathbb{H}_n)$, $1 \le p \le \infty$, the spaces of radial functions in $\mathcal{S}(\mathbb{H}_n)$ and in $L^p(\mathbb{H}_n)$, respectively. The space $L^1_{\mathrm{rad}}(\mathbb{H}_n)$ is a commutative, closed $*$-subalgebra of $L^1(\mathbb{H}_n)$. The Gelfand spectrum Σ of $L^1_{\mathrm{rad}}(\mathbb{H}_n)$ can be identified, as a measure space, with the space $\mathbb{N} \times \mathbb{R}$ equipped with the Godement–Plancherel measure μ defined by

$$\int_\Sigma F(\psi)\, d\mu(\psi) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{+\infty} \binom{m+n-1}{m} \int_{\mathbb{R}} F(m,\lambda)|\lambda|^n\, d\lambda.$$

The spherical Fourier transform of a function $f \in L^1_{\mathrm{rad}}(\mathbb{H}_n)$ is given by

$$\hat{f}(m,\lambda) = \int_{\mathbb{H}_n} f(g)\omega_{m,\lambda}(g)\, dg, \quad m \in \mathbb{N},\ \lambda \in \mathbb{R},$$

with

$$\omega_{m,\lambda}(z,s) = \binom{m+n-1}{m}^{-1} e^{i\lambda s} e^{-|\lambda||z|^2} L_m^{(n-1)}(2|\lambda||z|^2),$$

where $L_m^{(\alpha)}$ is the Laguerre polynomial of type $\alpha \in \mathbb{N}$ and degree $m \in \mathbb{N}$, defined by

$$L_m^{(\alpha)}(\tau) = \sum_{k=0}^{m} \frac{(-1)^k}{k!} \binom{m+\alpha}{k+\alpha} \tau^k, \quad \tau \in \mathbb{R}.$$

We have $\widehat{f_1 * f_2} = \widehat{f_1}\widehat{f_2}$ for any $f_1, f_2 \in L^1_{\mathrm{rad}}(\mathbb{H}_n)$. Since $\|\omega_{m,\lambda}\|_{L^\infty(\mathbb{H}_n)} = 1$, the spherical Fourier transform is bounded from $L^1_{\mathrm{rad}}(\mathbb{H}_n)$ to $L^\infty(\Sigma)$. Moreover, by the Godement–Plancherel theory, it extends uniquely to a unitary operator

$$\mathcal{G}\colon L^2_{\mathrm{rad}}(\mathbb{H}_n) \to L^2(\Sigma).$$

We still write $\hat{f}$ instead of $\mathcal{G}f$. If $f \in L^2_{\mathrm{rad}}(\mathbb{H}_n)$ and $\hat{f} \in L^1(\Sigma)$, the following inversion formula holds:

$$f(g) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{+\infty} \binom{m+n-1}{m} \int_{\mathbb{R}} \hat{f}(m,\lambda)\omega_{m,-\lambda}(g)|\lambda|^n\, d\lambda, \quad g \in \mathbb{H}_n. \tag{4}$$

The space $\mathcal{G}(\mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n))$ has been described first in [Ge2] and then in [BJR]. For our purposes, it is sufficient to remark that $\mathcal{G}(\mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)) \subset L^1(\Sigma)$. Moreover, if $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$, the functions Δf and $\mathcal{L} f$ are in $\mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$ and their spherical Fourier transforms are given by:

$$\widehat{\Delta f}(m,\lambda) = 4(2m+n)|\lambda|\hat{f}(m,\lambda), \tag{5}$$

$$\widehat{\mathcal{L} f}(m,\lambda) = (4(2m+n)|\lambda| + \lambda^2)\hat{f}(m,\lambda). \tag{6}$$

Both Δ and $\mathcal{L}$ are positive self-adjoint operators densely defined on $L^2(\mathbb{H}_n)$. So by the spectral theorem, for any bounded Borel function h on $[0,+\infty)$ the operators $h(\Delta)$ and $h(\mathcal{L})$ are bounded on $L^2(\mathbb{H}_n)$. Since the point 0 may be neglected in the

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spectral resolution (see [A, C]), we consider that the function h is defined on $\mathbb{R}_+$. If $f \in L^2_{\mathrm{rad}}(\mathbb{H}_n)$ the functions $h(\Delta)f$ and $h(\mathcal{L})f$ are in $L^2_{\mathrm{rad}}(\mathbb{H}_n)$ and their spherical Fourier transforms, by (5) and (6), are given by:

$$\widehat{h(\Delta)f}(m,\lambda) = h(4(2m+n)|\lambda|)\hat{f}(m,\lambda), \tag{7}$$

$$\widehat{h(\mathcal{L})f}(m,\lambda) = h(4(2m+n)|\lambda| + \lambda^2)\hat{f}(m,\lambda). \tag{8}$$

If $f \in \mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$, then by the previous remarks, the functions $h(\Delta)f$ and $h(\mathcal{L})f$ can be recovered from their spherical Fourier transforms by means of the inversion formula (4).

The operators $h(\Delta)$ and $h(\mathcal{L})$ commute with left translations. So by Schwartz's kernel theorem, which is valid also on $\mathbb{H}_n$ (see [KVW, Theorem 3.2]), they admit kernels in $\mathcal{S}'(\mathbb{H}_n)$ which we call H_Δ and $H_\mathcal{L}$, respectively, satisfying $h(\Delta)f = f * H_\Delta$ and $h(\mathcal{L})f = f * H_\mathcal{L}$ for any $f \in \mathcal{S}(\mathbb{H}_n)$. If h is the restriction on $\mathbb{R}_+$ of a function in $\mathcal{S}(\mathbb{R})$, then H_Δ and $H_\mathcal{L}$ are in $\mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$ (see [Ge2, Hu, M] for H_Δ, [V] for $H_\mathcal{L}$; see also [FMV, Corollary 7]), and their spherical Fourier transforms, by (7) and (8), are given by:

$$\widehat{H_\Delta}(m,\lambda) = h(4(2m+n)|\lambda|), \quad \widehat{H_\mathcal{L}}(m,\lambda) = h(4(2m+n)|\lambda| + \lambda^2).$$

3 Littlewood–Paley Decompositions and Besov Spaces

Let R be a non-negative function in $C^\infty(\mathbb{R})$ such that $\operatorname{supp} R \subset [\frac{1}{4}, 4]$ and

$$\sum_{j\in\mathbb{Z}} R(2^{-2j}\tau) = 1, \quad \tau > 0.$$

For any $j \in \mathbb{Z}$ we denote by φ_j and ψ_j the kernels of the operators $R(2^{-2j}\Delta)$ and $R(2^{-2j}\mathcal{L})$, respectively. The remarks at the end of Section 2 guarantee that $\varphi_j, \psi_j \in \mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$ and

$$\widehat{\varphi_j}(m,\lambda) = R(2^{2-2j}(2m+n)|\lambda|), \tag{9}$$

$$\widehat{\psi_j}(m,\lambda) = R(2^{-2j}(4(2m+n)|\lambda| + \lambda^2)). \tag{10}$$

If $j, k \in \mathbb{Z}$ with $|j-k| \geq 2$, then $\varphi_j * \varphi_k = \psi_j * \psi_k = 0$. Moreover we have the following.

Lemma 3.1 *For any $j \in \mathbb{Z}$ the sets*

$$U_j = \{k \in \mathbb{Z} : \varphi_j * \psi_k \neq 0\} \quad \textit{and} \quad V_j = \{k \in \mathbb{Z} : \psi_j * \varphi_k \neq 0\}$$

are finite, and $\min U_j \geq j-2$, $\max V_j \leq j+2$.

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Proof Fix $j \in \mathbb{Z}$ and $k \in U_j$. By (9) and (10) there exist $m \in \mathbb{N}$, $\lambda \in \mathbb{R}$ such that

$$R(2^{2-2j}(2m+n)|\lambda|)R(2^{-2k}(4(2m+n)|\lambda|+\lambda^2)) \neq 0.$$

Put $\xi = 4(2m+n)|\lambda|$ and $\eta = \lambda^2$. The pair (ξ, η) satisfies the following system of inequalities:

$$\begin{aligned} &\frac{1}{4} \leq 2^{-2j}\xi \leq 4, \\ &\frac{1}{4} \leq 2^{-2k}(\xi+\eta) \leq 4, \\ &0 \leq \eta \leq \frac{\xi^2}{16n^2}. \end{aligned} \tag{11}$$

On the other hand, it is easy to check that the system (11) admits solutions only if

$$2^{2j-4} \leq 2^{2k} \leq \frac{2^{4j+2}}{n^2} + 2^{2j+4}.$$

These conditions give the conclusion not only for U_j, but also for V_j; for the latter, it is sufficient to interchange the roles of j and k, noting that $k \in V_j$ if and only if $j \in U_k$. ∎

A direct application of the inversion formula (4) gives

$$\varphi_j(z,s) = 2^{Nj}\varphi_0(2^j z, 2^{2j}s), \quad j \in \mathbb{Z},\ (z,s) \in \mathbb{H}_n. \tag{12}$$

So

$$\|\varphi_j\|_{L^1(\mathbb{H}_n)} = \|\varphi_0\|_{L^1(\mathbb{H}_n)}, \quad j \in \mathbb{Z}. \tag{13}$$

On the other hand, despite the lack of homogeneity, by [FMV, Proposition 6] there exists $C > 0$ such that

$$\|\psi_j\|_{L^1(\mathbb{H}_n)} \leq C, \quad j \in \mathbb{Z}. \tag{14}$$

In this section, in order to carry on some results which are valid for both operators Δ and $\mathcal{L}$, we use the notation L to denote either Δ or $\mathcal{L}$. For any $u \in \mathcal{S}'(\mathbb{H}_n)$, if $L = \Delta$, we set $\Delta_j u = u * \varphi_j$, if $L = \mathcal{L}$, we set $\Delta_j u = u * \psi_j$. By standard arguments (see [FMV, Proposition 9]) we can deduce from (13) and (14) that

$$\|L^{\frac{\sigma}{2}}\Delta_j u\|_{L^p(\mathbb{H}_n)} \leq C2^{j\sigma}\|\Delta_j u\|_{L^p(\mathbb{H}_n)}, \quad \sigma \in \mathbb{R},\ j \in \mathbb{Z},\ 1 \leq p \leq \infty,\ u \in \mathcal{S}'(\mathbb{H}_n), \tag{15}$$

where both sides of (15) are allowed to be infinite.

By the spectral theorem, the following homogeneous Littlewood–Paley decomposition holds for any $f \in L^2(\mathbb{H}_n)$:

$$f = \sum_{j\in\mathbb{Z}} \Delta_j f \quad \text{in } L^2(\mathbb{H}_n). \tag{16}$$

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So

$$\|f\|_{L^\infty(\mathbb{H}_n)} \le \sum_{j\in\mathbb{Z}} \|\Delta_j f\|_{L^\infty(\mathbb{H}_n)}, \quad f \in L^2(\mathbb{H}_n), \tag{17}$$

where both sides of (17) are allowed to be infinite.

The methods of [St], together with any multiplier theorem for L (see [A]; see also [He, MS] for $L = \Delta$, [MRS1, MRS2] for $L = \mathcal{L}$), yield the following Littlewood–Paley theorem.

Proposition 3.2 *Let $1 < p < \infty$ and $u \in \mathcal{S}'(\mathbb{H}_n)$. The following facts are equivalent:*

(i) *$u \in L^p(\mathbb{H}_n)$;*

(ii) *$u = \sum_{j\in\mathbb{Z}} \Delta_j u$ in $\mathcal{S}'(\mathbb{H}_n)$ and $(\sum_{j\in\mathbb{Z}} |\Delta_j u|^2)^{\frac12} \in L^p(\mathbb{H}_n)$.*

Moreover, if $u \in L^p(\mathbb{H}_n)$, then

$$\|u\|_{L^p(\mathbb{H}_n)} \sim \Big\| \Big(\sum_{j\in\mathbb{Z}} |\Delta_j u|^2\Big)^{\frac12} \Big\|_{L^p(\mathbb{H}_n)}.$$

Remark For $L = \Delta$, using the homogeneity property (12), Proposition 3.2 has also been proved in [BGX, Proposition 2.3].

Let $q, r \in [1, \infty]$ and $\rho \in \mathbb{R}$. The homogeneous Besov space $\dot{B}^{\rho,q}_r(L)$ associated to the operator L is defined as follows:

$$\dot{B}^{\rho,q}_r(L) = \Big\{ u \in \mathcal{S}'(\mathbb{H}_n) : u = \sum_{j\in\mathbb{Z}} \Delta_j u \text{ in } \mathcal{S}'(\mathbb{H}_n) \\ \text{and } \{2^{j\rho}\|\Delta_j u\|_{L^r(\mathbb{H}_n)}\}_{j\in\mathbb{Z}} \in l^q(\mathbb{Z}) \Big\}.$$

In the following proposition, we collect all the needed properties about the spaces $\dot{B}^{\rho,q}_r(L)$.

Proposition 3.3 *Let $q, r \in [1, \infty]$ and $\rho < \frac{N}{r}$.*

(i) *The space $\dot{B}^{\rho,q}_r(L)$ is a Banach space endowed with the norm*

$$\|u\|_{\dot{B}^{\rho,q}_r(L)} = \big\| \{2^{j\rho}\|\Delta_j u\|_{L^r(\mathbb{H}_n)}\}_{j\in\mathbb{Z}} \big\|_{l^q(\mathbb{Z})}.$$

(ii) *The definition of $\dot{B}^{\rho,q}_r(L)$ does not depend on the choice of the function R in the Littlewood–Paley decomposition.*

(iii) *For any $u \in \mathcal{S}'(\mathbb{H}_n)$ and $\sigma > 0$ we have that $u \in \dot{B}^{\rho,q}_r(L)$ if and only if $L^{\frac{\sigma}{2}} u \in \dot{B}^{\rho-\sigma,q}_r(L)$, with*

$$\|u\|_{\dot{B}^{\rho,q}_r(L)} \sim \|L^{\frac{\sigma}{2}} u\|_{\dot{B}^{\rho-\sigma,q}_r(L)}.$$

(iv) *The inclusion $\dot{B}^{\rho,q}_r(L) \subset \mathcal{S}'(\mathbb{H}_n)$ is continuous.*

(v) *If $-\frac{N}{r'} < \rho < \frac{N}{r}$, then $\mathcal{S}(\mathbb{H}_n) \subset \dot{B}^{\rho,q}_r(L)$ with continuous inclusion.*

(vi) *If $q, r \in [1,\infty)$ and $-\frac{N}{r'} < \rho < \frac{N}{r}$, then $\mathcal{S}(\mathbb{H}_n)$ is dense in $\dot{B}_r^{\rho,q}(L)$.*

(vii) *If $q, r \in [1,\infty)$ and $-\frac{N}{r'} < \rho < \frac{N}{r}$, the dual space of $\dot{B}_r^{\rho,q}(L)$ is $\dot{B}_{r'}^{-\rho,q'}(L)$.*

(viii) *For all $q \in [1,\infty]$ and $\alpha \in [N-1, N]$ we have the continuous inclusions*

$$\dot{B}_{r_1}^{\rho_1,q}(\mathcal{L}) \subset \dot{B}_{r_2}^{\rho_2,q}(\mathcal{L}), \quad \frac{1}{r_1} - \frac{\rho_1}{\alpha} = \frac{1}{r_2} - \frac{\rho_2}{\alpha},\ \rho_1 \geq \rho_2,$$

$$\dot{B}_{r_1}^{\rho_1,q}(\Delta) \subset \dot{B}_{r_2}^{\rho_2,q}(\Delta), \quad \frac{1}{r_1} - \frac{\rho_1}{N} = \frac{1}{r_2} - \frac{\rho_2}{N},\ \rho_1 \geq \rho_2.$$

(ix) *For all $r \in [2,\infty)$ we have the continuous inclusion $\dot{B}_r^{0,2}(L) \subset L^r(\mathbb{H}_n)$.*

(x) *$\dot{B}_2^{0,2}(L) = L^2(\mathbb{H}_n)$ with equivalent norms.*

(xi) *For all $\vartheta, \rho_1, \rho_2, q_1, q_2, r_1, r_2$ satisfying $\vartheta \in [0,1]$, $q_i, r_i \in (1,\infty), \rho_i < \frac{N}{r_i}$, we have $[\dot{B}_{r_1}^{\rho_1,q_1}(L), \dot{B}_{r_2}^{\rho_2,q_2}(L)]_\vartheta = \dot{B}_r^{\rho,q}(L)$ with $\rho = (1-\vartheta)\rho_1 + \vartheta\rho_2$, $\frac{1}{q} = \frac{1-\vartheta}{q_1} + \frac{\vartheta}{q_2}$ and $\frac{1}{r} = \frac{1-\vartheta}{r_1} + \frac{\vartheta}{r_2}$.*

We omit the proof of Proposition 3.3. In fact, all the statements of the proposition are well known for the spaces $\dot{B}_r^{\rho,q}(\Delta)$ (see [BG, BGX, FV]) and the proofs for the spaces $\dot{B}_r^{\rho,q}(\mathcal{L})$ are analogous. The only properties really needed are estimates (14) and (15), Proposition 3.2, and the fact that the kernel of $h(\mathcal{L})$ is in $\mathcal{S}(\mathbb{H}_n)$ if $h \in \mathcal{S}(\mathbb{R})$ (see §2). Once we have these properties, we can prove Proposition 3.3 by the methods in [P], which do not involve any homogeneity property. More generally, we could define homogeneous Besov spaces and prove, with the same methods, an analogous proposition in the more general context of a nilpotent Lie group G endowed with a sub-Laplacian $L = -\sum_{j=1}^k X_j^2$, where $X_1, \dots, X_k$ are left-invariant vector fields on G which satisfy the Hörmander's condition, *i.e.,* they generate, together with their successive Lie brackets $[X_{i_1}, [\dots, X_{i_\alpha}] \cdots]$, the Lie algebra of G. For more details about properties of Besov spaces in this context, see [S1, S2, FMV], where nevertheless inhomogeneous Besov spaces are considered. Here we want to prove some continuous inclusions between the two kinds of homogeneous Besov spaces which we have introduced.

Proposition 3.4 *The following continuous inclusions hold:*

$$\dot{B}_r^{\rho,q}(\mathcal{L}) \subset \dot{B}_r^{\rho,q}(\Delta), \quad 1 \leq q \leq \infty,\ 1 \leq r < \infty,\ 0 < \rho < \frac{N}{r}; \tag{18}$$

$$\dot{B}_r^{\rho,q}(\Delta) \subset \dot{B}_r^{\rho,q}(\mathcal{L}), \quad 1 \leq q \leq \infty,\ 1 < r \leq \infty,\ -\frac{N}{r'} < \rho < 0. \tag{19}$$

Proof We only prove (18), since the proof of (19) is analogous. Fix $u \in \dot{B}_r^{\rho,q}(\mathcal{L})$, with $1 \leq q \leq \infty$, $1 \leq r < \infty$ and $0 < \rho < \frac{N}{r}$. Since $u = \sum_{j\in\mathbb{Z}} u * \psi_j$ in $\mathcal{S}'(\mathbb{H}_n)$, by Lemma 3.1 we have $u * \varphi_k = \sum_{j \geq k-2} u * \psi_j * \varphi_k$ in $\mathcal{S}'(\mathbb{H}_n)$ for any $k \in \mathbb{Z}$, and so

$$\begin{aligned} 2^{k\rho}\|u * \varphi_k\|_{L^r(\mathbb{H}_n)} &\leq 2^{k\rho} \sum_{j \geq k-2} \|u * \psi_j * \varphi_k\|_{L^r(\mathbb{H}_n)} \\ &\leq C \sum_{j \geq k-2} 2^{(k-j)\rho} 2^{j\rho} \|u * \psi_j\|_{L^r(\mathbb{H}_n)}, \end{aligned}$$

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by (13). Therefore, by Young's inequality

$$\Big\| \{2^{k\rho} \|u * \varphi_k\|_{L^r(\mathbb{H}_n)}\}_{k\in\mathbb{Z}} \Big\|_{l^q(\mathbb{Z})} \le C \|u\|_{\dot{B}^{\rho,q}_r(\mathcal{L})}.$$

We still have to prove that $u = \sum_{k\in\mathbb{Z}} u * \varphi_k$ in $\mathcal{S}'(\mathbb{H}_n)$. By Lemma 3.1, for any $f \in \mathcal{S}(\mathbb{H}_n)$ we have

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |\langle u * \psi_j * \varphi_k, f\rangle| &= \sum_{j\in\mathbb{Z}} \sum_{h=-\infty}^{2} |\langle u * \psi_j, f * \varphi_{j+h}\rangle| \\
&\le \sum_{h=-\infty}^{2} 2^{h\rho} \Big(\sum_{j\in\mathbb{Z}} 2^{j\rho} \|u * \psi_j\|_{L^r(\mathbb{H}_n)} 2^{-(j+h)\rho} \|f * \varphi_{j+h}\|_{L^{r'}(\mathbb{H}_n)} \Big) \\
&\le \sum_{h=-\infty}^{2} 2^{h\rho} \|u\|_{\dot{B}^{\rho,q}_r(\mathcal{L})} \|f\|_{\dot{B}^{-\rho,q'}_{r'}(\Delta)} < +\infty.
\end{aligned}$$

Note that $\varphi_k = \sum_{j\in\mathbb{Z}} \varphi_k * \psi_j$ in $\mathcal{S}(\mathbb{H}_n)$ for any $k \in \mathbb{Z}$, by (16) and Lemma 3.1. Therefore, since $u = \sum_{j\in\mathbb{Z}} u * \psi_j$ in $\mathcal{S}'(\mathbb{H}_n)$, by Fubini's theorem we have

$$\langle u, f\rangle = \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \langle u * \psi_j * \varphi_k, f\rangle = \sum_{k\in\mathbb{Z}} \langle u * \varphi_k, f\rangle, \quad f \in \mathcal{S}(\mathbb{H}_n). \qquad \blacksquare$$

However, with the exception of particular cases $\rho = 0$, $q = r = 2$ (see Proposition 3.3(x)), the spaces $\dot{B}^{\rho,q}_r(\Delta)$ and $\dot{B}^{\rho,q}_r(\mathcal{L})$ do not coincide. For example, by applying the Godement–Plancherel's formula and arguing as in the proof of Lemma 3.1, it is not difficult to check that for $j \to +\infty$ we have

$$\|\varphi_j\|_{\dot{B}^{\rho,q}_2(\Delta)} \sim 2^{j(\rho+\frac{N}{2})}, \quad \|\varphi_j\|_{\dot{B}^{\rho,q}_2(\mathcal{L})} \sim 2^{j(2\rho+\frac{N}{2})}, \quad 1 \le q \le \infty,\ 0 < \rho < \frac{N}{2}.$$

As a further evidence, in the following we will see that the spaces $\dot{B}^{\rho,q}_r(\Delta)$ and $\dot{B}^{\rho,q}_r(\mathcal{L})$ have very different behaviour with respect to Strichartz estimates for the solution of the Cauchy problem (1).

4 Dispersive Estimates

We begin by proving Proposition 1.1. Let us introduce the tools of the method; first we recall the stationary phase lemma [St, pp. 332–334] which will be the central argument.

Lemma 4.1 *Suppose $g, h \in C^\infty([a,b])$, with g real-valued and $h(b) = 0$. Suppose also $|g^{(k)}(x)| \ge \delta$ for any $x \in [a,b]$, with $k \in \mathbb{Z}_+$ and $\delta > 0$. If $k = 1$, we also require that g' is monotonic in $[a,b]$. Then there exists a constant $C_k > 0$, which depends only on k but not on a, b, g, h, δ, such that*

$$\Big| \int_a^b e^{-ig(x)} h(x)\, dx \Big| \le C_k \delta^{-\frac{1}{k}} \int_a^b |h'(x)|\, dx.$$

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Moreover, we will use the following properties of the Laguerre polynomials (see [BGX, EMOT]).

Lemma 4.2 *Fix $\alpha \in \mathbb{N}$. There exists $C_\alpha > 0$ such that for $\tau \geq 0$ and $m \in \mathbb{N}$ we have*

$$\left| L_m^{(\alpha)}(\tau) e^{-\frac{\tau}{2}} \right| \leq C_\alpha (m+1)^\alpha, \qquad \left| \tau \frac{d}{d\tau} \left(L_m^{(\alpha)}(\tau) e^{-\frac{\tau}{2}} \right) \right| \leq C_\alpha (m+1)^\alpha.$$

Finally, we will exploit the following estimates, which can be easily proved by comparing the sums with the corresponding integrals.

Lemma 4.3 *Fix $\beta \in \mathbb{R}$. There exists $C_\beta > 0$ such that for $0 < a < b$ and $n \in \mathbb{Z}_+$ we have*

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\geq a}} (2m+n)^\beta \leq C_\beta a^{\beta+1}, \quad \beta < -1; \tag{20}$$

$$\sum_{\substack{m\in\mathbb{N}\\ 2m+n\leq b}} (2m+n)^\beta \leq C_\beta b^{\beta+1}, \quad \beta > -1; \tag{21}$$

$$\sum_{\substack{m\in\mathbb{N}\\ a\leq 2m+n\leq b}} (2m+n)^{-1} \leq \log\left(C\frac{b}{a} \right). \tag{22}$$

We can now prove the following.

Proposition 4.4 *There exists a constant $C > 0$, which depends only on n, such that for any $\rho \in [N - \frac{3}{2}, N - \frac{1}{2}]$, $j \in \mathbb{Z}$ and $t \in \mathbb{R}^*$ we have*

$$\| e^{-it\sqrt{\mathcal{L}}} \psi_j \|_{L^\infty(\mathbb{H}_n)} \leq C |t|^{-\frac{1}{2}} 2^{j\rho}.$$

Proof Fix $t \in \mathbb{R}^*$, $j \in \mathbb{Z}$ and $(z,s) \in \mathbb{H}_n$. By (4), (8) and (10), putting $\sigma = \frac{s}{t}$ and $M = 2m + n$ inside the sum over m, we have

$$e^{-it\sqrt{\mathcal{L}}} \psi_j(z,s) = \frac{2^{n-1}}{\pi^{n+1}} \sum_{m=0}^{+\infty} \int_{\mathbb{R}} e^{-it(\sigma\lambda + \sqrt{4M|\lambda| + \lambda^2})} R(2^{-2j}(4M|\lambda| + \lambda^2)) \times e^{-|\lambda||z|^2} L_m^{(n-1)}(2|\lambda||z|^2) |\lambda|^n \, d\lambda.$$

Performing the change of variable $x = 2^{-2j} M\lambda$, we obtain

$$e^{-it\sqrt{\mathcal{L}}} \psi_j(z,s) = \frac{2^{n-1}}{\pi^{n+1}} 2^{Nj} \sum_{m=0}^{+\infty} \int_{\mathbb{R}} e^{-it2^{2j} g_{j,\sigma,m}(x)} h_{j,z,m}(x) \, dx,$$

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where

$$g_{j,\sigma,m}(x) = \frac{1}{M}\Big(\sigma x + \sqrt{2^{2-2j}M^2|x| + x^2}\Big), \tag{23}$$

$$h_{j,z,m}(x) = R\Big(4|x| + \frac{2^{2j}x^2}{M^2}\Big)\, e^{-\frac{2^{2j}|x||z|^2}{M}} L_m^{(n-1)}\Big(\frac{2^{1+2j}|x||z|^2}{M}\Big)\frac{|x|^n}{M^{n+1}}. \tag{24}$$

So

(25)

$$\operatorname{supp} h_{j,z,m} \subset \Big\{x \in \mathbb{R} : \frac{1}{4} \le 4|x| + \frac{2^{2j}x^2}{M^2} \le 4\Big\} = \{x \in \mathbb{R} : a_{j,m} \le |x| \le b_{j,m}\Big\},$$

where

$$a_{j,m} = \frac{1}{8(1+\sqrt{1+2^{2j-4}M^{-2}})}, \qquad b_{j,m} = \frac{2}{1+\sqrt{1+2^{2j}M^{-2}}}.$$

In particular

$$b_{j,m} \le \min\{1, 2^{1-j}M\}. \tag{26}$$

Note that $g_{j,\sigma,m}(-x) = g_{j,-\sigma,m}(x)$ and $h_{j,z,m}(-x) = h_{j,z,m}(x)$. Therefore, by symmetry we can consider only the integrals

$$I_m = \int_{a_m}^{b_m} e^{-it2^{2j}g_m(x)} h_m(x)\,dx$$

where we write g_m, h_m, a_m, b_m for $g_{j,\sigma,m}, h_{j,z,m}, a_{j,m}, b_{j,m}$, respectively. We prove that

$$\sum_{m=0}^{+\infty} |I_m| \le \begin{cases} C|t|^{-\frac{1}{2}}2^{-\frac{3}{2}j} & j \ge 0, \\ C|t|^{-\frac{1}{2}}2^{-\frac{j}{2}} & j < 0. \end{cases} \tag{27}$$

For $x \in [a_m, b_m]$, by (23) we have

$$g_m'(x) = \frac{1}{M}\Big(\sigma + \sqrt{1 + \frac{2^{2-4j}M^4}{2^{2-2j}M^2x + x^2}}\Big), \tag{28}$$

$$g_m''(x) = -2^{2-4j}M^3(2^{2-2j}M^2x + x^2)^{-\frac{3}{2}}. \tag{29}$$

Note that by (25) we have

$$2^{-2-2j}M^2 \le 2^{2-2j}M^2x + x^2 \le 2^{2-2j}M^2, \quad x \in [a_m, b_m]. \tag{30}$$

So (29) and (30) yield

$$2^{-1-j} \le |g_m''(x)| \le 2^{5-j}, \quad x \in [a_m, b_m]. \tag{31}$$

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Furthermore, by Lemma 4.2 and (26), one can verify that

$$\|h_m'\|_{L^1([a_m,b_m])} \le \begin{cases} C2^{-nj}M^{n-2} & M \le 2^j, \\ CM^{-2} & M > 2^j. \end{cases} \tag{32}$$

So, by Lemma 4.1 with $k = 2$, we obtain

$$|I_m| \le \begin{cases} C|t|^{-\frac{1}{2}}2^{-(n+\frac{1}{2})j}M^{n-2} & M \le 2^j, \\ C|t|^{-\frac{1}{2}}2^{-\frac{j}{2}}M^{-2} & M > 2^j. \end{cases} \tag{33}$$

For $j < 0$, (27) follows directly from (33). For $n \ge 2$ and $j \ge 0$, (27) still follows from (33) by applying Lemma 4.3 separately to the sums $\sum_{M \le 2^j} |I_m|$ and $\sum_{M > 2^j} |I_m|$. But for $n = 1$ and $j \ge 0$, this argument does not work, since we cannot apply (21) to the sum $\sum_{M \le 2^j} |I_m|$.

So, from now on we assume $n = 1$ and $j \ge 0$. We divide $\mathbb{N}$ into five (possibly empty) disjoint subsets:

$$A_1 = \{m \in \mathbb{N} : M > 2^j\},$$

$$A_2 = \{m \in \mathbb{N} : M \le 2^j, M \le |t|^{-\frac{1}{2}}2^{\frac{j}{2}}\},$$

$$A_3 = \{m \in \mathbb{N} : M \le 2^j, M > |t|^{-\frac{1}{2}}2^{\frac{j}{2}}, \sigma \ge -\sqrt{1 + 2^{-1-2j}M^2}\},$$

$$A_4 = \{m \in \mathbb{N} : M \le 2^j, M > |t|^{-\frac{1}{2}}2^{\frac{j}{2}}, \sigma \le -\sqrt{1 + 2^{5-2j}M^2}\},$$

$$A_5 = \{m \in \mathbb{N} : M \le 2^j, M > |t|^{-\frac{1}{2}}2^{\frac{j}{2}}, -\sqrt{1 + 2^{5-2j}M^2} < \sigma < -\sqrt{1 + 2^{-1-2j}M^2}\}.$$

Then our assertion reads:

$$\sum_{m \in A_r} |I_m| \le C|t|^{-\frac{1}{2}}2^{-\frac{3}{2}j}, \quad r = 1, \dots, 5,\ A_r \ne \varnothing. \tag{34}$$

We prove (34) separately for each r, each time using Lemma 4.3 , *i.e.,* we will use (20) for $r = 1, 3, 4$, (21) for $r = 2$ and (22) for $r = 5$. The case $r = 1$ can be treated as for $n \ge 2$. For $r = 2$, we estimate $\sum_{M \le |t|^{-\frac{1}{2}}2^{\frac{j}{2}}} |I_m|$ by means of the inequality

$$|I_m| \le C\|h_m\|_{L^1([a_m,b_m])} \le C2^{-2j},$$

which follows from (24), (26) and Lemma 4.2. For $r = 3, 4$ we estimate

$$\sum_{M > |t|^{-\frac{1}{2}}2^{\frac{j}{2}}} |I_m|$$

by means of Lemma 4.1 applied with $k = 1$, using (32) and the estimates

$$g_m'(x) \ge \frac{1}{M}\left(\sqrt{1 + 2^{-2j}M^2} - \sqrt{1 + 2^{-1-2j}M^2}\right) \ge C2^{-2j}M, \quad m \in A_3,$$

$$-g_m'(x) \ge \frac{1}{M}\left(\sqrt{1 + 2^{5-2j}M^2} - \sqrt{1 + 2^{4-2j}M^2}\right) \ge C2^{-2j}M, \quad m \in A_4,$$

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which are consequences of (28) and (30). For $r = 5$, we note that $A_5 \neq \varnothing$ implies $\sigma < -1$ and $M \in J = (2^{j-\frac{5}{2}}\sqrt{\sigma^2-1}, 2^{j+\frac{1}{2}}\sqrt{\sigma^2-1})$. Then we estimate $\sum_{M\in J}|I_m|$ by means of Lemma 4.1 applied with $k = 2$. ■

From Proposition 4.4 we can obtain the following by the same proof as in [BGX, pp. 114–115], [FV, Corollary 10].

Corollary 4.5 *For $\rho \in [N-\frac{3}{2}, N-\frac{1}{2}]$ there exists a constant $C_\rho > 0$ such that*

(35) $$\big\| e^{-it\sqrt{\mathcal{L}}} f \big\|_{L^\infty(\mathbb{H}_n)} \le C_\rho |t|^{-\frac{1}{2}} \|f\|_{\dot{B}_1^{\rho,1}(\mathcal{L})}, \quad f \in \mathcal{S}(\mathbb{H}_n),\ t \in \mathbb{R}^*,$$

(36) $$\big\| e^{-it\sqrt{\mathcal{L}}} f \big\|_{\dot{B}_\infty^{-1,1}(\mathcal{L})} \le C_\rho |t|^{-\frac{1}{2}} \|f\|_{\dot{B}_1^{\rho-1,1}(\mathcal{L})}, \quad f \in \mathcal{S}(\mathbb{H}_n),\ t \in \mathbb{R}^*.$$

The proof of the dispersive inequality is now straightforward.

Proof of Proposition 1.1 By (35) we obtain

$$\|\cos t\sqrt{\mathcal{L}} u_0\|_{L^\infty(\mathbb{H}_n)} \le C|t|^{-\frac{1}{2}} \|u_0\|_{\dot{B}_1^{\rho,1}(\mathcal{L})}$$

and by (17), (15) and (36) we obtain

$$\begin{aligned}\Big\| \frac{\sin t\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}} u_1 \Big\|_{L^\infty(\mathbb{H}_n)} &\le \sum_{j\in\mathbb{Z}} \Big\| \frac{\sin t\sqrt{\mathcal{L}}}{\sqrt{\mathcal{L}}} u_1 * \psi_j \Big\|_{L^\infty(\mathbb{H}_n)} \\ &\le \sum_{j\in\mathbb{Z}} 2^{-j} \|\sin t\sqrt{\mathcal{L}} u_1 * \psi_j\|_{L^\infty(\mathbb{H}_n)} \le C|t|^{-\frac{1}{2}} \|u_1\|_{\dot{B}_1^{\rho-1,1}(\mathcal{L})}. \end{aligned}$$ ■

5 Strichartz Inequalities

We can now prove Theorem 1.2 and Corollary 1.3.

Proof of Theorem 1.2 By (2) we can write

$$\partial_t v(t) = -\frac{e^{it\sqrt{\mathcal{L}}} - e^{-it\sqrt{\mathcal{L}}}}{2i} \sqrt{\mathcal{L}} u_0 + \frac{e^{it\sqrt{\mathcal{L}}} + e^{-it\sqrt{\mathcal{L}}}}{2} u_1,$$

where $\sqrt{\mathcal{L}} u_0$ and u_1 both belong to $L^2(\mathbb{H}_n)$. Analogously, by (3)

$$\partial_t w(t) = \int_0^t \frac{e^{i(t-\sigma)\sqrt{\mathcal{L}}} + e^{-i(t-\sigma)\sqrt{\mathcal{L}}}}{2} f(\sigma)\, d\sigma.$$

So

$$\begin{aligned} &\|v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1,2}(\mathcal{L}))} + \|\partial_t v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1-1,2}(\mathcal{L}))} \\ &\qquad \le C\big(\|\sqrt{\mathcal{L}}\, v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1-1,2}(\mathcal{L}))} + \|\partial_t v\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1-1,2}(\mathcal{L}))} \big) \\ &\qquad \le C\big(\|e^{-it\sqrt{\mathcal{L}}} \sqrt{\mathcal{L}} u_0\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1-1,2}(\mathcal{L}))} + \|e^{-it\sqrt{\mathcal{L}}} u_1\|_{L^{p_1}_{\mathbb{R}}(\dot{B}_{r_1}^{\rho_1-1,2}(\mathcal{L}))} \big) \end{aligned}$$

and

$$\begin{aligned}\|w\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1,2}(\mathcal{L}))} + \|\partial_t w\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1-1,2}(\mathcal{L}))} &\le C\big(\|\sqrt{\mathcal{L}}\,w\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1-1,2}(\mathcal{L}))} + \|\partial_t w\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1-1,2}(\mathcal{L}))}\big)\\ &\le C\Big(\Big\|\int_0^t e^{i(t-\sigma)\sqrt{\mathcal{L}}} f(\sigma)\,d\sigma\Big\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1-1,2}(\mathcal{L}))}\\ &\qquad + \Big\|\int_0^t e^{-i(t-\sigma)\sqrt{\mathcal{L}}} f(\sigma)\,d\sigma\Big\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1-1,2}(\mathcal{L}))}\Big).\end{aligned}$$

Therefore, if we substitute $\rho_1 - 1$ for ρ_1 in the following theorem, Theorem 1.2 easily follows.

Theorem 5.1 *Let $r_1, r_2 \in [2,\infty]$. Let $\rho_1, \rho_2 \in \mathbb{R}$ and $p_1, p_2 \in [1,\infty]$ such that*
(i) $\frac{2}{p_i} = \frac{1}{2} - \frac{1}{r_i}$ *for $i = 1,2$;*
(ii) $-(N-\frac{1}{2})(\frac{1}{2}-\frac{1}{r_i}) \le \rho_i \le -(N-\frac{3}{2})(\frac{1}{2}-\frac{1}{r_i})$ *for $i = 1,2$.*
Let r_i', p_i' such that $\frac{1}{r_i'} + \frac{1}{r_i} = 1$ and $\frac{1}{p_i'} + \frac{1}{p_i} = 1$ for $i = 1,2$. Then for every interval I which contains 0, the following estimates are satisfied:

$$\|e^{-it\sqrt{\mathcal{L}}}u_0\|_{L_{\mathbb{R}}^{p_1}(\dot B_{r_1}^{\rho_1,2}(\mathcal{L}))} \le C\|u_0\|_{L^2(\mathbb{H}_n)},$$

$$\Big\|\int_0^t e^{\pm i(t-\sigma)\sqrt{\mathcal{L}}} f(\sigma)\,d\sigma\Big\|_{L_I^{p_1}(\dot B_{r_1}^{\rho_1,2}(\mathcal{L}))} \le C\|f\|_{L_I^{p_2'}(\dot B_{r_2'}^{-\rho_2,2}(\mathcal{L}))},$$

where the constant $C > 0$ depends neither on u_0, f nor on the interval I.

We omit the proof of Theorem 5.1. In fact, once we have obtained Proposition 4.4, the procedure is classical and a good reference is given, for example, in the papers by Ginibre and Velo [GV] or by Ginibre [Gi]. A detailed presentation in this framework is also given in [FV].

Proof of Corollary 1.3 Let us remark first that for $\rho_1 \ge 0$, (viii) and (ix) of Proposition 3.3 imply

$$\dot B_{r_1}^{\rho_1,2}(\mathcal{L}) \subset \dot B_{r_{\min}}^{0,2}(\mathcal{L}) \cap \dot B_{r_{\max}}^{0,2}(\mathcal{L}) \subset L^{r_{\min}}(\mathbb{H}_n) \cap L^{r_{\max}}(\mathbb{H}_n), \tag{37}$$

where $\frac{1}{r_{\min}} = \frac{1}{r_1} - \frac{\rho_1}{N}$ and $\frac{1}{r_{\max}} = \frac{1}{r_1} - \frac{\rho_1}{N-1}$. If we take $\rho_1 = -(N-\frac{3}{2})(\frac{1}{2}-\frac{1}{r_1}) + 1$ in Theorem 1.2, we have $\rho_1 \ge 0$ if and only if $r_1 \le \frac{2(2N-3)}{2N-7}$. Taking into account also the condition $r_1 \ge 2$, which corresponds to $\rho_1 \le 1$, we obtain by (37) the extremal spaces

$$\begin{aligned}u = v + w &\in L_I^\infty(\dot B_2^{1,2}(\mathcal{L})) \cap L_I^{2N-3}\big(\dot B^{0,2}_{\frac{2(2N-3)}{2N-7}}(\mathcal{L})\big)\\ &\subset L_I^\infty\big(L^{\frac{2N}{N-2}}(\mathbb{H}_n) \cap L^{\frac{2(N-1)}{N-3}}(\mathbb{H}_n)\big) \cap L_I^{2N-3}\big(L^{\frac{2(2N-3)}{2N-7}}(\mathbb{H}_n)\big).\end{aligned}$$

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On the other hand, taking $\rho_1 = -(N-\frac{1}{2})(\frac{1}{2}-\frac{1}{r_1})+1$ in Theorem 1.2, we have $\rho_1 \geq 0$ if and only if $r_1 \leq \frac{2(2N-1)}{2N-5}$. The other bound $r_1 \geq 2$ still corresponds to $\rho_1 \leq 1$, and therefore we obtain the extremal spaces

$$u = v + w \in L^\infty_I(\dot{B}^{1,2}_2(\mathcal{L})) \cap L^{2N-1}_I\big(\dot{B}^{0,2}_{\frac{2(2N-1)}{2N-5}}(\mathcal{L})\big)$$

$$\subset L^\infty_I\big(L^{\frac{2N}{N-2}}(\mathbb{H}_n) \cap L^{\frac{2(N-1)}{N-3}}(\mathbb{H}_n)\big) \cap L^{2N-1}_I\big(L^{\frac{2(2N-1)}{2N-5}}(\mathbb{H}_n)\big).$$

By interpolation we obtain $u \in L^p_I(L^r(\mathbb{H}_n))$ with $0 \leq \frac{2}{p} \leq \frac{1}{2} - \frac{1}{r}$ and $(N-1)\big(\frac{1}{2}-\frac{1}{r}\big) - 1 \leq \frac{1}{p} \leq N\big(\frac{1}{2}-\frac{1}{r}\big) - 1$. ■

6 About the Sharpness of the Dispersive Estimates

We end this paper by discussing the sharpness of the dispersive estimate obtained in Proposition 1.1. Let us define the functions $v_j \in \mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$, $j \in \mathbb{Z}$, by

$$\widehat{v_j}(m,\lambda) = \begin{cases} R(2^{-2j}(4n\lambda+\lambda^2)) & \text{if } m=0, \lambda>0,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 6.1 *For any $\rho \in \mathbb{R}$ there exists $C_\rho > 0$ such that*

$$\|v_j\|_{\dot{B}^{\rho,1}_1(\mathcal{L})} \leq C_\rho 2^{j\rho}, \quad j \in \mathbb{Z}.$$

Proof We only have to prove the uniform estimate $\|v_j\|_{L^1(\mathbb{H}_n)} \leq C$, $j \in \mathbb{Z}$. Indeed

$$\widehat{v_j * \psi_k}(m,\lambda) = \begin{cases} R(2^{-2j}(4n\lambda+\lambda^2))R(2^{-2k}(4n\lambda+\lambda^2)) & \text{if } m=0, \lambda>0,\\ 0 & \text{otherwise,}\end{cases}$$

implies $v_j * \psi_k = 0$ if $|j-k| \geq 2$. Therefore by (14)

$$\|v_j\|_{\dot{B}^{\rho,1}_1(\mathcal{L})} = \sum_{k=j-1}^{j+1} 2^{k\rho}\|v_j*\psi_k\|_{L^1(\mathbb{H}_n)} \leq C_\rho \|v_j\|_{L^1(\mathbb{H}_n)} 2^{j\rho},$$

where C_ρ depends only on ρ.

Let us estimate $\|v_j\|_{L^1(\mathbb{H}_n)}$:

$$|v_j(z,s)| = \frac{2^{n-1}}{\pi^{n+1}}\Big|\int_{\lambda_1}^{\lambda_2} e^{-i\lambda s}R(2^{-2j}(4n\lambda+\lambda^2))e^{-\lambda|z|^2}\lambda^n\,d\lambda\Big|, \tag{38}$$

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where $\lambda_1 = \sqrt{4n^2 + 2^{2j-2}} - 2n$ and $\lambda_2 = \sqrt{4n^2 + 2^{2j+2}} - 2n$. Then for $s \neq 0$

$$
\begin{aligned}
(39) \qquad |v_j(z,s)| &= \frac{2^{n-1}}{\pi^{n+1}} \Big| \int_{\lambda_1}^{\lambda_2} \big(\frac{d}{d\lambda} e^{-i\lambda s}\big) \frac{R(2^{-2j}(4n\lambda + \lambda^2)) e^{-\lambda |z|^2} \lambda^n}{is} \, d\lambda \Big| \\
&= \frac{2^{n-1}}{\pi^{n+1}} \frac{1}{|s|} \Big| \int_{\lambda_1}^{\lambda_2} e^{-i\lambda s} \frac{d}{d\lambda} (R(2^{-2j}(4n\lambda + \lambda^2)) e^{-\lambda |z|^2} \lambda^n) \, d\lambda \Big| \\
&= \frac{2^{n-1}}{\pi^{n+1}} \frac{1}{s^2} \Big| \int_{\lambda_1}^{\lambda_2} e^{-i\lambda s} \frac{d^2}{d\lambda^2} (R(2^{-2j}(4n\lambda + \lambda^2)) e^{-\lambda |z|^2} \lambda^n) \, d\lambda \Big| .
\end{aligned}
$$

So we have two possible ways to estimate $|v_j(z,s)|$: using (38)

$$
(40) \qquad |v_j(z,s)| \leq \begin{cases} C 2^{j(n+1)} e^{-\frac{2^j |z|^2}{C}} & j \geq 0, \\ C 2^{2j(n+1)} e^{-\frac{2^{2j} |z|^2}{C}} & j < 0, \end{cases}
$$

or using (39)

$$
(41) \qquad |v_j(z,s)| \leq \begin{cases} \frac{C}{s^2} 2^{j(n-1)} (1 + 2^j |z|^2 + 2^{2j} |z|^4) e^{-\frac{2^j |z|^2}{C}} & j \geq 0, \\ \frac{C}{s^2} 2^{2j(n-1)} (1 + 2^{2j} |z|^2 + 2^{4j} |z|^4) e^{-\frac{2^{2j} |z|^2}{C}} & j < 0. \end{cases}
$$

For $j \geq 0$, we have by (40)

$$
\begin{aligned}
\int_{\{|s|<2^{-j}, z\in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds &\leq C 2^{j(n+1)} \Big(\int_{\{|s|<2^{-j}\}} ds \Big) \Big(\int_{\mathbb{C}^n} e^{-\frac{2^j |z|^2}{C}} \, dz \Big) \\
&= C 2^j \Big(\int_{\{|s|<2^{-j}\}} ds \Big) \Big(\int_{\mathbb{C}^n} e^{-\frac{|w|^2}{C}} \, dw \Big) \leq C,
\end{aligned}
$$

and by (41),

$$
\begin{aligned}
&\int_{\{|s|\geq 2^{-j}, z\in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds \\
&\qquad \leq C 2^{j(n-1)} \Big(\int_{\{|s|\geq 2^{-j}\}} \frac{1}{s^2} \, ds \Big) \Big(\int_{\mathbb{C}^n} (1 + 2^j |z|^2 + 2^{2j} |z|^4) e^{-\frac{2^j |z|^2}{C}} \, dz \Big) \\
&\qquad = C 2^{-j} \Big(\int_{\{|s|\geq 2^{-j}\}} \frac{1}{s^2} \, ds \Big) \Big(\int_{\mathbb{C}^n} (1 + |w|^2 + |w|^4) e^{-\frac{|w|^2}{C}} \, dw \Big) \leq C.
\end{aligned}
$$

Therefore,

$$
\|v_j\|_{L^1(\mathbb{H}_n)} = \int_{\{|s|<2^{-j}, \, z\in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds + \int_{\{|s|\geq 2^{-j}, z\in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds \leq C.
$$

Similarly, for $j < 0$, we have

$$
\|v_j\|_{L^1(\mathbb{H}_n)} = \int_{\{|s|<2^{-2j}, z\in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds + \int_{\{|s|\geq 2^{-2j}, \in\mathbb{C}^n\}} |v_j(z,s)| \, dz ds \leq C.
$$

∎

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By the definition of the functions v_j we have

$$\cos(t\sqrt{\mathcal{L}})v_j(0,\sigma_j t) = C2^{Nj}\int_0^{+\infty} e^{-it2^{2j}g_j(x)}h_j(x)\,dx + C2^{Nj}\int_0^{+\infty} e^{-it2^{2j}\tilde{g}_j(x)}h_j(x)\,dx, \tag{42}$$

where σ_j is a constant depending only on j, $g_j = g_{j,\sigma_j,0}$ and $h_j = h_{j,0,0}$ are the functions defined in (23) and (24) respectively, and

$$\tilde{g}_j(x) = \frac{1}{n}\left(\sigma_j x - \sqrt{2^{2-2j}n^2x + x^2}\right). \tag{43}$$

Lemma 6.2 *For any $j \in \mathbb{Z}$, let η_j be a function in $C^2(\mathbb{R})$ with $\operatorname{supp}\eta_j \subset [a_j, b_j]$ and let γ_j be a real-valued function in $C^4([a_j, b_j])$ with $\gamma_j'(x_j) = 0$ for some $x_j \in (a_j, b_j)$ and $\gamma_j''(x) \neq 0$ for any $x \in [a_j, b_j]$. Then there exists $T_j > 0$ such that*

$$\Big|\int_{a_j}^{b_j} e^{-it2^{2j}\gamma_j(x)}\eta_j(x)\,dx\Big| \geq \frac{\sqrt{\pi}}{2}t^{-\frac{1}{2}}2^{-j}|\gamma_j''(x_j)|^{-\frac{1}{2}}|\eta_j(x_j)|, \quad t > T_j.$$

Proof It is not restrictive to suppose $\gamma_j(x_j) = 0$ and $\eta_j(x_j) \neq 0$. Let ξ_j be the function defined by

$$\xi_j(x) = \begin{cases} -\sqrt{\frac{2\gamma_j(x)}{\gamma_j''(x_j)}} & x \in [a_j, x_j], \\ \sqrt{\frac{2\gamma_j(x)}{\gamma_j''(x_j)}} & x \in (x_j, b_j]. \end{cases}$$

It is not hard to check that $\xi_j \in C^3([a_j, b_j])$, $\xi_j' > 0$ on $[a_j, b_j]$ and $\xi_j'(x_j) = 1$. Performing the change of variable $y = \xi_j(x)$,

$$\int_{a_j}^{b_j} e^{-it2^{2j}\gamma_j(x)}\eta_j(x)\,dx = \int_{\xi_j(a_j)}^{\xi_j(b_j)} e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2}\Phi_j(y)\,dy,$$

where $\Phi_j \in C^2$, $\Phi_j(y) = \eta_j(\xi_j^{-1}(y))(\xi_j^{-1})'(y)$, $\operatorname{supp}\Phi_j \subset [\xi_j(a_j), \xi_j(b_j)]$, and $\Phi_j(0) = \eta_j(x_j)$. We can write

$$\int_{\xi_j(a_j)}^{\xi_j(b_j)} e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2}\Phi_j(y)\,dy = J_{j,t} + K_{j,t},$$

where

$$J_{j,t} = \int_{-\infty}^{+\infty} e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2}e^{-y^2}\Phi_j(0)\,dy = \frac{\sqrt{\pi}\eta_j(x_j)}{\sqrt{|1 + it2^{2j}\frac{\gamma_j''(x_j)}{2}|}}\,e^{-\frac{i}{2}\arctan(t2^{2j}\frac{\gamma_j''(x_j)}{2})}$$

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and

$$K_{j,t} = \int_{-\infty}^{+\infty} e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2}(\Phi_j(y) - e^{-y^2}\Phi_j(0))\,dy$$
$$= -\int_{-\infty}^{+\infty} \frac{d}{dy}(e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2})\frac{\Phi_j(y) - e^{-y^2}\Phi_j(0)}{it2^{2j}\gamma_j''(x_j)y}\,dy$$
$$= \frac{1}{it2^{2j}\gamma_j''(x_j)} \int_{-\infty}^{+\infty} e^{-it2^{2j}\frac{\gamma_j''(x_j)}{2}y^2}\frac{d}{dy}(\frac{\Phi_j(y) - e^{-y^2}\Phi_j(0)}{y})\,dy.$$

Therefore,

$$\Big|\int_{a_j}^{b_j} e^{-it2^{2j}\gamma_j(x)}\eta_j(x)\,dx\Big| \geq |J_{j,t}|\Big|1 - \frac{|K_{j,t}|}{|J_{j,t}|}\Big|,$$

and, since $y \mapsto (\Phi_j(y) - e^{-y^2}\Phi_j(0))/y$ is a function in $C^1(\mathbb{R})$ whose derivative is in $L^1(\mathbb{R})$, (as can be verified by direct calculation), we have

$$\frac{|K_{j,t}|}{|J_{j,t}|} \leq \frac{\sqrt{|1 + it\,2^{2j}\frac{\gamma_j''(x_j)}{2}|}}{\sqrt{\pi}\,|\eta_j(x_j)|}\frac{C_j}{|t\,2^{2j}\gamma_j''(x_j)|} \leq C_j' t^{-\frac{1}{2}} \leq \frac{1}{2}, \quad t \geq 4(C_j')^2,$$

where C_j and C_j' are positive constants depending on j but not on t. Thus we obtain

$$\Big|\int_{a_j}^{b_j} e^{-it2^{2j}\gamma_j(x)}\eta_j(x)\,dx\Big| \geq \frac{\sqrt{\pi}}{2}t^{-\frac{1}{2}}2^{-j}|\gamma_j''(x_j)|^{-\frac{1}{2}}|\eta_j(x_j)|, \quad t > T_j. \qquad \blacksquare$$

Returning to (42), for any $j \in \mathbb{Z}$ we can fix $x_j > 0$ such that $4x_j + \frac{2^{2j}x_j^2}{n^2} = 1$ and $\sigma_j < 0$ such that $g_j'(x_j) = 0$. By Lemma 6.2 and (31) we obtain the following lower estimates for $t > T_j$:

$$\Big|\int_0^{+\infty} e^{-it2^{2j}g_j(x)}h_j(x)\,dx\Big| \geq \begin{cases} Ct^{-\frac{1}{2}}2^{-(n+\frac{1}{2})j} & \text{if } j \geq 0, \\ Ct^{-\frac{1}{2}}2^{-\frac{j}{2}} & \text{if } j < 0. \end{cases} \tag{44}$$

In order to estimate the last integral in (42), we first remark that $\tilde{g}_j{}'(x) < 0$ for any $x \in \operatorname{supp} h_j \subset [a_j, b_j]$. Performing the change of variable $y = \tilde{g}_j(x)$,

$$\Big|\int_{a_j}^{b_j} e^{-it2^{2j}\tilde{g}_j(x)}h_j(x)\,dx\Big| = \Big|\int_{\tilde{g}_j(a_j)}^{\tilde{g}_j(b_j)} e^{-it2^{2j}y}H_j(y)\,dy\Big|,$$

where $H_j \in C^\infty$ and $H_j(y) = h_j(\tilde{g}_j{}^{-1}(y))(\tilde{g}_j{}^{-1})(y)$, $\operatorname{supp} H_j \subset [\tilde{g}_j(b_j), \tilde{g}_j(a_j)]$. Then, for any $j \in \mathbb{Z}$ there exist C_j, $T_j' > 0$ such that

$$\Big|\int_0^{+\infty} e^{-it2^{2j}\tilde{g}_j(x)}h_j(x)\,dx\Big| = |\widehat{H}_j(t2^{2j})| \leq C_j t^{-1}, \quad t > T_j'. \tag{45}$$

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By (42), (44), and (45), there exists $T_j'' > 0$ such that for $t > T_j''$:

$$\| \cos(t\sqrt{\mathcal{L}})v_j\|_{L^\infty(\mathbb{H}_n)} \geq \begin{cases} Ct^{-\frac{1}{2}}2^{(N-n-\frac{1}{2})j} & \text{if } j \geq 0, \\ Ct^{-\frac{1}{2}}2^{(N-\frac{1}{2})j} & \text{if } j < 0. \end{cases} \tag{46}$$

Sharpness in t Estimates (46) give, for instance, $\| \cos t\sqrt{\mathcal{L}}v_0\|_{L^\infty(\mathbb{H}_n)} \geq Ct^{-\frac{1}{2}}$, $t > T_0$. So the decay in t in Proposition 1.1 cannot be improved.

Sharpness in ρ Suppose that for some $\rho \in \mathbb{R}$ the estimate $\| \cos t\sqrt{\mathcal{L}}f\|_{L^\infty(\mathbb{H}_n)} \leq C_\rho |t|^{-\frac{1}{2}}\|f\|_{\dot{B}_1^{\rho,1}(\mathcal{L})}$ holds for any $f \in \mathcal{S}(\mathbb{H}_n)$. In particular, by Lemma 6.1,

$$\| \cos t\sqrt{\mathcal{L}}v_j\|_{L^\infty(\mathbb{H}_n)} \leq C_\rho |t|^{-\frac{1}{2}}2^{j\rho}, \quad j \in \mathbb{Z}.$$

Estimates (46) force $\rho \in [N - n - \frac{1}{2}, N - \frac{1}{2}]$.

Final remarks We would like to emphasise that there is no hope of obtaining a dispersive inequality as in Proposition 1.1 with the spaces $\dot{B}_r^{\rho,q}(\Delta)$. Let us define the functions $w_j \in \mathcal{S}_{\mathrm{rad}}(\mathbb{H}_n)$, $j \in \mathbb{Z}$, by

$$\widehat{w_j}(m,\lambda) = \begin{cases} R(2^{2-2j}n\lambda) & \text{if } m = 0, \lambda > 0, \\ 0 & \text{otherwise.} \end{cases}$$

By the inversion formula (4)

$$\begin{aligned} w_j(z,s) &= \frac{2^{n-1}}{\pi^{n+1}} \int_0^{+\infty} e^{-i\lambda s} R(2^{2-2j}n\lambda) e^{-\lambda |z|^2} \lambda^n \, d\lambda \\ &= \frac{2^{n-1}}{\pi^{n+1}} 2^{Nj} \int_0^{+\infty} e^{-i\nu 2^{2j}s} R(4n\nu) e^{-\nu |2^j z|^2} \nu^n \, d\nu \\ &= 2^{Nj} w_0(2^j z, 2^{2j}s). \end{aligned}$$

Therefore $\|w_j\|_{L^1(\mathbb{H}_n)} = \|w_0\|_{L^1(\mathbb{H}_n)}$. This implies, as for the functions v_j (see the proof of Lemma 6.1), that $\|w_j\|_{\dot{B}_1^{\rho,1}(\Delta)} \leq C_\rho 2^{j\rho}$, where C_ρ depends only on ρ. By the definition of w_j we have

$$\cos t\sqrt{\mathcal{L}}w_j(0,\sigma_j t) = C2^{Nj} \int_0^{+\infty} e^{-it2^{2j}g_j(x)} k(x)\, dx + C2^{Nj} \int_0^{+\infty} e^{-it2^{2j}\tilde{g}_j(x)} k(x)\, dx,$$

where σ_j is a constant depending only on j, $g_j = g_{j,\sigma_j,0}$ and $\tilde{g}_j$ are the functions defined in (23) and (43), respectively, and $k(x) = R(4x)\frac{x^n}{n^{n+1}}$. For any $j \in \mathbb{Z}$ we fix $x_j = \frac{1}{4}$ and $\sigma_j < 0$ such that $g_j'(\frac{1}{4}) = 0$. Arguing as before (see the proof of (46)) we obtain for $t > T_j$

$$\| \cos t\sqrt{\mathcal{L}}w_j\|_{L^\infty(\mathbb{H}_n)} \geq \begin{cases} Ct^{-\frac{1}{2}}2^{(N+1)j} & \text{if } j \geq 0, \\ Ct^{-\frac{1}{2}}2^{(N-\frac{1}{2})j} & \text{if } j < 0. \end{cases}$$

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These estimates imply that there is no $\rho \in \mathbb{R}$ for which

$$\| \cos t\sqrt{\mathcal{L}} f\|_{L^\infty(\mathbb{H}_n)} \leq C|t|^{-\frac{1}{2}}\|f\|_{\dot{B}_1^{\rho,1}(\Delta)}$$

for any $f \in \mathcal{S}(\mathbb{H}_n)$.

As a conclusion we would like to remark that analysing the wave equation related to the Kohn Laplacian Δ with the spaces $\dot{B}_r^{\rho,q}(\mathcal{L})$, we obtain the dispersive inequality for the wave semigroup: for any $\rho \in [N - \frac{3}{2}, N - \frac{1}{2}]$,

$$\left\| e^{-it\sqrt{\Delta}} f \right\|_{L^\infty(\mathbb{H}_n)} \leq C_\rho |t|^{-\frac{1}{2}} \|f\|_{\dot{B}_1^{\rho,1}(\mathcal{L})}, \qquad f \in \mathcal{S}(\mathbb{H}_n), t \in \mathbb{R}^*.$$

This result does not give Proposition 1.1 (unless $u_1 = 0$) because estimate (15) does not hold with $L = \Delta$ and $\Delta_j u = u * \psi_j$.

Finally, for the Schrödinger equation related to the full Laplacian, by Proposition 3.4 and [FV, Corollary 10] we have the dispersive estimate

$$\left\| e^{-it\mathcal{L}} f \right\|_{L^\infty(\mathbb{H}_n)} \leq C t^{-\frac{1}{2}} \|f\|_{\dot{B}_1^{N-2,1}(\mathcal{L})}, \quad f \in \mathcal{S}(\mathbb{H}_n), t > 0. \tag{47}$$

By a direct computation as in Section 4, the estimate (47) cannot be improved. So by analysing it with the spaces $\dot{B}_r^{\rho,q}(\mathcal{L})$, the behaviour of the Schrödinger operator $e^{-it\mathcal{L}}$ is the same as in [FV] with the spaces $\dot{B}_r^{\rho,q}(\Delta)$.

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Dipartimento di Ingegneria Gestionale e dell'Informazione
Università di Bergamo,
Viale Marconi 5
I-24044 Dalmine (BG)
Italy
e-mail: gfurioli@unibg.it

Dipartimento di Scienze Chimiche, Fisiche e Matematiche
Università dell'Insubria
Via Valleggio 11
I-22100 Como
Italy
e-mail: melzi@uninsubria.it

Dipartimento di Matematica
Università di Genova
Via Dodecaneso 35
I-16146 Genova
Italy
e-mail: veneruso@dima.unige.it